An Omnibus Test for Univariate and Multivariate Normality

By JURGEN A. DOORNIK† and HENRIK HANSEN

Nuffield College, Oxford — *University of Copenhagen, Denmark*

November 24, 1994

SUMMARY

We suggest an easy to use version of the omnibus test for normality using skewness and kurtosis based on Shenton and Bowman (1977) which controls well for size. A multivariate version is introduced. Size and power are investigated in comparison with four other tests for multivariate normality. The alternative hypothesis in the power simulations is the whole Johnson system of distributions.

Keywords: JOHNSON SYSTEM; KURTOSIS; MULTIVARIATE NORMALITY TEST; SKEWNESS; UNIVARIATE NORMALITY TEST; WILSON-HILFERTY TRANSFORMATION

1. INTRODUCTION

Testing for normality is a common procedure in much applied work and many tests have been proposed, see for example the overviews in Mardia (1980), D'Agostino (1982) and Small (1985). The need for testing normality in a multivariate setting is discussed, *inter alia*, by Gnanadesikan (1977, §5.4.2), Cox and Small (1978) and recently Cox and Wermuth (1994).

A test frequently used is the sum of squares of the standardized sample skewness and kurtosis, which is asymptotically distributed as a χ^2-variate. For small samples the transformation to approximate χ^2 as given in Bowman and Shenton (1975) is cumbersome, and many computer programs only report the asymptotic version of the test.

The next section bases a more convenient test statistic on work by Shenton and Bowman (1977) (a possibility already mentioned by Pearson, D'Agostino and Bowman, 1977), who give the kurtosis a conditional gamma distribution. The test statistic uses transformed skewness and kurtosis, and is simple to implement. A multivariate version of the test is introduced in §3. Size and power comparisons with several other multivariate tests are given in §5 and §6.

All results in this paper are based on random samples. In practice, however, the tests will also be applied to regression residuals and residuals from time series models.

†*Address for correspondence*: Nuffield College, Oxford OX1 1NF, UK

Limited numerical results in Pierce and Gray (1982), Pierce (1985) and Lütkepohl and Schneider (1989) suggest that the size of the tests is largely unaffected, but there could be some loss of power.

2. THE UNIVARIATE OMNIBUS TEST

Let $(x_1, \ldots, x_n)$ be a sample of n independent observations on a 1–dimensional random variable with mean μ and variance σ^2. Write $\mu_i = E(X-\mu)^i$, so that $\sigma^2 = \mu_2$. The skewness and kurtosis are defined as:

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

Sample counterparts are defined by:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,\ m_i = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^i,\ \sqrt{b_1} = \frac{m_3}{m_2^{3/2}} \text{ and } b_2 = \frac{m_4}{m_2^2}.$$

Bowman and Shenton (1975) consider the test ($\underset{a}{\sim}$ denotes 'asymptotically distributed as'):

$$E_p^a = \frac{n\left(\sqrt{b_1}\right)^2}{6} + \frac{n\left(b_2-3\right)^2}{24} \underset{a}{\sim} \chi^2(2)$$

unsuitable except in very large samples. The statistics $\sqrt{b_1}$ and b_2 are not independently distributed (although uncorrelated), and the sample kurtosis especially approaches normality very slowly. They proceed to derive a test based on approximating the distribution of $\sqrt{b_1}$ and b_2 by the Johnson system (see e.g. Kendall, Stuart and Ord, 1987, §6.27–36, or Pearson and Hartley, 1972, §18), assuming independence. The Johnson S_U approximation to $\sqrt{b_1}$ is given in D'Agostino (1970). The kurtosis part is less convenient, as it involves iteration to find the parameters of the fitting distribution (owing to a Johnson S_B being used for $n < 25$).

The test considered here derives from Shenton and Bowman (1977), who give b_2 (conditional on $b_2 > 1 + b_1$) a gamma distribution. The distribution of $\sqrt{b_1}$ is still based on Johnson S_U. Let z_1 and z_2 denote the transformed skewness and kurtosis, where the transformation creates statistics which are much closer to standard normal. The test statistic is ($\underset{app}{\sim}$ denotes 'approximately distributed as'):

$$E_p = z_1^2 + z_2^2 \underset{app}{\sim} \chi^2(2).$$

The transformation for the skewness is as in D'Agostino (1970); the kurtosis is transformed from a gamma distribution to χ^2, which is then translated into standard normal using the Wilson-Hilferty cubed root transformation. The precise form of z_1 and z_2 is given in the appendix, together with a numerical example. The advantage of this statistic is that is easy to implement and requires only tables of the χ^2 distribution. Note that the formulae break down for $n \leq 7$.

3. THE MULTIVARIATE OMNIBUS TEST

Let $X' = (X_1, \ldots, X_n)$ be a $p \times n$ matrix of n observations on a p-dimensional vector with sample mean and covariance $\bar{X} = n^{-1}(X_1 + \ldots + X_n)$ and $S = n^{-1}\check{X}'\check{X}$ where $\check{X}' = (X_1 - \bar{X}, \ldots, X_n - \bar{X})$.

Create a matrix with the reciprocals of the standard deviation on the diagonal:

$$V = \operatorname{diag}\left(S_{11}^{-1/2}, \ldots, S_{pp}^{-1/2}\right),$$

and form the correlation matrix $C = VSV$. Define the $p \times n$ matrix of transformed observations:

$$R' = H\Lambda^{-1/2}H'V\check{X}',$$

with $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, the matrix with the eigenvalues of C on the diagonal. The columns of H are the corresponding eigenvectors, such that $H'H = I_p$ and $\Lambda = H'CH$. Using population values for C and V, a multivariate normal can thus be transformed into independent standard normals; using sample values this is only approximately so.

We may now compute univariate skewness and kurtosis of each transformed n–vector of observations. Defining $B_1' = (\sqrt{b_{11}}, \ldots, \sqrt{b_{1p}})$, $B_2' = (b_{21}, \ldots, b_{2p})$ and ι as a p-vector of ones, the test statistic:

$$\frac{nB_1'B_1}{6} + \frac{n(B_2 - 3\iota)'(B_2 - 3\iota)}{24} \underset{a}{\sim} \chi^2(2p)$$

will again require large samples.

The proposed multivariate statistic is:

$$E_p = Z_1'Z_1 + Z_2'Z_2 \underset{app}{\sim} \chi^2(2p),$$

where $Z_1' = (z_{11}, \ldots, z_{1p})$ and $Z_2' = (z_{21}, \ldots, z_{2p})$ are determined by (1) and (2) given in the appendix. So after transformation of the data to approximately independent standard normals, the univariate test is applied to each dimension. Using the correlation matrix, rather than the covariance, makes the test scale invariant. Basing the square root of C on the eigenvectors gives invariance to ordering (using Choleski decomposition to compute the square root is inadvisable, as it makes the test sensitive to the ordering of the variables). In practice it will be useful to inspect all elements in Z_1 and Z_2 and maybe even compute the transformed skewness and kurtosis of the original data vectors.

If the rank of C is less than p, some eigenvalues will be zero. In that case the following procedure can be used. Select the eigenvectors corresponding to the p^* non-zero eigenvalues, G say, and create a new data matrix $X^* = XG$. This will be an $n \times p^*$ matrix. Now compute E_p using X^* and base the tests on p^* degrees of freedom. Simulations showed that this works satisfactorily.

4. FOUR OTHER MULTIVARIATE TESTS

Four other tests for multivariate normality are used in our size and power comparisons. Whereas §3 suggests to first transform to approximate independent normality and then compute marginal skewness and kurtosis coefficients, Small (1980) took the opposite route: weight the marginal skewness and kurtosis coefficients of the raw variables by their approximate correlations. Let B_1 and B_2 denote the vectors of sample skewness and kurtosis of the original data, which has sample correlation matrix $C = (c_{ij})$. Small (1980) used the Bowman and Shenton (1975) correction, but here we use §2 to compute the transformed skewness and kurtosis of the original data vectors obtaining Z_1 and Z_2, and:

$$Q_p = Z_1' U_1^{-1} Z_1 + Z_2' U_2^{-1} Z_2 \underset{app}{\sim} \chi^2(2p),$$

where $U_1 = (c_{ij}^3)$ and $U_2 = (c_{ij}^4)$.

The next test is that proposed by Mardia (1970). Create the $n \times n$ matrix:

$$D = (d_{ij}) = \check{X} S^{-1} \check{X}',$$

and define multivariate measures of skewness and kurtosis as:

$$b_{1p}^* = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}^3 \text{ and } b_{2p}^* = \frac{1}{n} \sum_{i=1}^{n} d_{ii}^4.$$

An omnibus test based on these measures is:

$$M_p = \frac{n b_{1p}^*}{6} + \frac{n \left(b_{2p}^* - p(p+2)\right)^2}{8p(p+2)} \underset{a}{\sim} \chi^2 \left(\frac{p(p+1)(p+2)}{6} + 1 \right).$$

A large sample will be required for the χ^2 distribution to hold.

Recently, Mudholkar, McDermott and Srivastava (1992) proposed a new test of multivariate normality based on the correlation between the normalized diagonal of D and jackknifed estimates of its variance. At various stages the statistic is transformed and empirically adjusted, resulting in a statistic which is approximately normally distributed. We square this statistic to obtain the χ^2 equivalent:

$$Z_p = \left(\frac{z_p + A_1(p)/n - A_2(p)/n^2}{\{B_1(p)/n - B_2(p)/n^2\}^{\frac{1}{2}}} \right)^2 \underset{app}{\sim} \chi^2(1).$$

The original paper shows how to obtain A_i and B_i. To obtain a correct size for the test statistic, we found that we had to add $\mu_{n,p} = A_1(p)/n - A_2(p)/n^2$ to z_p rather than subtract it.

Finally, we consider the multivariate Shapiro-Wilk test as proposed by Royston (1983). This combines the univariate tests (see Royston, 1982) creating a statistic H which has an approximate χ^2 distribution with non-integer degrees of freedom e, where $1 \le e \le p$. For convenience we translate this into a $\chi^2(1)$ using the Wilson-Hilferty approximation:

$$w = \left\{ \left(\frac{H}{e} \right)^{\frac{1}{3}} - 1 + \frac{2}{9e} \right\} \left(\frac{9e}{2} \right)^{\frac{1}{2}}, \quad W_p = w^2 \underset{app}{\sim} \chi^2(1).$$

5. SIZE OF THE TESTS

To investigate the size of the E_p test we simulated rejection frequencies under the null of normality, using $\chi^2(2p)$ critical values. The reported simulations were done in Ox (a C++ like matrix language), using the internal random number generator. The computations were repeated in Gauss, resulting in minor descrepancies owing to a different random number generator. Each table is based on common random numbers. We report the approximate Monte Carlo standard error as MCSE $= \sqrt{\{q(1-q)/M\}}$ where M is the number of replications and q the tail probability of interest.

Table 1 contrasts E_p with its asymptotic version (E_p^a) and the form suggested by Anscombe and Glynn (1983), labelled E_p^V. The latter uses the Pearson type V distribution for the kurtosis and a Johnson S_U for the skewness. Table 1 shows that this does not control the size as well as E_p. Figure 1 illustrates the benefits of taking the dependence into account.

Tables 2 and 3 give the size for some multivariate cases. In Table 2 the simulations are under the null of independent normality, with all tests showing acceptable size, although W_p shows that the Wilson-Hilferty approximation which we adopted only works moderately well in the bivariate case. Table 3 is for the bivariate normal case, with various correlation coefficients. It shows that Q_p is more sensitive to ρ than E_p. Both M_p and Z_p are invariant to ρ.

6. POWER COMPARISONS

For the power simulations we compute rejection frequencies at 5% under the alternative distribution, using a sample size of 50 observations. The critical values for E_p, Z_p and W_p are taken from the χ^2 table; the 5% critical values for M_p are found by simulation: 18.2 for $p=3$, $n=50$ and 10.0 for $p=2$, $n=50$.

In the first set of experiments we use the entire Johnson system as the set of alternative distributions. This results in power simulations forming a two-dimensional grid over the skewness–kurtosis plane. A similar approach was adopted by Baringhaus, Danschke and Henze (1989) who considered four univariate tests, one of them the test of Bowman and Shenton (1975). Because of symmetry we only consider positive skewness. Remember that $\beta_2 > 1+\beta_1$ always; power in that part of the skewness–kurtosis plane which does not satisfy this relation are set to zero, resulting in the flat area in the far corner. Along the edge $\beta_2 = 1+\beta_1$ are the bimodal distributions. The normal distribution is at $(\sqrt{\beta_1}, \beta_2) = (0,3)$; the exponential at $(2,9)$, just outside the graphs.

The experiments use three independent random variables, and can be summarized as follows:

label	n	M	p	X_1	X_2	X_3
P12	50	1000	3	Johnson	standard normal	standard normal
P30	50	1000	3	Johnson	Johnson	Johnson

The graphs are presented for P12 only, for the test statistics E_p, Z_p and M_p; P12 for Q_p very closely resembles that of E_p and is omitted. We see from Figs. 2–4 that E_p has the best power properties. Z_p has virtually no power against any of these alternatives. Z_p focuses only on the elements of D which form the kurtosis component of M_p, so low power against skewed alternatives is expected, however, power in the kurtosis dimension is absent too. Figure 5 shows the difference of power between E_p and W_p, and confirms an established result for the univariate case, namely that the Shapiro-Wilk test has higher power against platykurtic distributions, and lower power against leptokurtic distributions. Or more precisely: Shapiro-Wilk has higher power against alternatives in the bimodal area of the Johnson S_B system, but lower against S_U. The graphs for the P30 simulations are not presented, but give the same overall picture: all powers have improved, but the relative ranking remains the same.

We experimented with an alternative form of the proposed test statistic E_p using only one degree of freedom for the kurtosis component: the p χ^2 components of the kurtosis are added up, giving (non-integer) degrees of freedom $\alpha = pa + c\sum b_{1i}$ for $\chi = 2k\sum(b_{2i} - 1 - b_{1i})$, to which the Wilson-Hilferty transformation is applied (these replace the α and χ in (2), given in the appendix). This alternative test results in somewhat better size properties, but a loss of power in low kurtosis directions (a property shared by M_p which also has only one kurtosis component). This form could be useful if all kurtosis components show departure in the same direction.

In the next set of power experiments, the alternative is the generalized Burr-Pareto-logistic distribution with normal marginals, see Cook and Johnson (1986). The parameters α and β correspond to the notation used there. The case of $\alpha \to \infty$ and $\beta = 0$ corresponds to independent normals, whereas $\alpha \to \infty$ for $\beta \neq 0$ is Morgenstern's bivariate uniform distribution, still with normal marginals. We see in Table 4 that Mardia's test performs best here, followed by E_p. The most salient feature, however, is the complete lack of power of the Q_p and W_p tests against a non-normal bivariate distribution with normal marginals. It is interesting to note that the test suggested by Malkovich and Afifi (1973), which consists of that linear combination of the variables which maximizes skewness or kurtosis, will also fail in this situation. The W_p^* statistic is an alternative form of the multivariate Shapiro-Wilk test, where the data are first transformed to approximate independent normality as for E_p; then the p univariate Shapiro-Wilk tests are squared and added up, resulting in a $\chi^2(p)$ test statistic.

We conclude that E_p is the preferred test of the four considered. It is both simple, has correct size and good power properties. However, E_p could potentially be used in conjunction with M_p.

ACKNOWLEDGEMENTS

We wish to thank Sir David Cox, David Hendry, Chris Orme and Neil Shephard for helpful comments. Financial support from the UK Economic and Social Science Research Council is gratefully acknowledged by Jurgen Doornik.

APPENDIX

The transformation for the skewness $\sqrt{b_1}$ into z_1 is as in D'Agostino (1970):

$$
\begin{aligned}
\beta &= \frac{3(n^2+27n-70)(n+1)(n+3)}{(n-2)(n+5)(n+7)(n+9)}, \\
\omega^2 &= -1+\{2(\beta-1)\}^{\frac{1}{2}}, \\
\delta &= \frac{1}{\left\{\log(\sqrt{\omega^2})\right\}^{\frac{1}{2}}}, \\
y &= \sqrt{b_1}\left\{\frac{\omega^2-1}{2}\frac{(n+1)(n+3)}{6(n-2)}\right\}^{\frac{1}{2}}, \\
z_1 &= \delta\log\left\{y+(y^2+1)^{\frac{1}{2}}\right\}.
\end{aligned}
\tag{1}
$$

The kurtosis b_2 is transformed from a gamma distribution to χ^2, which is then translated into standard normal z_2 using the Wilson-Hilferty cubed root transformation:

$$
\begin{aligned}
\delta &= (n-3)(n+1)(n^2+15n-4), \\
a &= \frac{(n-2)(n+5)(n+7)(n^2+27n-70)}{6\delta}, \\
c &= \frac{(n-7)(n+5)(n+7)(n^2+2n-5)}{6\delta}, \\
k &= \frac{(n+5)(n+7)(n^3+37n^2+11n-313)}{12\delta}, \\
\alpha &= a+b_1c, \\
\chi &= (b_2-1-b_1)2k, \\
z_2 &= \left\{\left(\frac{\chi}{2\alpha}\right)^{\frac{1}{3}}-1+\frac{1}{9\alpha}\right\}(9\alpha)^{\frac{1}{2}}.
\end{aligned}
\tag{2}
$$

We base a numerical example on the iris data of Fisher (1936), which was used for the same purpose by Small (1980) and Fang and Wang (1994, example 6.5). Using the 50 observations for *Iris Setosa* on sepal length, sepal width, petal length and petal width, we find for each individual variable: skewness (0.11645, 0.039921, 0.10318, 1.2159) and kurtosis (2.6542, 3.7442, 3.8046, 4.4343). Applying the transformation to approximate normality discussed in §3 yields skewness $B_1' = (0.19965, -0.17132, 0.15837, 1.1610)$ and kurtosis $B_2' = (2.8221, 4.1994, 3.9722, 4.5793)$. Application of (1) and (2) then yields $Z_1' = (0.63839, -0.54876, 0.50762, 3.1862)$ and $Z_2' = (0.24687, 2.5423, 2.2381, -1.3278)$. We see that after rotation, skewness is a problem in one dimension, whereas excess kurtosis is evident in two dimensions. The resulting test statistics are: $E_p = 24.4145$, $Q_p = 24.0387$, $M_p = 27.3413$, $Z_p = 1.12299$. Both E_p and Q_p are significant at less than 1%, whereas the Z_p value corresponds to the 29 per cent point, and M_p to 10% (based on simulation results). In this case the non-normality of petal width seems to be the main factor, as without that variable non-normality is not rejected at the 5% level, with test-statistics of 10.8174 (9%), 8.74419 (19%), 12.9066 (20%) and 1.06251 (30%) respectively. The same conclusion was reached by Royston (1983).

REFERENCES

Anscombe, G. J. and Glynn, D. F. (1983). Distribution of the kurtosis statistic b_2 for normal samples. *Biometrika* **70**, 227–234.

Baringhaus, L., Danschke, R. and Henze, N. (1989). Recent and classical tests for normality — A comparative study. *Comm. Statist. B* **18**, 363–379.

Bowman, K. O. and Shenton, L. R. (1975). Omnibus test contours for departures from normality based on $\sqrt{b_1}$ and b_2. *Biometrika* **62**, 243–250.

Cook, R. D. and Johnson, M. E. (1986). Generalized Burr-Pareto-logistic distributions with applications to a uranium exploration data set. *Technometrics* **28**, 123–131.

Cox, D. R. and Small, N. J. H. (1978). Testing multivariate normality. *Biometrika* **65**, 263–272.

Cox, D. R. and Wermuth, N. (1994). Tests of linearity, multivariate normality and the adequacy of linear scores. *Appl. Statist.* **43**, 347–355.

D'Agostino, R. B. (1970). Transformation to normality of the null distribution of g_1. *Biometrika* **57**, 679–681.

—— (1982). Departures from normality, testing for. In Kotz, S., Johnson, N. L. and Read, C. B. (eds.), *Encyclopedia of Statistical Sciences*, vol. 2. Amsterdam: North-Holland.

Fang, K.-T. and Wang, Y. (1994). *Number-theoretic Methods in Statistics.* London: Chapman and Hall.

Fisher, R. A. (1936). The use of multiple measurements in taxonomic problems. *Annals of Eugenics*, **7**, 179–188. Reprinted in R.A. Fisher (1950). *Contributions to Mathematical Sciences.* London: John Wiley and Sons.

Gnanadesikan, R. (1977). *Methods for Statistical Data Analysis of Multivariate Observations.* London: John Wiley and Sons.

Kendall, M. G., Stuart, A. and Ord, J. K. (1987). *Advanced Theory of Statistics* 5th edn., vol. 1. London: Charles Griffin and Co.

Lütkepohl, H. and Schneider, W. (1989). Testing for normality of autoregressive time series. *Comp. Statist. Quart.* **2**, 151–168.

Malkovich, J. F. and Afifi, A. A. (1973). Testing for normality of autoregressive time series. *J. Am. Statist. Ass.* **68**, 176–179.

Mardia, K. V. (1970). Measures of multivariate skewness and kurtosis. *Biometrika* **57**, 519–530.

—— (1980). Tests of univariate and multivariate normality. In Krishnaiah, P. R. (ed.), *Handbook of Statistics*, vol. 1, ch. 9. Amsterdam: North-Holland.

Mudholkar, G. S., McDermott, M. and Srivastava, D. K. (1992). A test of p–variate normality. *Biometrika* **79**, 850–854.

Pearson, E. S., D'Agostino, R. B. and Bowman, K. O. (1977). Test for departure from normality: Comparison of powers. *Biometrika* **64**, 231–246.

Pearson, E. S. and Hartley, H. O. (1972). *Biometrika Tables for Statisticians*, vol. II. Cambridge: Cambridge University Press.

Pierce, D. A. (1985). Testing normality of errors in regression models. *Biometrika* **72**, 293–297.

Pierce, D. A. and Gray, R. J. (1982). Testing normality of errors in regression models. *Biometrika* **69**, 233–236.

Royston, J. P. (1982). An extension of Shapiro and Wilk's W test for normality to large samples. *Appl. Statist.* **31**, 115–124.

—— (1983). Some techniques for assessing multivariate normality based on the Shapiro–Wilk W. *Appl. Statist.* **32**, 121–133.

Shenton, L. R. and Bowman, K. O. (1977). A bivariate model for the distribution of $\sqrt{b_1}$ and b_2. *J. Am. Statist. Ass.* **72**, 206–211.

Small, N. J. H. (1980). Marginal skewness and kurtosis in testing multivariate normality. *Appl. Statist.* **29**, 85–87.

—— (1985). Multivariate normality, testing for. In Kotz, S., Johnson, N. L. and Read, C. B. (eds.), *Encyclopedia of Statistical Sciences*, vol. 6. Amsterdam: North-Holland.

Table 1. *Empirical size of* E_p^a, E_p^V *and* E_p *for* $p = 1$, $M = 10\,000$.

	E_p^a				E_p^V				E_p			
n	20%	10%	5%	1%	20%	10%	5%	1%	20%	10%	5%	1%
10	.033	.017	.010	.002	.158	.093	.057	.024	.204	.101	.048	.007
20	.062	.036	.022	.010	.165	.091	.055	.018	.185	.092	.046	.011
50	.094	.055	.035	.012	.177	.095	.055	.017	.178	.088	.046	.011
150	.138	.070	.044	.018	.191	.098	.055	.016	.182	.092	.047	.012
MCSE	.004	.003	.002	.001	.004	.003	.002	.001	.004	.003	.002	.001

Table 2. *Empirical size of* E_p, Q_p, Z_p *and* W_p *for* $p = 2, 3, 6$, $M = 10\,000$.

	E_p				Q_p				Z_p				W_p			
n, p	20%	10%	5%	1%	20%	10%	5%	1%	20%	10%	5%	1%	20%	10%	5%	1%
20,2	.184	.097	.049	.011	.183	.097	.046	.009	.225	.110	.051	.007	.190	.089	.038	.005
50,2	.179	.092	.051	.015	.177	.096	.050	.014	.208	.104	.054	.009	.192	.089	.039	.007
150,2	.181	.096	.049	.013	.181	.096	.049	.014	.204	.103	.051	.010	.192	.091	.038	.006
20,3	.184	.093	.048	.011	.185	.093	.046	.010	.220	.109	.049	.007	.195	.094	.044	.008
50,3	.180	.093	.051	.015	.180	.095	.052	.015	.215	.107	.050	.006	.201	.093	.045	.009
150,3	.182	.095	.054	.016	.183	.097	.053	.016	.200	.102	.050	.010	.205	.103	.047	.008
20,6	.179	.093	.048	.012	.179	.091	.048	.011	.217	.109	.052	.008	.200	.102	.052	.010
50,6	.177	.092	.052	.012	.180	.096	.051	.013	.218	.106	.052	.008	.208	.099	.051	.009
150,6	.182	.094	.052	.015	.182	.096	.053	.016	.198	.097	.045	.009	.207	.105	.053	.011

Table 3. *Empirical size of E_p, Q_p and W_p for $p = 2$, $n = 50$, $M = 10\,000$.*

	E_p				Q_p				W_p			
ρ	20%	10%	5%	1%	20%	10%	5%	1%	20%	10%	5%	1%
.99	.180	.091	.048	.013	.212	.114	.065	.021	.152	.058	.029	.006
.9	.178	.095	.050	.013	.200	.109	.062	.019	.153	.065	.027	.007
.5	.180	.096	.052	.013	.182	.097	.055	.015	.192	.090	.041	.006
0	.179	.092	.051	.015	.177	.096	.050	.014	.192	.089	.039	.007

Table 4. *Power at 5% against generalized Burr-Pareto-logistic with normal marginals for $p = 2$, $n = 50$, $M = 5\,000$.*

α	β	E_p	Q_p	M_p	Z_p	W_p	W_p^*	α	β	E_p	Q_p	M_p	Z_p	W_p	W_p^*
.1	−1	.955	.139	.993	.544	.027	.912	1	−1	.053	.054	.248	.064	.046	.062
.1	−.5	.954	.145	.993	.573	.028	.906	1	−.5	.070	.052	.238	.066	.046	.069
.1	0	.953	.153	.990	.597	.026	.906	1	0	.085	.053	.251	.065	.044	.083
.1	.5	.952	.162	.991	.621	.025	.905	1	.5	.105	.052	.268	.068	.046	.099
.1	1	.955	.179	.991	.643	.026	.911	1	1	.134	.048	.288	.073	.043	.116
.2	−1	.774	.068	.941	.260	.039	.685	2	−1	.041	.052	.144	.058	.041	.051
.2	−.5	.780	.070	.940	.285	.035	.692	2	−.5	.042	.052	.133	.060	.040	.050
.2	0	.788	.075	.935	.315	.032	.700	2	0	.047	.050	.132	.059	.041	.052
.2	.5	.799	.084	.936	.349	.029	.715	2	.5	.057	.051	.142	.064	.043	.058
.2	1	.815	.095	.938	.387	.028	.731	2	1	.073	.052	.146	.064	.041	.065
.5	−1	.212	.053	.543	.084	.049	.184	10	−1	.058	.048	.053	.050	.039	.056
.5	−.5	.242	.054	.542	.088	.048	.197	10	−.5	.047	.047	.055	.050	.039	.053
.5	0	.272	.055	.552	.106	.047	.218	10	.0	.045	.048	.057	.050	.037	.049
.5	.5	.311	.056	.577	.124	.046	.251	10	.5	.050	.051	.061	.053	.037	.051
.5	1	.369	.057	.611	.144	.043	.296	10	1	.062	.053	.057	.053	.039	.053

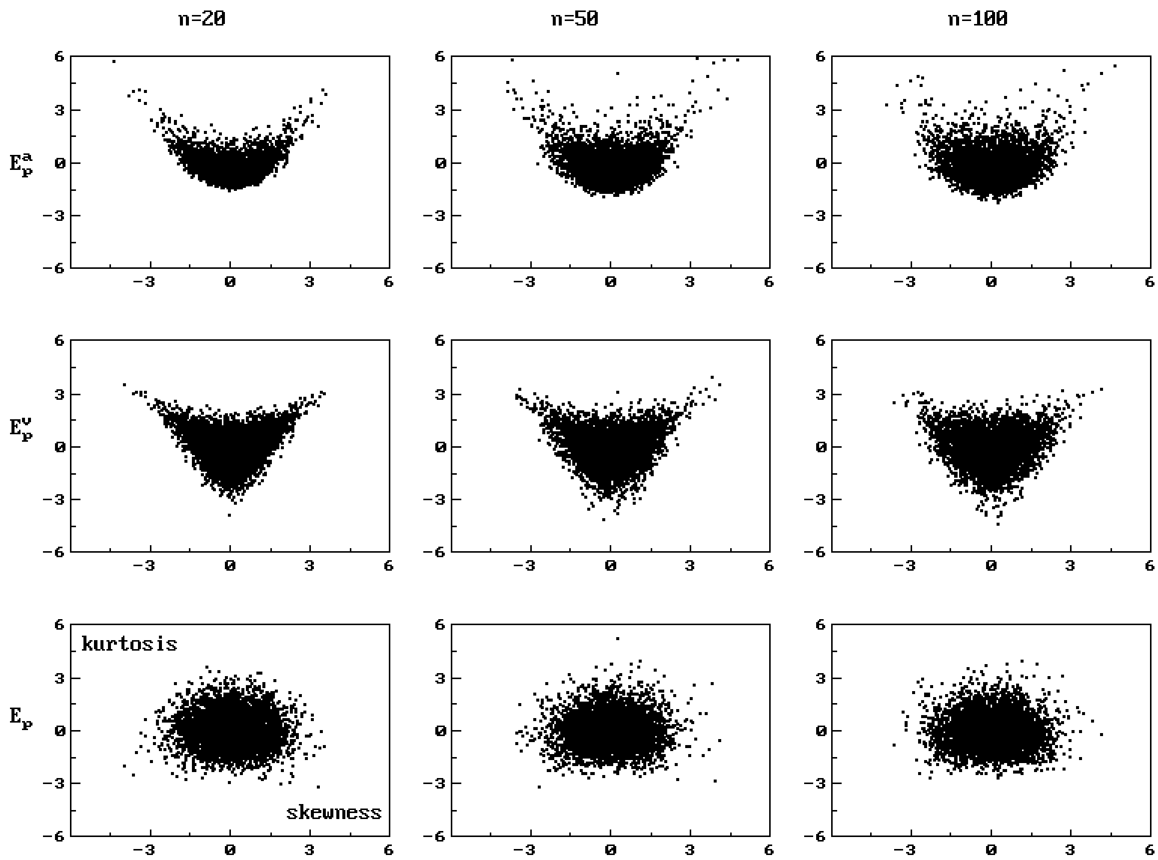

Fig. 1. Cross-plots of transformed skewness and kurtosis for 5000 samples of sizes $n = 20, 50, 100$ from the standard normal.

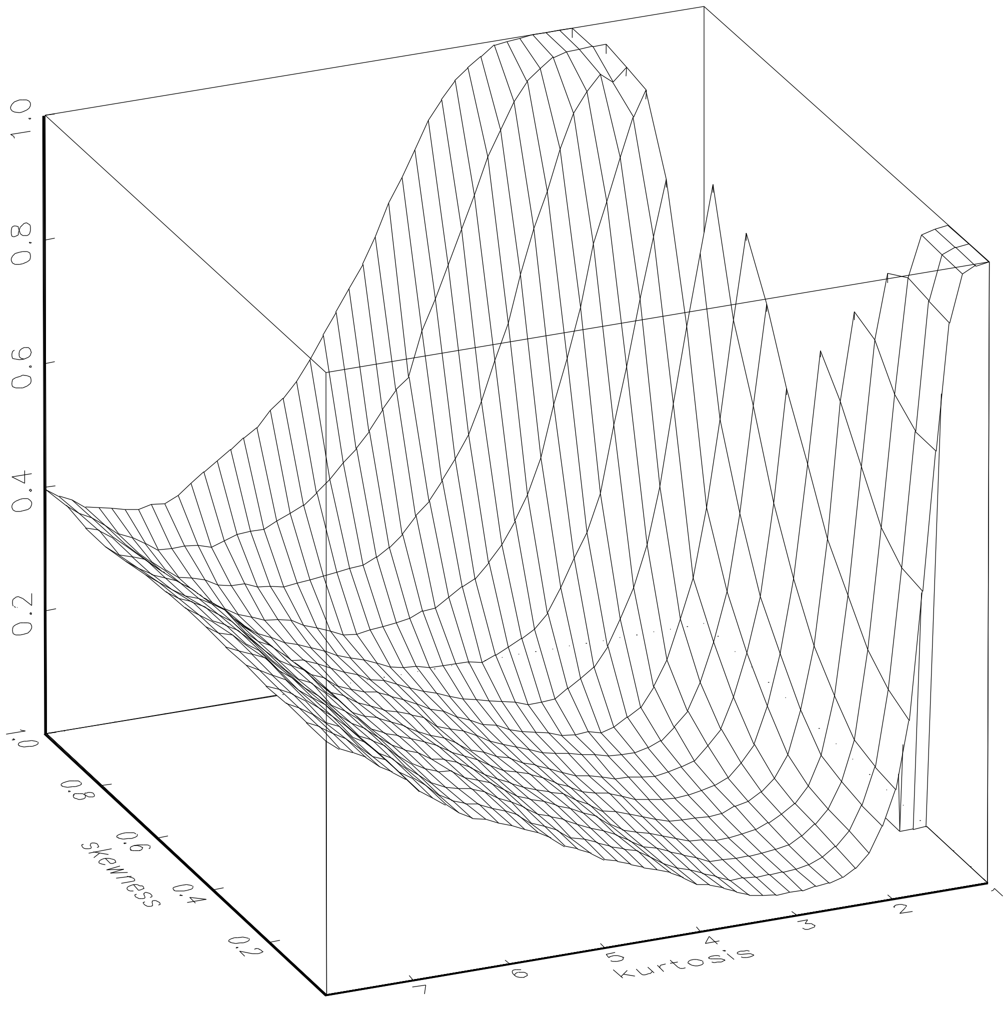

Fig. 2. P12 for E_p: power against two standard normal marginals and the Johnson system for the third marginal.

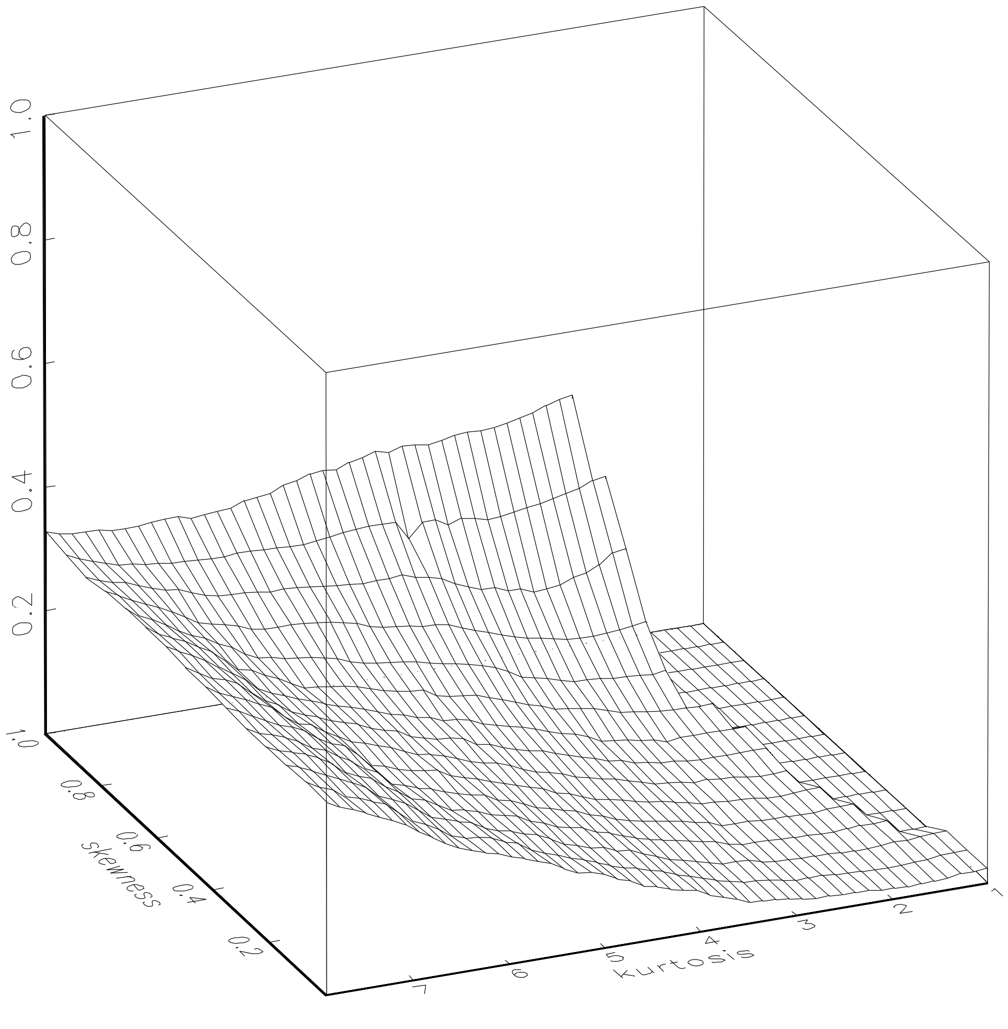

Fig. 3. P12 for M_p: power against two standard normal marginals and the Johnson system for the third marginal.

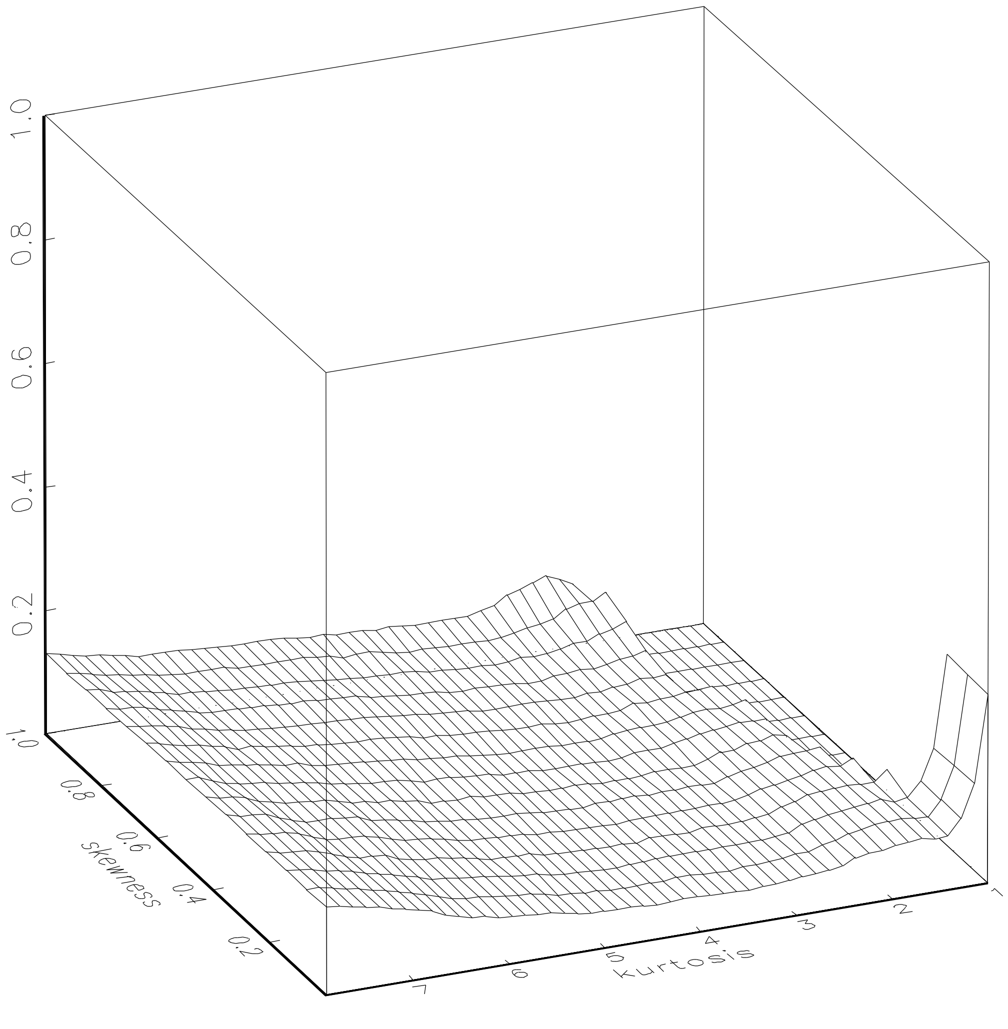

Fig. 4. P12 for Z_p: power against two standard normal marginals and the Johnson system for the third marginal.

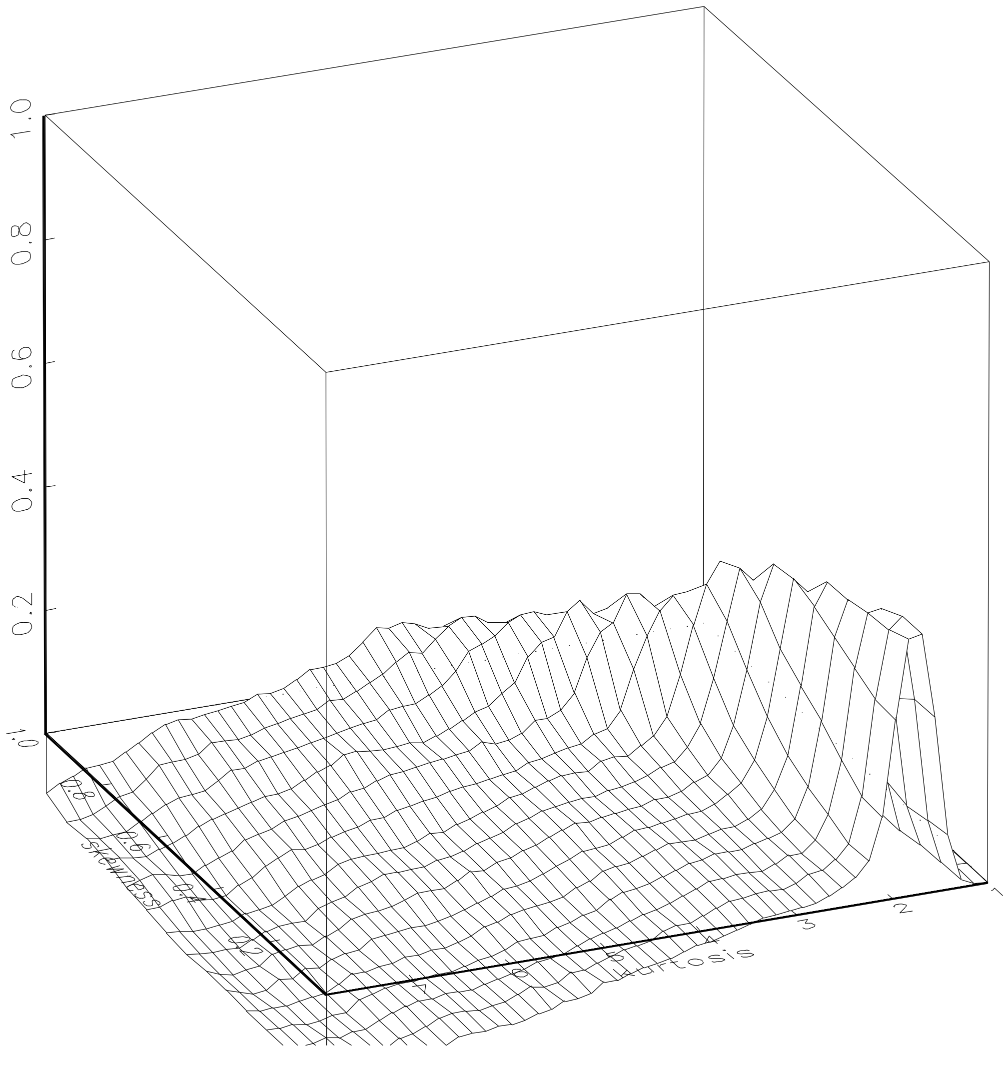

Fig. 5. P12 for $W_p - E_p$: difference in power against two standard normal marginals and the Johnson system for the third marginal.