Dynamic Pollution Games*

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Abstract

This paper extends and challenges Mäler and de Zeeuw's (1996) acid rain linear-quadratic differential game. First, it generalises their model to other, degradable pollutants, and as a consequence questions the normative significance of 'critical loads'. Secondly, it contradicts their numerical result in the acid rain game that steady state pollution stocks in the linear Markovian equilibrium are an order of magnitude greater than the open-loop levels. Instead, it is shown that the difference between the linear subgame perfect and commitment equilibria depends critically on the nature of pollution transportation. Finally, it allows players to use non-linear Markovian strategies, and hence achieve partial cooperation in equilibrium (see Tsutsui and Mino (1990)). A central contribution of the paper is the numerical solution for non-linear equilibria in a model with multiple state variables and asymmetric players.

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1. Introduction

Pollution of the environment is, in many cases, a game. Often more than one party is involved; each party realises that it has some influence on the welfare of both itself and others (i.e. there is strategic interaction); and parties' interests conflict. By way of illustration, consider the specific example of acid rain within Europe. A relatively small number of parties is involved (twelve countries account for ninety percent of sulphur emissions in Europe); the emissions of a country fall on both itself and neighbouring countries; and each country would prefer all other countries to reduce their emissions, while maintaining its own levels of sulphur production.

The outcome of environmental pollution games are well-understood when the problem is static in nature i.e. damage is related to the flow of pollution. If the basic game structure is Prisoners' Dilemma (as has been argued by e.g. Mäler (1990)), then cooperation to reduce pollution is unsustainable. Although cooperation would yield the Pareto-efficient outcome, each country faces a strict incentive to defect from cooperation, and free-ride on the pollution abatement efforts of others. In equilibrium, all players defect. Extensions to this basic story have shown (within a one-shot game) that limited cooperation among a sub-set of players can be sustained in equilibrium (see Barrett (1994) and Carraro and Siniscalco (1993)); and (using repeated game results) that the Pareto-efficient outcome can be achieved if countries realise that they will be playing the game more than once, if they are sufficiently patient, and if they are uncertain about when the game will end (see Fudenberg and Tirole (1991)).

There is another class of environmental problems, however, of increasing concern, in which the stock (rather than the flow) of pollution gives rise to environmental damage. Examples include climate change (atmospheric carbon dioxide accumulation), ozone depletion (chlorofluorocarbon build-up), biodiversity loss, acid rain, and fish wars (clearly, this list is not exhaustive). In these games, the players care about the stock of pollution i.e. there is a state variable in the game which is payoff-relevant. The players' environment is not constant, as it was in the static or repeated version, but instead evolves endogenously as a result of their actions. The appropriate framework for the analysis of these environmental problems is dynamic game theory.

Mäler and de Zeeuw (1996) analyse the "dynamic acid rain game". They model the game in continuous time, with n players minimising the present discounted value of emissions abatement costs and pollution damages over time. The emissions of each player contribute to the acid stock of that player, and also to the pollution stocks of other countries. The authors examine the cooperative solution and the open-loop and linear feedback Nash equilibria of the game. Using linear-quadratic functional form assumptions, they show in a numerical example that steady state pollution levels are substantially higher (by an order of magnitude) when players are constrained to use subgame perfect strategies. Their model provides a normative economic interpretation of 'critical loads'[1]: emissions in the steady state must be such that depositions equal critical loads for all players.

This paper extends and challenges this analysis in three ways. First, it generalises the model to other, degradable pollutants. Acid rain is a highly persistent pollutant: Newbery (1990) cites evidence that current levels of soil acidification in the U.K. originate in part from the Industrial Revolution. Other pollutants, however, degrade. In these cases, critical loads are no longer sufficient to define steady state emission levels and therefore lose their normative significance. Secondly, the paper allows for the use of non-linear feedback strategies by players. Tsutsui and Mino (1990) show, in a model with a single state variable and symmetric players, that such strategies permit partial cooperation in equilibrium. Here, there are multiple state variables and asymmetric players; an important contribution of the paper is the numerical solution of Tsutsui and Mino's theory to this more complicated setting. Finally, Mäler and de Zeeuw's numerical results on the effect of linear Markov strategies is challenged. Using the same two country example of acid rain and parameter values as Mäler and de Zeeuw, it is shown that the linear feedback steady state acid stocks are almost identical to the open-loop levels.[2] A further numerical example (of groundwater pollution between neighbouring countries) indicates that the nature of pollution transportation is critically important in determining the effect of strategic behaviour.

[1]Defined as the maximal exposure to a pollutant that an ecological system can adjust to without suffering long term damage.

[2]Aart de Zeeuw has confirmed in private correspondence that Mäler and de Zeeuw's result is due to a programming error.

Section 2 develops the general dynamic pollution game. Sections 2.1 summarises the workings of Mäler and de Zeeuw in deriving the cooperative solution, the open-loop non-cooperative equilibrium, and the linear feedback (Markovian) equilibrium, extending the model to degradable pollutants. Section 2.2 derives the equations for the non-linear Markovian equilibria of the game. Section 3 solves the model numerically for two cases: acid rain between Ireland and the U.K.; and groundwater pollution between two neighbouring countries. Section 4 concludes. The appendix contain details of the numerical routines used to solve the model.

2. The Model

This section develops a model of dynamic pollution games. It is in almost all respects the same as Mäler and de Zeeuw (1996), so many of the details are omitted. There is only one difference in the model: pollution degradability is allowed for. The differential equation governing the accumulation of the pollutant is:

$$\begin{aligned} \frac{ds_i}{dt} = \dot{s}_i(t) &= d_i(t) - c_i - \tau_i s_i(t), \\ &= \sum_{j=1}^{n} a_{ij} e_j(t) - c_i - \tau_i s_i(t), \end{aligned} \tag{1}$$

for the n players, $i = 1, ..., n$. The left-hand side of this equation denotes the rate of change in the pollution stock s_i with time t at each of the players' locations. The right-hand side has three terms. The first is the flow of depositions at time t, $d_i(t)$; this equals emissions from country j, multiplied by the coefficient a_{ij} (a number less than 1 which indicates the fraction of j's emissions which land at i) summed over all countries $j = 1, ..., n$. The second is the assimilative capacity of the environment: if depositions are above this capacity, then there is a net contribution to the pollution stock. c_i is known as the critical load of the environment. The last term represents the natural degradation of the pollution stock: if $d_i = c_i$, then $\dot{s}_i = -\tau_i s_i$, and so τ_i measures the half-life of the pollutant. Some pollutants are extremely persistent (e.g., CFCs in the lower atmosphere), and will have a τ close to zero. Others are relatively volatile (e.g. organic solvents), and will have higher τ; in the limit, if the pollutant is entirely flow in its effect, $\tau = \infty$, and the game becomes static.[3]

The objective is to minimise the sum of the costs of abating emissions of pollution and the damage caused by the stock of pollution. The cost and damage functions are assumed quadratic, so that country i's objective is to:

$$\min_{e_i(t)} \int_0^\infty \mathrm{e}^{-rt} \left[\frac{\gamma_i}{2}(\overline{e}_i - e_i(t))^2 + \frac{\delta_i}{2} s_i(t)^2 \right] dt,$$

where r is the discount rate, common to all players. $\overline{e}_i$, γ_i and δ_i are positive parameters defined to ensure interior solutions to the minimisation problem.

Finally, the following notation will be used. $\mathbf{s}$ (respectively $\mathbf{e}$) is an n-vector of pollution stocks (emissions) for the n players. The matrix of coefficients $\{a_{ij}\}$ will be denoted $\mathbf{A}$, and referred to as the transportation matrix. $\mathbf{A_d}$ is an n-by-n diagonal matrix with the coefficient a_{ii} as the ith diagonal element. $\mathbf{a}_i$, an n-vector, is the ith column of the transportation matrix $\mathbf{A}$. $\mathbf{c}$ is an n-vector of critical loads. $\overline{\mathbf{e}}$ is an n-vector of the parameters $\{\overline{e}_i\}$. τ is an n-by-n diagonal matrix with non-zero elements equal to the coefficients $\{\tau_i\}$. $\mathbf{\Gamma}$ (respectively $\mathbf{\Delta}$) is a n-by-n diagonal matrix with the ith diagonal element equal to γ_i (resp. δ_i). Finally, $\mathbf{\Delta}_i$ is an n-by-n matrix consisting of zeros except for the ith diagonal element, which is the damage parameter δ_i.

The rest of this section analyses four solutions to the game: the cooperative outcome (in which all players coordinate to minimise aggregate abatement costs and damages), the Nash open-loop, the linear Markov perfect, and the non-linear Markov perfect equilibria (closed-loop equilibria will not be considered).

[3] Equation (1) is clearly a simplification: it assumes linearity (and therefore ignores potentially important issues such as irreversibilities), stationarity (a_{ij}, c_i and τ_i are constant over time), and determinism (there is no randomness involved). It will be seen, however, that even this simple formulation is sufficiently rich to capture many of the aspects of the problem which are of interest.

The cooperative equilibrium is the solution to the problem:

$$\begin{aligned} \min_{\{e_i(t)\}} \quad & \int_0^\infty \mathrm{e}^{-rt} \sum_{i=1}^{n} \left[\frac{\mathbf{e}^\top \mathbf{\Gamma} \mathbf{e}}{2} + \frac{\mathbf{s}^\top \mathbf{\Delta} \mathbf{s}}{2} \right] dt \\ \text{s.t.} \quad \dot{\mathbf{s}}(t) &= \mathbf{A}\mathbf{e}(t) - \mathbf{c} - \tau \mathbf{s}(t), \\ \mathbf{s}(0) &= \text{given}. \end{aligned}$$

It is straightforward to show that the system has a unique steady state that is saddle-point stable. The steady state values are:

$$\mathbf{e}^C = \mathbf{A}^{-1}\left(\mathbf{c} + \tau \mathbf{s}^C\right), \tag{2}$$

$$\mathbf{s}^C = \left[\mathbf{\Gamma}^{-1}\mathbf{A}^\top (r+\tau)^{-1}\mathbf{\Delta} + \mathbf{A}^{-1}\tau\right]^{-1}\left(\overline{\mathbf{e}} - \mathbf{A}^{-1}\mathbf{c}\right). \tag{3}$$

The form of equation (2) shows that different solutions will have different steady state emissions, unless $\tau = \mathbf{0}$. In the latter case, as Mäler and de Zeeuw (1996) argue, critical loads have a normative significance since they define precisely the steady state emissions of all equilibria: $\mathbf{e} = \mathbf{A}^{-1}\mathbf{c}$.[4] But when $\tau > \mathbf{0}$ i.e the pollutant is degradable, critical loads are no longer sufficient to define steady state emissions. The steady state requires no net addition to pollution stocks; the emission levels consistent with this are a function of the pollution stock when $\tau > \mathbf{0}$.

The cooperative solution is useful only for benchmark purposes. Each player faces a strict incentive to free-ride on the abatement efforts of the others; in equilibrium, therefore, cooperation is unsustainable, and each player will act to minimise its own (rather than aggregate) costs and damages:

$$\min_{e_i(t)} \quad \int_0^\infty \mathrm{e}^{-rt} \left[\frac{\gamma_i}{2}(\overline{e}_i - e_i(t))^2 + \frac{\delta_i}{2} s_i(t)^2 \right] dt$$

[4]This provides a justification for the long-term objective of the Second Sulphur Protocol, signed in June 1994 by over twenty European countries, to reduce depositions of acid rain to critical load levels (see article 2, paragraph 1).

$$\begin{aligned} \text{s.t.} \qquad \dot{s}_i &= \sum_{j=1}^{n} a_{ij} e_j - c_i - \tau_i s_i, \\ s_i(0) &= \text{given}, \qquad i = 1, ..., n. \end{aligned}$$

Again, there is a unique, saddle-point stable steady state given by:

$$\mathbf{e}^{OL} = \mathbf{A}^{-1} \left(\mathbf{c} + \tau \mathbf{s}^{OL} \right), \tag{4}$$

$$\mathbf{s}^{OL} = \left[\mathbf{\Gamma}^{-1} \mathbf{A_d} (r + \tau)^{-1} \mathbf{\Delta} + \mathbf{A}^{-1} \tau \right]^{-1} \left(\overline{\mathbf{e}} - \mathbf{A}^{-1} \mathbf{c} \right). \tag{5}$$

Comparison of equations (3) and (5) shows that non-cooperative steady state pollution stocks are at least as great as cooperative levels. Obviously, they are equal if the matrix $\mathbf{A}$ is diagonal i.e. there are no externalities between players. When $\mathbf{A}$ has non-zero off-diagonal elements, the free-riding incentives mentioned earlier mean that pollution stocks at one or more locations are larger in the non-cooperative setting. Equation (4) then implies that non-cooperative steady state emissions are at least as great as cooperative levels.

The usual criticism of the open-loop equilibrium is that it is not subgame perfect; the feedback solution does not suffer from this weakness, and is usually preferred for this reason. In addition, Markovian strategies are renegotiation-proof (see Farrell and Maskin (1989)), unlike general closed-loop strategies (e.g. punishment strategies in repeated games). The linear feedback equilibrium s attractive from the viewpoint of analysis and realism: analytically, because it seems natural that the value function of a linear-quadratic game should be quadratic; realistically, because the simplicity of the linear strategies make their use by players of limited rationality easier to envisage.

Let the value function for each player i be denoted:

$$V_i(\mathbf{s}(t)) = \min_{\{e_i(T), T \geq t\}} \int_t^\infty \mathrm{e}^{-rt} \left[C_i(e_i(T)) + D_i(s_i(T)) \right] dT,$$

where $V_i : \Re_+^n \rightarrow \Re$ is a continuously differentiable function. Standard dynamic programming techniques (see Başar and Olsder (1982)) require the value function to satisfy the Hamilton-Bellman-Jacobi (HBJ) equation:

$$rV_i = \min_{e_i} \left[C_i(e_i) + D_i(s_i) + \mathbf{V'}_i^\top(\mathbf{s})(\mathbf{A}\mathbf{e} - \mathbf{c} - \tau\mathbf{s}) \right] \tag{6}$$

where $\mathbf{V'}_i(\mathbf{s})$ is an n-vector whose jth element is the partial derivative $\frac{\partial V_i(\mathbf{s})}{\partial s_j}$. It is natural to 'guess' a quadratic solution to (6):

$$V_i(\mathbf{s}) = \frac{1}{2}\mathbf{s}^\top \mathbf{S}_i \mathbf{s} + \mathbf{k}_i^\top \mathbf{s} + m_i,$$

where $\mathbf{S}_i$ is an n-by-n symmetric matrix with continuously differentiable entries, $\mathbf{k}_i$ is a continuously differentiable n-vector, and m_i is a continuously differentiable function. Substitution of this guess into equation (6), and carrying out the minimisation on the right-hand side, gives:

$$\begin{aligned} r\left[\frac{1}{2}\mathbf{s}^\top \mathbf{S}_i \mathbf{s} + \mathbf{k}_i^\top \mathbf{s} + m_i\right] \quad &= \quad \frac{1}{2\gamma_i}\left(\mathbf{S}_i\mathbf{s} + \mathbf{k}_i\right)^\top \mathbf{a}_i\mathbf{a}_i^\top \left(\mathbf{S}_i\mathbf{s} + \mathbf{k}_i\right)^\top + \frac{\mathbf{s}^\top \mathbf{\Delta}_i \mathbf{s}}{2} + \\ &\quad \left(\mathbf{S}_i\mathbf{s} + \mathbf{k}_i\right)^\top \left[\mathbf{A}\overline{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\left(\mathbf{S}_j\mathbf{s} + \mathbf{k}_j\right) - \tau\mathbf{s}\right]. \end{aligned}$$

Equating coefficients on quadratic, linear and constant terms gives the following three equations for the coefficients $\mathbf{S}_i, \mathbf{k}_i$ and $m_i,\ i = 1, ..., n$:

$$0 \quad = \quad \mathbf{\Delta}_i - \mathbf{S}_i\frac{\mathbf{a}_i\mathbf{a}_i^\top}{\gamma_i}\mathbf{S}_i - \mathbf{S}_i\left(\frac{r}{2} + \tau + \sum_{j\neq i}\frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\mathbf{S}_j\right) - \left(\frac{r}{2} + \tau + \sum_{j\neq i}\mathbf{S}_j\frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\right)\mathbf{S}_i, \tag{7}$$

$$0 \quad = \quad -(r+\tau)\mathbf{k}_i + \mathbf{S}_i(\mathbf{A}\overline{\mathbf{e}} - \mathbf{c}) - \mathbf{S}_i\sum_{j\neq i}\frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\mathbf{k}_j - \left(\sum_{j\neq i}\mathbf{S}_j\frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\right)\mathbf{k}_i - \mathbf{S}_i\frac{\mathbf{a}_i\mathbf{a}_i^\top}{\gamma_i}\mathbf{k}_i, \tag{8}$$

$$0 \quad = \quad -rm_i + (\mathbf{A}\overline{\mathbf{e}} - \mathbf{c})^\top\mathbf{k}_i - \left(\sum_{j\neq i}\mathbf{k}_j\frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\right)\mathbf{k}_i - \mathbf{k}_i^\top\frac{\mathbf{a}_i\mathbf{a}_i^\top}{\gamma_j}\mathbf{k}_i. \tag{9}$$

Equation (7) defines a system of coupled algebraic Riccati equations. It can be shown that a solution exists, but that uniqueness cannot be guaranteed (see Lancaster (1995)). These equations cannot be solved analytically, and will be examined in section 3. Numerical solution shows that only one of the solutions yields a stable node. Equation (8) defines a set of coupled algebraic tracking equations, which can be solved easily once the $\{\mathbf{S}_i\}$ are known, as they are linear in the $\{\mathbf{k}_i\}$. The coefficients $\{m_i\}$ follow directly from equation (9).

The steady state expressions are:

$$
\begin{aligned}
\mathbf{e}^{LF} &= \mathbf{A}^{-1}\left(\mathbf{c} + \tau \mathbf{s}^{LF}\right), && (10) \\
\mathbf{s}^{LF} &= \left(\sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mathbf{S}_j + \tau\right)^{-1} \left(\mathbf{A}\overline{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mathbf{k}_j\right). && (11)
\end{aligned}
$$

The relative sizes of $\mathbf{s}^{LF}$ and $\mathbf{s}^{OL}$ will depend on the 'dynamic conjectural variations' implicit in the Markov perfect equilibrium (see Dockner (1992)). Since emissions are strategic substitutes See Bulow et al. (1985)), the dynamic conjectural variations in the linear Markovian equilibrium will be negative. In words: each player perceives that its rivals will reduce their emissions as rivals' pollution stocks rise. Each player has an incentive, therefore, to increase pollution stocks in order to decrease rivals' emissions; in equilibrium, each player therefore increases its emissions, and so pollution stocks rise relative to the zero conjectural variation, open-loop equilibrium. The extent of the increase in steady state pollution stocks is an empirical question, since there is an offsetting factor: while each player increases emissions to influence its rivals, it realises also that it will increase its own pollution stock (and hence damages) by doing so.

2.2. Non-linear Markov Perfect Equilibria

The 'guess' of a quadratic value function (and hence strategies which are linear in the state variable) may well ignore other solutions. For example, consider the value function and resulting emission rules:

$$
\begin{aligned}
V_i(\mathbf{s}) &= \frac{1}{2}\mathbf{s}^\top \mathbf{S}_i \mathbf{s} + \mathbf{k}_i^\top \mathbf{s} + m_i + h_i(\mathbf{s}), \\
e_i &= \overline{e}_i - \frac{\mathbf{a}_i^\top}{\gamma_i}\left(\mathbf{S}_i \mathbf{s} + \mathbf{k}_i + \mathbf{h}_i'(\mathbf{s})\right),
\end{aligned}
$$

where $h_i : \Re^n_+ \rightarrow \Re$ is some non-linear continuously differentiable function of $\mathbf{s}$ and $\mathbf{h}'_i(\mathbf{s})$ is an n-vector whose jth element is the derivative $\frac{\partial h_i(\mathbf{s})}{\partial s_j}$.[5]

The existence of non-linear solutions to the HBJ equation (6) is of interest because of the possibility it raises that some degree of cooperation may be sustainable in equilibrium. Tsutsui and Mino (1990) show in a 2-player symmetric game with a single state variable that (i) there exist multiple non-linear Markovian equilibria; and (ii) steady state levels of the state variable which are arbitrarily close to the cooperative level can be supported as a non-linear equilibrium. They show that as players become more patient, at least one non-linear differentiable Markovian equilibrium moves closer (in terms of the level of the equilibrium state variable) to the cooperative outcome. In the limit, as the discount factor goes to one (i.e. players are infinitely patient), a non-cooperative Markovian and the cooperative equilibria coincide. Unlike in a repeated game, players need not consider the entire history of the game in order to sustain the cooperative equilibrium – strategies can be Markovian, and the current state can therefore be viewed as a kind of sufficient statistic. (Dockner and Long (1993) and Feichtinger and Wirl (1993) apply this result to the case of a global pollutant e.g. climate change and a dynamic 'battle of the sexes' game, respectively.) The result is generalised in Dutta and Sundaram (1993), who show that 'tragedies of the common' – excessive depletion of a commonly-owned resource, or insufficient accumulation of capital – need not occur in Markov perfect equilibria. Indeed, non-cooperative equilibria may involve steady state levels of the state variable that exceed the cooperative level. The outcome of the game is determined by the form of the Markovian strategy; Dutta and Sundaram show that the use of continuously differentiable or monotonic discontinuous strategies always leads to a tragedy. Finally, Dutta (1995a) provides a 'Folk theorem'[6] for stochastic games:[7] all

[5]Alternatively, the non-linear value function could be written $V_i(\mathbf{s}) = h^*_i(\mathbf{s})$, for some non-linear function $h^*_i(\mathbf{s})$. The two approaches are analytically equivalent; the former is adopted to highlight the terms that arise from non-linear solutions.

[6]The Folk theorem for repeated games (i.e. dynamic games with no state variable) states that any set of feasible, individually rational payoffs for the stage game can be supported as a subgame perfect equilibrium of the supergame. Three factors are important in this theorem: (i) the game must not have a commonly known end-date; (ii) players must not discount the future too heavily; and (iii) punishing strategies must be credible. See e.g. Fudenberg and Tirole (1991).

[7]Stochastic games are a generalisation of dynamic games which allow the state variable differential equation to be stochastic.

strictly individually rational long-run average payoffs in a stochastic game can be approximated by an equilibrium discounted average payoff, for sufficiently large discount factors i.e. sufficiently patient players. The result requires that non-Markovian strategies are allowed.[8]

The game analysed in this paper is linear-quadratic, and the equilibria of interest are Markov perfect. The model would appear, therefore, to be similar to Tsutsui and Mino (1990); there are significant differences, however, between their model and this one. Here there are n state variables (rather than one), which means that the HBJ equation is a partial, rather than ordinary, differential equation; consequently, it is much more difficult to solve. Moreover, the players in this game are asymmetric (potentially in all dimensions – the costs and damages they face, and the effect they have on each others' pollution stocks); the simplifying assumption of symmetry used by Tsutsui and Mino is inappropriate here.

2.2.1. Solving the Hamilton-Bellman-Jacobi Equation

Substituting the form of the value function into the HBJ equation gives:

$$\begin{aligned} r\left[\frac{1}{2}\mathbf{s}^\top \mathbf{S}_i\mathbf{s} + \mathbf{k}_i^\top \mathbf{s} + h_i(\mathbf{s}) + m_i\right] &= (\mathbf{S}_i\mathbf{s} + \mathbf{k}_i + \mathbf{h}_i'(\mathbf{s}))^\top \frac{\mathbf{a}_i\mathbf{a}_i^\top}{\gamma_i}(\mathbf{S}_i\mathbf{s} + \mathbf{k}_i + \mathbf{h}_i'(\mathbf{s})) + \frac{\mathbf{s}^\top \boldsymbol{\Delta}_i\mathbf{s}}{2} + \\ & (\mathbf{S}_i\mathbf{s} + \mathbf{k}_i + \mathbf{h}_i'(\mathbf{s}))^\top \left[\mathbf{A}\bar{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\left(\mathbf{S}_j\mathbf{s} + \mathbf{k}_j + \mathbf{h}_j'(\mathbf{s})\right) - \tau\mathbf{s}\right]. \end{aligned}$$

Matching terms involving h_i and $\mathbf{h}_i'$:

$$\begin{aligned} rh_i(\mathbf{s}) &= \frac{1}{2\gamma_i}(2\mathbf{S}_i\mathbf{s} + 2\mathbf{k}_i + \mathbf{h}_i'(\mathbf{s}))^\top \mathbf{a}_i\mathbf{a}_i^\top \mathbf{h}_i'(\mathbf{s}) + \\ & \mathbf{h}_i'^\top(\mathbf{s})\left[\mathbf{A}\bar{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\left(\mathbf{S}_j\mathbf{s} + \mathbf{k}_j + \mathbf{h}_j'(\mathbf{s})\right) - \tau\mathbf{s}\right] - \\ & (\mathbf{S}_i\mathbf{s} + \mathbf{k}_i)^\top \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\mathbf{h}_j'(\mathbf{s}). \end{aligned} \tag{12}$$

[8] Dutta (1995b) provides a counter-example to the general conclusion that infinitely-patient players can achieve the cooperative outcome in a dynamic game. He shows that players must be neither too impatient nor too patient to support the first-best.

With equations (7)-(9), equation (12) – a system of non-linear first-order partial differential equations for the $\{h_i(\mathbf{s})\}$ – solves the HBJ equation (6) for non-linear solutions. The system must be solved numerically; results are reported in section 3 and the solution method explained in the appendix.

Steady state emissions and pollution stocks are given by:

$$\begin{aligned} \mathbf{e}^{NLF} &= \mathbf{A}^{-1}\left(\mathbf{c} + \tau \mathbf{s}^{NLF}\right), & (13) \\ \left(\sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mathbf{S}_j + \tau\right) \mathbf{s}^{NLF} + \sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mathbf{h}_j'(\mathbf{s}^{NLF}) &= \mathbf{A}\bar{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mathbf{k}_j. & (14) \end{aligned}$$

A few comments can be made on the general nature of the non-linear Markov equilibrium. First, existence of an equilibrium is guaranteed: a solution to equations (7)-(9) exists, from previous arguments; and, since the $\{h_i(\mathbf{s})\}$ are assumed continuously differentiable, a solution to equation (12) exists (at least within some neighbourhood of the linear feedback steady state stock $\mathbf{s}^{LF}$); see Courant and Hilbert (1989). The numerical analysis of the next section will confirm this. Analytical comparison of steady state levels with other non-cooperative equilibria is not possible. Clearly, the linear Markovian equilibrium is included in the set of non-linear equilibria (set $h_i(\mathbf{s}) \equiv 0, \ \forall i, \mathbf{s}$). There are many other non-linear equilibria possible, however (see below for more on this point); these equilibria can involve steady state pollution stocks which are higher or lower than the corresponding linear Markovian equilibrium. Since the $\{h_i(\mathbf{s})\}$ are continuously differentiable, Dutta and Sundaram's (1993) result suggests that the Markovian steady state pollution stocks will always be at least as great as the cooperative levels. This cannot be shown formally, however, since Dutta and Sundaram's proof relies on there being a common resource i.e. a single state variable. Extension of the result to the multiple state variable case awaits further research.

2.2.2. Non-uniqueness of the Non-linear Equilibrium

Two factors give rise to multiplicity of equilibria when non-linear strategies are allowed: the reversibility terms $\mathbf{c} + \tau\mathbf{s}$ in the state variable differential equation;

and player asymmetry. To see this, consider the simplest case where the n players are symmetric (so that $\overline{e}_i = \overline{e}, \gamma_i = \gamma, \delta_i = \delta, a_{ii} = a, a_{ij} = a'$ for all $i, j \neq i$) and $c_i = \tau_i = 0$. The latter means that steady state emissions must equal zero. Equating the marginal benefits and costs of the last unit of emissions provides a natural boundary condition in this case.[9] The marginal benefit of the last ton of emissions is $\gamma\overline{e}$. The marginal cost is the present value of marginal damages from the resulting pollution stock $\int_0^\infty \mathrm{e}^{-rt} aD'(s^*)dt = \frac{a\delta s^*}{r}$. Therefore the steady state pollution stock $s^* = \frac{r\gamma\overline{e}}{a\delta}$: the stock is uniquely determined in this case, and multiple equilibria are not possible.

This argument makes clear that the natural boundary condition will not apply if players are asymmetric, since in this case there is no one marginal benefit and cost trade-off to determine s^*. (Wirl and Dockner (1995) present a model in which player asymmetry generates multiplicity.) And if pollution is reversible (i.e. either $c > 0$ or $\tau > 0$), then again the marginal benefit/cost argument cannot be used. (see, for example, Karp and Livernois (1994) for a model with multiplicity where $c > 0$ and $\tau = 0$; and Dockner and Long (1993) for the converse case.) Equilibrium multiplicity can also be understood in terms of dynamic conjectural variations. Each player cares about the slopes of rivals' emission rules, as well as the level of emissions. A range of emission levels can be supported by appropriate equilibrium beliefs about these slopes, where the beliefs must be consistent with the equilibrium emission rules used by the players. See Karp and Livernois (1994) for further discussion.

Mathematically, the problem is one of an 'incomplete transversality condition' (see Tsutsui and Mino (1990)). The first-order condition for optimality and the state variable differential equation together mean that

$$\dot{\mathbf{s}} = \mathbf{A}\overline{\mathbf{e}} - \mathbf{c} - \tau\mathbf{s} - \sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mu_j \tag{15}$$

where μ_j is an n-vector of co-state variables. It can be shown that the μ_i must satisfy

[9] Karp (1992) uses a similar argument.

the following differential equation:

$$\dot{\mu}_i = (\mathbf{S}_i + \mathbf{h}_i''(\mathbf{s})) \left[\mathbf{A}\overline{\mathbf{e}} - \mathbf{c} - \tau \mathbf{s} - \sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mu_j \right], \quad i = 1, ..., n, \tag{16}$$

where $\mathbf{h}_i'' : \Re_+^n \rightarrow \Re^{n \times n}$. Equations (15) and (16) are a pair of differential equations, with boundary conditions provided by the transversality conditions:

$$\dot{\mathbf{s}} = \dot{\mu}_i = 0 \text{ at some } \mathbf{s}^*, \mu_i^*,$$

for $i = 1, ..., n$. Both of these transversality conditions imply that

$$\tau \mathbf{s}^* + \sum_j \frac{\mathbf{a}_j \mathbf{a}_j^\top}{\gamma_j} \mu_j^* = \mathbf{A}\overline{\mathbf{e}} - \mathbf{c};$$

i.e. the transversality conditions alone are not sufficient to determine $(\mathbf{s}^*, \mu_i^*)$, since $\mathbf{h}_i''(\mathbf{s}^*)$ is indeterminate.

Nor can stability considerations determine $\mathbf{h}_i''(\mathbf{s}^*)$ precisely. Expanding the state variable differential equation around the steady state using a first-order Taylor series gives

$$\begin{aligned} \dot{\mathbf{s}} &\approx \left. \frac{d\mathbf{s}}{dt} \right|_{\mathbf{s}=\mathbf{s}^*} + \left[\frac{d}{d\mathbf{s}} \left(\frac{d\mathbf{s}}{dt} \right) \bigg|_{\mathbf{s}=\mathbf{s}^*} \right]^\top (\mathbf{s} - \mathbf{s}^*) \\ &= \left[\mathbf{A} \left. \frac{d\mathbf{e}}{d\mathbf{s}} \right|_{\mathbf{s}=\mathbf{s}^*} - \tau \right]^\top (\mathbf{s} - \mathbf{s}^*). \end{aligned}$$

Stability requires the eigenvalues of the n-by-n matrix $\left(\mathbf{A} \frac{d\mathbf{e}}{d\mathbf{s}} \Big|_{\mathbf{s}=\mathbf{s}^*} - \tau \right)$ either to be negative, or to straddle zero. While this condition places some restriction on the $\{\mathbf{h}_i''(\mathbf{s})\}$ (since $\frac{de_i}{d\mathbf{s}} = -\left(\mathbf{S}_i + \mathbf{h}_i''(\mathbf{s}) \right) \frac{\mathbf{a}_i}{\gamma_i}$), it does not determine them completely.

3. Numerical Solution of the Model

This section describes numerical solution of the dynamic model of section 2. Two examples are presented: acid rain between Ireland and the U.K.; and groundwater pollution between neighbouring countries. These examples are chosen for two reasons.

First, Mäler and de Zeeuw (1996) have solved the Ireland/U.K. acid rain game for the cooperative, open-loop and linear Markov non-cooperative equilibria; it will be instructive to compare the results of this paper with theirs.[10] Secondly, the example of groundwater pollution provides a stark contrast to the acid rain example, as will be seen below.

3.1. Example 1: Acid Rain between Ireland and the U.K.

The specific parameter values for Ireland and the U.K., in that order, are:

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 0.2143 & 0.0057 \\ 0.0786 & 0.2619 \end{pmatrix}, \\
\mathbf{c} &= \begin{pmatrix} 166 \\ 366 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
\overline{\mathbf{e}} &= \begin{pmatrix} 105,690 \\ 91,600 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
\delta &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

Units are hundreds of tonnes of sulphur per annum (these can be converted to kilotonnes of SO_2 by multiplying by a factor of 0.2). The common discount rate is set initially at 4%.

τ is set to zero since sulphur stocks are very persistent. This means that steady state emissions are the same in all solutions:

$$\mathbf{e}^C = \mathbf{e}^{OL} = \mathbf{e}^{LF} = \mathbf{e}^{NLF} = \mathbf{A}^{-1}\mathbf{c} = \begin{pmatrix} 743 \\ 1,174 \end{pmatrix}.$$

[10] This example is also of interest because acid rain is a major economic concern: the recently-signed Second Sulphur Protocol commits Europe to an emissions reduction target that will cost 25 billion deutschemark per annum (International Institute for Applied Systems Analysis (IIASA) estimate). Although the acid rain problem between these two countries its small, due to prevailing south-westerly winds, these two countries are the only ones in Europe for whom background depositions from other countries are less than critical loads.

Solution	**Steady state acid stock**
Cooperative	$\begin{pmatrix} 14,640 \\ 13,492 \end{pmatrix}$
Open-loop	$\begin{pmatrix} 19,589 \\ 13,811 \end{pmatrix}$
Linear Markov	$\begin{pmatrix} 19,669 \\ 13,856 \end{pmatrix}$
Non-linear Markov A	$\begin{pmatrix} 20,500 \\ 14,000 \end{pmatrix}$
Non-linear Markov B	$\begin{pmatrix} 19,200 \\ 13,750 \end{pmatrix}$

Table 1: Steady State Acid Stocks

Steady state acid stocks are reported in table 1 for the four alternative solutions. (All equilibria quoted in the following tables are stable, either as saddle points or as stable nodes.) The first two entries of the table relate to the cooperative and open-loop solutions; they confirm Mäler and de Zeeuw's (1996) numerical results. As expected, open-loop pollution stocks are greater than cooperative levels. The extent to which this is true is very asymmetric: for example, Ireland's steady state acid stock is increased by a third; on the other hand, the U.K.'s stock is very similar in the two solutions.

The playing of linear feedback strategies increases steady state acid stocks, as intuition would suggest. The size of the increase is somewhat surprising, however – stocks in the linear Markovian equilibrium differ from open-loop levels by no more than 0.4%. Note that this is in sharp contrast to the results quoted by Mäler and de Zeeuw (1996), where the steady state acid stocks in the linear Markov equilibrium are substantially higher than open-loop levels for both of the countries (Mäler and de Zeeuw's figures are 369,100 and 304,300). The inefficiency of the linear feedback equilibrium relies on the dynamic conjectures implied by Markovian strategies – each player perceives that its rivals will reduce their emissions as pollution stocks rise. In equilibrium, all players increase emissions (relative to the open-loop levels) to exploit this fact. The results of this section imply that the dynamic conjectural variations of the two countries are near zero, and so each pays little (but some) attention to the fact that the other will decrease emissions as acid stocks rise. The intuition for this comes from inspection of the transportation matrix. Diagonal elements are considerably larger than off-diagonal elements (i.e. most of country i's emissions are deposited on country i); so any benefits from influencing rivals' behaviour are largely outweighed by increased damages. This conclusion depends, of course, on the countries and the pollutant in the example. The transportation matrix for Ireland and the U.K. has small off-diagonal elements, leading to the similarity in steady state acid stocks in the equilibria. The next numerical example shows a case where there is a large difference between the linear Markov and open-loop equilibria.

The two figures quoted for the non-linear Markov entry indicate that there are multiple equilibria with different steady state acid stocks. Equilibrium A involves steady state acid stocks which exceed the levels in the linear Markov equilibrium; equilibrium B has lower pollution stocks than either the linear Markovian or open-loop equilibria. Tsutsui and Mino (1990) show that the welfare of the players in the various non-linear equilibria varies monotonically with the level of the state variable. Their analytical result cannot be applied immediately to this more complicated setting with multiple state variables and asymmetric players. The numerical solutions confirm, however, that the value functions of both players are higher in equilibrium A than in the linear feedback solution (i.e. the non-linear equilibrium is Pareto inferior), while they are lower in equilibrium B. In principle, it should be possible to find the 'least competitive' (i.e. involving lowest steady state pollution stocks) non-linear equilibrium which is stable. It is expected (following Tsutsui and Mino (1990)) that the steady state pollution stocks in this equilibrium will lie above the cooperative levels. Unfortunately, it has not yet been possible to identify this equilibrium, and this task is the subject of ongoing research.

3.2. Example 2: Groundwater Pollution between Neighbouring Countries

There are two major differences between the next example and the acid rain game studied above. First, the pollutant in this section is degradable ($\tau > \mathbf{0}$) while there is no buffering capacity in the environment ($\mathbf{c} = \mathbf{0}$). Secondly, the parameter values are chosen specifically to ensure that there is a substantial difference (unlike in the previous example) between the steady state pollution stocks of the open-loop and linear feedback equilibria. There is an upper bound, however, on the difference between linear feedback and open-loop steady state stocks. The bound arises from the requirement that steady state emissions, given by $\mathbf{e} = \mathbf{A}^{-1}\mathbf{taus}$, should be non-negative. For $\mathbf{s}^{LF}$ to be much larger than $\mathbf{s}^{C}$ requires the off-diagonal elements of $\mathbf{A}$ to be large. But this lowers steady state emissions, which eventually become negative as the $\{a_{ij}\}$, $j \neq i$ increase. The parameter values used are such that the difference between $\mathbf{s}^{LF}$ and $\mathbf{s}^{C}$ is as large as possible while ensuring non-negative emissions.

The parameter values used are:

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 0.20 & 0.15 \\ 0.10 & 0.25 \end{pmatrix}, & & \\
\mathbf{c} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \tau &= \begin{pmatrix} 0.050 \\ 0.075 \end{pmatrix}, \\
\overline{\mathbf{e}} &= \begin{pmatrix} 105,690 \\ 91,600 \end{pmatrix}, & \gamma &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
\delta &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & r &= 0.04.
\end{aligned}
$$

All parameter values are fictitious. But the values used are consistent with e.g. groundwater pollution by pesticides between neighbouring countries who share a water table. The off-diagonal elements in the transportation matrix are large, reflecting the high mobility of chemicals in water. The degradability values correspond to chemical half-lives of between 9 and 14 years (c.f. the half-life of many herbicides in the anaerobic environment of groundwater of 10 years). A zero critical load is a reasonable description of groundwater, in which there are no buffering processes to decrease the stock of pesticides. (For more on the physical chemistry properties of pesticides in groundwater, see Conway and Pretty (1991).)

The resulting steady state emissions and pollution stock levels are reported in table 2. As in the previous example, and in line with intuition, open-loop pollution stocks are greater than cooperative levels: by almost 20% for the first country, by a factor of 3 for the second. And again, pollution stocks in the linear feedback equilibrium are higher than in the open-loop, reflecting the negative dynamic conjectures embodied in the playing of linear Markovian strategies. In this example, pollution stocks are increased significantly (by around 14% for both countries) by this effect. This can be attributed to the greater off-diagonal elements in the transportation matrix. Previously, the fraction of emissions falling on the emitting country was a factor of 3 (Ireland) or 38 (the U.K.) higher than the fraction exported. In this example, the corresponding factors are 1.3 and 2.5. Strategic effects are no longer out-weighed by the damages that the player inflicts on itself – the benefits and costs of strategic behaviour are more evenly balanced.

The asymmetry of the players and degradation of the pollutant mean that non-linear equilibria exist. Two such equilibria are reported in the table. The first (A) has steady state pollution stocks lying above the linear feedback levels. This equilibrium in Pareto inferior to the linear feedback solution. Equilibrium B has steady state pollution stocks which are lower than the open-loop levels; this equilibrium is Pareto preferred to the linear feedback. Both non-linear equilibria are stable, but neither is the 'least competitive' stable equilibrium.

Solution	**Steady state pollution stock**	**Steady state emissions**
Cooperative	$\begin{pmatrix} 38,256 \\ 12,798 \end{pmatrix}$	$\begin{pmatrix} 9,549 \\ 20 \end{pmatrix}$
Open-loop	$\begin{pmatrix} 45,674 \\ 37,704 \end{pmatrix}$	$\begin{pmatrix} 4,193 \\ 9,634 \end{pmatrix}$
Linear Markov	$\begin{pmatrix} 52,280 \\ 42,784 \end{pmatrix}$	$\begin{pmatrix} 4,919 \\ 10,867 \end{pmatrix}$
Non-linear Markov A	$\begin{pmatrix} 55,000 \\ 43,000 \end{pmatrix}$	$\begin{pmatrix} 5,821 \\ 10,571 \end{pmatrix}$
Non-linear Markov B	$\begin{pmatrix} 45,000 \\ 37,700 \end{pmatrix}$	$\begin{pmatrix} 4,132 \\ 9,657 \end{pmatrix}$

Table 2: Steady State Results for Groundwater Pollution

The steady state emissions are given in the second column of the table. Unlike in the previous example, emissions differ between the various solutions. This confirms that critical loads are no longer sufficient to describe steady state emissions. Emissions are not monotonic in the steady state pollution stock. This can be seen most clearly for the first country, whose emissions fall (from 9,549 to 4,193) between cooperative and non-cooperative solution, despite the rise in both countries' pollution stocks. (Compare also the second country's emissions in the linear and non-linear A Markovian equilibria.) This is a direct consequence of the size of the off-diagonal elements of the matrix **A**.

4. Conclusions

Pollution rarely respects boundaries imposed by man; consequently, in many cases, more than one party is involved in and affected by environmental degradation. This paper has followed Mäler and de Zeeuw (1996) by modelling such situations as a game; and since the pollutant is assumed to be persistent, as a dynamic game.

The paper differs from Mäler and de Zeeuw in several key ways. First, it extends their model to allow for pollutant degradability. Secondly, it analyses non-linear (as well as linear) Markovian equilibria in a model with multiple state variables and asymmetric players. Finally, it derives numerical solutions which are dramatically different from theirs. Three main conclusions can be drawn from the analysis:

1. The concept of critical load has normative significance only for persistent pollutants (with $\tau = 0$). In this case, steady state emissions are the same in all equilibria (they are set so that depositions equal critical loads). For degradable pollutants ($\tau > 0$), however, equilibria with different steady state pollution stocks will have different steady state emissions.

2. The numerical examples indicate that the difference between the steady states of the open-loop and linear Markov equilibria depends critically on the nature of pollution transportation. Acid rain is distributed very asymmetrically; as a consequence, there is little difference in steady state pollution when players are constrained to subgame perfect strategies.

3. Some degree of cooperation can be sustained in equilibrium, even when players condition strategies on only the current state (rather than entire history) of the game. Numerical solution finds non-linear Markovian equilibria with steady state pollution stocks below open-loop levels. These equilibria are Pareto preferred.

APPENDIX

Numerical Solution for Non-linear Markovian Equilibria

This appendix details how the system of non-linear first-order partial differential equations (12) can be solved numerically to find the non-linear components to the players' value functions. Equation (12), for reference, is:

$$\begin{aligned} rh_i(\mathbf{s}) &= \frac{1}{2\gamma_i}\left(2\mathbf{S}_i\mathbf{s} + 2\mathbf{k}_i + \mathbf{h}_i'(\mathbf{s})\right)^\top \mathbf{a}_i\mathbf{a}_i^\top \mathbf{h}_i'(\mathbf{s}) + \\ &\quad \mathbf{h}_i'^\top(\mathbf{s})\left[\mathbf{A}\overline{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\left(\mathbf{S}_j\mathbf{s} + \mathbf{k}_j + \mathbf{h}_j'(\mathbf{s})\right) - \tau\mathbf{s}\right] - \\ &\quad (\mathbf{S}_i\mathbf{s} + \mathbf{k}_i)^\top \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\mathbf{h}_j'(\mathbf{s}). \end{aligned} \tag{A1}$$

(The $\{\mathbf{S}_i\}$ and $\{\mathbf{k}_i\}$ are given by equations (7)-(9).) Let s^* denote a steady state, then

$$\left(\sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\mathbf{S}_j + \tau\right)\mathbf{s}^* + \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\mathbf{h}_j'(\mathbf{s}^*) = \mathbf{A}\overline{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}\mathbf{k}_j. \tag{A2}$$

Substitution of this equation into (A1) gives

$$\begin{aligned} rh_i(\mathbf{s}^*) &= \left(\mathbf{S}_i\mathbf{s}^* + \mathbf{k}_i + 0.5\mathbf{h}_i'(\mathbf{s}^*)\right)^\top \frac{\mathbf{a}_i\mathbf{a}_i^\top}{\gamma_i}\mathbf{h}_i'(\mathbf{s}^*) - \\ &\quad (\mathbf{S}_i\mathbf{s}^* + \mathbf{k}_i)^\top\left(\mathbf{A}\overline{\mathbf{e}} - \mathbf{c} - \sum_j \frac{\mathbf{a}_j\mathbf{a}_j^\top}{\gamma_j}(\mathbf{S}_j\mathbf{s} + \mathbf{k}_j) - \tau\mathbf{s}^*\right). \end{aligned} \tag{A3}$$

Approximate the function $h_i(\mathbf{s})$ by a first order Taylor series expansion around $\mathbf{s}^*$, using the fact that $h_i(\mathbf{s}^{LF}) = 0$:

$$\begin{aligned} h_i(\mathbf{s}) &\approx h_i(\mathbf{s}^*) + (\mathbf{s} - \mathbf{s}^*)^\top\mathbf{h}_i'(\mathbf{s}^*) + \ldots \\ \text{i.e.}\quad h_i(\mathbf{s}^{LF}) &= h_i(\mathbf{s}^*) + (\mathbf{s}^{LF} - \mathbf{s}^*)^\top\mathbf{h}_i'(\mathbf{s}^*) \\ \Rightarrow\quad h_i(\mathbf{s}^*) &= -(\mathbf{s}^{LF} - \mathbf{s}^*)^\top\mathbf{h}_i'(\mathbf{s}^*). \end{aligned}$$

Substituting this into equation (A3) gives

$$\mathbf{h}_i'^{\top}(\mathbf{s}^*)\frac{\mathbf{a}_i\mathbf{a}_i^{\top}}{2\gamma_i}\mathbf{h}_i'(\mathbf{s}^*)+\left((\mathbf{S}_i\mathbf{s}^*+\mathbf{k}_i)^{\top}\frac{\mathbf{a}_i\mathbf{a}_i^{\top}}{\gamma_i}+r(\mathbf{s}^{LF}-\mathbf{s}^*)^{\top}\right)\mathbf{h}_i'(\mathbf{s}^*)-$$

$$(\mathbf{S}_i\mathbf{s}^*+\mathbf{k}_i)^{\top}\left(\mathbf{A}\overline{\mathbf{e}}-\mathbf{c}-\sum_j\frac{\mathbf{a}_j\mathbf{a}_j^{\top}}{\gamma_j}(\mathbf{S}_j\mathbf{s}^*+\mathbf{k}_j)-\tau\mathbf{s}^*\right) = 0. \tag{A4}$$

Equation (A4) is a non-coupled system of non-linear equations in the constant vectors $\{\mathbf{h}_i'(\mathbf{s}^*)\}$. But the equation is under-determined: there are n conditions for n^2 unknowns. This is (part of) the incomplete transversality condition problem identified by Tsutsui and Mino (1990). There is, therefore, a degree of arbitrariness in the solutions to equation (A4). The equations are solved by a Newton-Raphson procedure (using matrix derivative rules that can be found in Magnus and Neudecker (1988)) written in GAUSS; copies of the programs are available from the author on request.

The solutions must, however, be stable. Stability imposes conditions on $\{\mathbf{h}_i''(\mathbf{s}^*)\}$. The second derivative of $h_i(\mathbf{s})$ can be found by differentiation equation (A1) with respect to $\mathbf{s}$, to give auxiliary equations. To lighten notation, consider the 2-player auxiliary equation (the extension to n players is straightforward):

$$\begin{aligned}
&\left(r+\sum_j\frac{\mathbf{a}_j\mathbf{a}_j^{\top}}{\gamma_j}\mathbf{S}_j\right)\mathbf{h}_i'(\mathbf{s})+\mathbf{S}_i\frac{\mathbf{a}_j\mathbf{a}_j^{\top}}{\gamma_j}\mathbf{h}_j'(\mathbf{s}) \quad = \\
&\Bigg[\left(\left((\mathbf{S}_i\mathbf{s}+\mathbf{k}_i+\mathbf{h}_i'(\mathbf{s}))^{\top}\frac{\mathbf{a}_i\mathbf{a}_i^{\top}}{\gamma_i}\right)\otimes\mathbf{I}_2\right)+\left(\left(\frac{\mathbf{a}_i\mathbf{a}_i^{\top}}{2\gamma_i}\mathbf{h}_i'(\mathbf{s})\right)\otimes\mathbf{I}_2\right)^{\top}+ \\
&\left(\left(\mathbf{A}\overline{\mathbf{e}}-\mathbf{c}-\sum_j\frac{\mathbf{a}_j\mathbf{a}_j^{\top}}{\gamma_j}\left(\mathbf{S}_j\mathbf{s}+\mathbf{k}_j+\mathbf{h}_j'(\mathbf{s})\right)\right)\otimes\mathbf{I}_2\right)^{\top}-\left(\left((\mathbf{S}_i\mathbf{s}+\mathbf{k}_i)^{\top}\frac{\mathbf{a}_i\mathbf{a}_i^{\top}}{\gamma_i}\right)\otimes\mathbf{I}_2\right)\Bigg]\mathbf{h}_i''(\mathbf{s})- \\
&\left(\left((\mathbf{S}_i\mathbf{s}+\mathbf{k}_i)^{\top}\frac{\mathbf{a}_j\mathbf{a}_j^{\top}}{\gamma_j}\right)\otimes\mathbf{I}_2\right)\mathbf{h}_j''(\mathbf{s}),
\end{aligned} \tag{A5}$$

for $i=1,2,\ j\neq i$. $\otimes$ denotes a Cartesian product and $\mathbf{I}_2$ is the two dimensional identity matrix. The coupled system of auxiliary equations, evaluated at $\mathbf{s}^*$, can be solved directly to find $\{\mathbf{h}_i''(\mathbf{s}^*)\}$, which are n^2-by-1 vectors.

It is important to note the local nature of this solution method: $\{h_i(\mathbf{s})\}$ and their derivatives have been calculated at the (steady state) level $\mathbf{s}^*$, not elsewhere. ($h_i(\mathbf{s})$ can be approximated, however, using a Taylor series expansion.) $\mathbf{s}^*$ cannot be too far away from

$\mathbf{s}^{LF}$, since a first order Taylor series expansion has been used in the numerical solution. Numerically, the solution must be checked against the steady state condition of equation (A2). The Taylor series approximation means that equation (A2) will not be satisfied exactly; once the discrepancy exceeds a tolerance parameter, the solution is rejected.

The last check made on the solution is that the non-linear equilibrium is Pareto preferred to the linear feedback equilibrium. If it is not, then the solution is rejected.

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