Modelling by Lévy Processess for Financial Econometrics

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ABSTRACT This paper reviews some recent work in which Lévy processes are used to model and analyse time series from financial econometrics. A main feature of the paper is the use of positive Ornstein-Uhlenbeck (OU) type processes inside stochastic volatility processes. The basic probability theory associated with such models is discussed in some detail.

1. Introduction

This paper reviews some recent work in which Lévy processes are used to model and analyse financial time series at the "microscopic" tick-by-tick scale and at the larger "macroscopic" time scale of hourly or daily data. The models aim to incorporate as many as possible of the main stylized features of financial series, be they stock prices, foreign exchange rates or interest rates, while maintaining mathematical tractability.

On the macroscopic scales prototypical stochastic volatility models for one-dimensional variates are of the form

$$\mathrm{d}x^*(t) = \left\{\mu + \beta\sigma^2(t)\right\}\mathrm{d}t + \sigma(t)\mathrm{d}w(t) \tag{1.1}$$

in the case of log stock prices or log foreign exchange rates, and of the form

$$\mathrm{d}x(t) = \left\{\mu + \beta\sigma^2(t) - \phi x(t)\right\}\mathrm{d}t + \sigma(t)\mathrm{d}w(t) \tag{1.2}$$

for short interest rates[1]. In both cases, $w(t)$ is Brownian motion and $\sigma^2(t)$ is a non-negative stationary stochastic process, for simplicity assumed independent of $w(t)$. The solution $x^*(t)$ of (1.1) is a stochastic process of "integrated" type whereas the solutions of (1.2), in which we are interested, are stationary. Models of the type (1.1) and (1.2) are referred to as *SV* (stochastic volatility) models and *SSV* (stationary stochastic volatility) models, respectively.

Of particular interest are cases where $\sigma^2(t)$ is of OU type (Ornstein-Uhlenbeck type) or is a superposition of such processes — such models were introduced in this context by Barndorff-Nielsen and Shephard (1999). In the former instance, σ^2 satisfies a stochastic differential equation of the form

$$\mathrm{d}\sigma^2(t) = -\lambda\sigma^2(t)\mathrm{d}t + \mathrm{d}z(\lambda t)$$

where $z(t)$ is a Lévy process with positive increments; thus $z(t)$ is a subordinator[2].

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[1] In the interest rate context it may make sense to replace the Brownian motion $w(t)$ by a Levy process with non-negative increments (a subordinator), in order to guarantee (so long as μ and β are non-negative) that the process x(t) is non-negative.

[2] The rather unusal timing in the subordinator has been selected so that as λ changes the marginal distribution of $\sigma^2(t)$ does not.

Some aspects of studies of turbulence will also be discussed, motivated by the fact that there are several striking similarities - as well as important differences - between key empirical features of observational series in finance on the one hand and turbulence on the other. See in particular Section 2, which summarizes and compares the stylized traits in question. Sections 3-9 review the probabilistic theory needed for the construction and analysis of the financial models, and in Section 10 we take up the discussion of those models, necessarily also in summary form. The final Section 11 considers some further issues and possible future work.

In the sequel we shall use the following notation for cumulant transforms of a random variate x

$$\mathrm{C}\{\zeta \ddagger x\} = \log \mathrm{E}\{e^{i\zeta x}\}, \quad \mathrm{K}\{\theta \ddagger x\} = \log \mathrm{E}\{e^{\theta x}\} \quad \text{and} \quad \bar{\mathrm{K}}\{\theta \ddagger x\} = \log \mathrm{E}\{e^{-\theta x}\} \tag{1.3}$$

with straightforward extensions of the notation to more general random variates.

2. Stylized features of finance and turbulence

A number of characteristic features of observational series from finance and from turbulence are summarised in table 1. The features are widely recognized as being essential for understanding and modelling within these two, quite different, subject areas. In finance the observational series concerned consist of values of assets such as stocks or (logarithmic) stock returns or exchange rates, while in turbulence the series typically give the velocities or velocity derivatives (or differences), in the mean wind direction of a large Reynolds number wind field. For some typical examples of empirical probability densities of logarithmic asset returns see Eberlein and Keller (1995) and Shephard (1996), while for velocity differences in large Reynolds number wind fields see, for instance, Barndorff-Nielsen (1998a).

	Finance	Turbulence
varying activity	volatility	intermittency
semiheavy tails	+	+
asymmetry	+	+
aggregational Gaussianity	+	+
0 autocorrelation	+	−
quasi long range dependence	+	+
scaling/selfsimilarity	[+]	+

TABLE 1. Stylised features.

A very characteristic trait of time series from turbulence as well as finance is that there seems to be a kind of switching regime between periods of relatively small random fluctuations and periods of high 'activity'. In turbulence this phenomenon is known as intermittency, see e.g. Frisch (1995, Ch. 8) for a thorough discussion, whereas in finance one speaks of stochastic volatility or conditional heteroscedasticity. For the integrated log-price process $x^*(t)$ in finance a basic expression of the volatility is given by the *quadratic variation* process $x^*[t]$, which is defined as

$$x^*[t] = \text{p-}\lim \sum \{x^*(t_{i+1}^r) - x^*(t_i^r)\}^2 \tag{2.1}$$

for any sequence of partitions $t_0^r = 0 < t_1^r < \cdots < t_{m_r}^r = t$ with $\sup_i\{t_{i+1}^r - t_i^r\} \to 0$ for $r \to \infty$. For SV models

$$x^*[t] = \int_0^t \sigma^2(u)\mathrm{d}u = \sigma^{2*}(t),$$

the integrated volatility of the process. A similar concept, called intermittency, is used in the turbulence literature. If we write $u = u(x,t)$ as the velocity at position x in the mean direction of the wind field at time t, then intermittency is defined as the energy dissipation rate per unit mass around position ξ:

$$e_r(\xi) = r^{-1}\int_{\xi-r/2}^{\xi+r/2}\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

measured over a symmetric interval of length r and centred at ξ (since u is considered as being stationary in time, t has been suppressed in the notation for e_r).

The term "semiheavy tails", in table 1, is intended to indicate that the data suggest modelling by probability distributions whose densities behave, for $x \to \pm\infty$, as

$$const.\,|x|^{\rho_\pm}\exp(-\sigma_\pm|x|)$$

for some $\rho_+, \rho_- \in \mathbf{R}$ and $\sigma_+, \sigma_- \geq 0$.

Velocity differences in turbulence show an inherent asymmetry consistent with Kolmogorov's modified theory of homogeneous high Reynolds number turbulence (cf. Barndorff-Nielsen (1986)). Distributions of financial asset returns are generally rather close to being symmetric around 0, but there is a definite tendency towards a dynamic version of asymmetry stemming from the fact that the market is prone to react differently to positive as opposed to negative returns, cf. for instance Shephard (1996, Subsection 1.3.4)). This reaction pattern, or at least part of it, is referred to as a 'leverage effect' whereby increased volatility tends to be associated with negative returns.

By aggregational Gaussianity is meant the fact that long term aggregation of financial asset returns, in the sense of summing the returns over longer periods, will lead to approximately normally distributed variates, and similarly in the turbulence context[3]. For illustrations of this, see for instance Eberlein and Keller (1995) and Barndorff-Nielsen (1986).

The estimated autocorrelation functions based on log price differences on stocks or currencies are generally (closely) consistent with an assumption of zero autocorrelation.

Nevertheless, this type of financial data exhibit 'quasi long range dependence' which manifests itself *inter alia* in the empirical autocorrelation functions of the absolute values or the squares of the returns, which stay positive for many lags.

For discussions of scaling phenomena in turbulence we refer to Frisch (1995). As regards finance, see Barndorff-Nielsen and Prause (1999) and Mantegna and Stanley (2000), and references given there. The latter work also, more broadly, discusses relations between finance and turbulence.

In addition, it is relevant to mention the one-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2},$$

which relates the velocity changes in time to velocity changes in position. This nonlinear partial differential equation may be viewed as a 'toy model' version of the Navier-Stokes equations of fluid

[3]However, in turbulence a small skewness generally persists, in agreement with Kolmogorov's theory of isotropic turbulence.

dynamics and, as such, has been the subject of extensive analytical and numerical studies, see for instance Frisch (1995, pp. 142-3), Barndorff-Nielsen and Leonenko (2000) and Bertoin (2000), and references given there. In finance, Burgers equation has turned up in work by Hodges and Carverhill (1993) and Hodges and Selby (1997). However, the interpretation of the equation in finance does not appear to have any relation to the role of the equation in turbulence.

3. Lévy processes and random fields

In this section we recall the definition of a Lévy process and a random field. The latter will be used extensively in the development of long-range dependent processes in Section 9 of this paper. We then study the cumulant generating functions of integrals defined with respect to Lévy processes and fields.

Recall that a Lévy process is a continuous in probability, cadlag stochastic process $z = \{z(t)\}_{t \geq 0}$ with independent and stationary increments and $z(0) = 0$. For such a process we have $\mathrm{C}\{\zeta \ddagger z(t)\} = t\mathrm{C}\{\zeta \ddagger z(1)\}$ and $z(1)$ has the Lévy-Khintchine representation

$$\mathrm{C}\{\zeta \ddagger z(1)\} = ia\zeta - \frac{1}{2}b\zeta^2 + \int_{\mathbf{R}} \left\{ e^{i\zeta x} - 1 - i\zeta\tau(x) \right\} U(\mathrm{d}x) \tag{3.1}$$

where the Lévy measure U is such that it has no atom at 0 and

$$\int_{\mathbf{R}} \min\{1, x^2\} U(\mathrm{d}x) < \infty$$

and where τ is a centering function that we choose as

$$\tau(x) = \begin{cases} x & \text{for} \quad |x| \leq 1 \\ \frac{x}{|x|} & \text{for} \quad |x| > 1. \end{cases} \tag{3.2}$$

Correspondingly, the process $z(t)$ is a sum $z(t) = at + \sqrt{b}w(t) + z_0(t)$ of a drift term at and where $w(t)$ and $z_0(t)$ are independent processes, $w(t)$ being Brownian motion and $z_0(t)$ a Lévy process with $\mathrm{C}\{\zeta \ddagger z_0(1)\} = \int_{\mathbf{R}} \left\{ e^{i\zeta x} - 1 - i\zeta\tau(x) \right\} U(\mathrm{d}x)$. If $b = 0$, and we shall mainly consider such cases, then z is said to be a Lévy *jump process* and if also $a = 0$ it is a Lévy *pure jump process*. If z has only non-negative increments, then it is a *subordinator* (cf. Bertoin (1996, Ch. III)).

A Lévy field on a region $\mathcal{S}$ is a random measure z on $\mathcal{S}$ such that the values of z corresponding to a countable number of disjoint (measurable) subsets of $\mathcal{S}$ are independent and such that for every subset A of $\mathcal{S}$ the random variable $z(A)$ is infinitely divisible. Hence, for such a field, z will have infinitesimal Lévy-Khintchine representation

$$\mathrm{C}\{\zeta \ddagger z(\mathrm{d}\omega)\} = i\zeta a(\mathrm{d}\omega) - \frac{1}{2}\zeta^2 b(\mathrm{d}\omega) + \int_{\mathbf{R}} \left\{ e^{i\zeta x} - 1 - i\zeta\tau(x) \right\} \nu(\mathrm{d}x; \mathrm{d}\omega) \tag{3.3}$$

where the (generalized) Lévy measure ν satisfies $\nu(\{0\}; A) = 0$ and

$$\int_{\mathbf{R}} \min\{1, |x|^2\} \nu(\mathrm{d}x; A) < \infty \tag{3.4}$$

for all A. The quantity (a, b, ν) is referred to as the characteristic triplet of the field z.

In the present paper we will only consider cases where z has no Gaussian components, i.e. the measure b is identically 0. Then z is termed a *Lévy jump field.*

There is no essential loss of generality in assuming that ν factorizes as

$$\nu(\mathrm{d}x;\mathrm{d}\omega) = U(\mathrm{d}x;\omega)\upsilon(\mathrm{d}\omega) \tag{3.5}$$

for some measure υ on $\mathcal{S}$. We shall use the notation

$$U^{-}(x;\omega) = U((-\infty,x];\omega) \quad \text{and} \quad U^{+}(x;\omega) = U([x,\infty);\omega) \tag{3.6}$$

for the tail masses of the measure $U(\cdot;\omega)$. Furthermore, in most of the cases to be considered, the Lévy measure $U(\cdot;\omega)$ is, for each ω, absolutely continuous with respect to Lebesgue measure on $\mathcal{S}$, with a density $u(x;\omega)$. When $U(\cdot;\omega)$ does not depend on ω we write $U(\mathrm{d}x)$, $U^{+}(x)$, etc.

Any Lévy process $z(t)$ induces a Lévy random measure z on $\mathbf{R}_{+}$, where $\mathbf{R}_{+} = (0,\infty)$, starting from the definition

$$z((a,b]) = z(b) - z(a). \tag{3.7}$$

Conversely, if z is a Lévy random measure on $\mathcal{S} = \mathbf{R}_{+}$ with characteristic triplet of the form $(0,0,U(\mathrm{d}x)\mathrm{d}\omega)$ then the prescription

$$z(t) = z((0,t]) \tag{3.8}$$

determines a Lévy process.

Functions f on $\mathcal{S}$ can, under suitable regularity conditions[4], be integrated with respect to the random field z. We use the notation

$$f \bullet z = \int_{\mathcal{S}} f(\omega) z(\mathrm{d}\omega) \tag{3.9}$$

for the integrals. In the Lévy process case (3.7)-(3.8) $f \bullet z$ coincides with the usual stochastic integral of f with respect to the process $z(t)$ and we write

$$f \bullet z = \int_{\mathbf{R}_{+}} f(t) z(\mathrm{d}t)$$

For Lévy processes a key result for many calculations is embodied in the formula

$$\mathrm{C}\{\zeta \ddagger f \bullet z\} = \int_{\mathbf{R}_{+}} \mathrm{C}\{\zeta f(\tau) \ddagger z(1)\}\mathrm{d}\tau \tag{3.10}$$

This rather well-known result follows essentially from the following formal calculation, using product

[4] A brief review of the relevant mathematical theory is given in Barndorff-Nielsen and Pérez-Abreu (1999).

integration and the independent scattering property of z:

$$\begin{aligned}
\exp\{\mathrm{C}\{\zeta \ddagger f \bullet z\}\} &= \mathrm{E}\left\{\exp\left(i\zeta \int_{\mathbf{R}_+} f dz\right)\right\} \\
&= \mathrm{E}\left[\prod_{\tau\in\mathbf{R}_+} \exp\{i\zeta f(\tau)\mathrm{d}z(\tau)\}\right] \\
&= \prod_{\tau\in\mathbf{R}_+} \mathrm{E}\left[\exp\{i\zeta f(\tau)\mathrm{d}z(\tau)\}\right] \\
&= \prod_{\tau\in\mathbf{R}_+} \exp\left[\mathrm{C}\{\zeta f(\tau) \ddagger \mathrm{d}z(\tau)\}\right] \\
&= \prod_{\tau\in\mathbf{R}_+} \exp\left[\mathrm{C}\{\zeta f(\tau) \ddagger z(1)\}\mathrm{d}\tau\right] \\
&= \exp\left[\int_{\mathbf{R}_+} \mathrm{C}\{\zeta f(\tau) \ddagger z(1)\}\mathrm{d}\tau\right].
\end{aligned}$$

More generally, for any Lévy random measure z with characteristic triplet $(0, 0, U(\mathrm{d}x)\upsilon(\mathrm{d}\omega))$ we have

$$\mathrm{C}\left\{\zeta \ddagger \int_{\mathcal{S}} f dz\right\} = \int_{\mathcal{S}} \kappa\{\zeta f(\omega)\}\, \upsilon(\mathrm{d}\omega) \tag{3.11}$$

where

$$\kappa(\zeta) = \int_{\mathbf{R}} \left\{e^{i\zeta x} - 1 - i\zeta\tau(x)\right\} U(\mathrm{d}x) \tag{3.12}$$

is the cumulant function of an infinitely divisible random variable with characteristic triplet $(0, 0, U)$.

Now consider the case where z is nonnegative. For such fields, integrals $f \bullet z$ of functions f with respect to z have representation (see Jacod and Shiryaev (1987, Ch. II, §4C))

$$f \bullet z = f \bullet a + \int_{\mathbf{R}_+}\int_{\mathcal{S}} f(\omega)xH(\mathrm{d}x;\mathrm{d}\omega) \tag{3.13}$$

with $a'(\mathrm{d}w)$ a measure on $\mathcal{S}$ and where $H(\mathrm{d}x;\mathrm{d}\omega)$ follows the Poisson law with mean $\nu(\mathrm{d}x;\mathrm{d}\omega)$. In this case the cumulant functional of the field is of the form

$$\mathrm{C}\{f \ddagger z\} = \mathrm{C}\{1 \ddagger f \bullet z\} = if \bullet a' + \int_{\mathbf{R}_+}\int_{\mathcal{S}} \left\{e^{if(\omega)x} - 1\right\} \nu(\mathrm{d}x;\mathrm{d}\omega) \tag{3.14}$$

Suppose in particular that υ (in (3.5)) is a probability measure which we therefore denote by π. Then equation (3.14) may be rewritten as

$$\bar{\mathrm{K}}\{\theta \ddagger f \bullet z\} = -\theta f \bullet a' - \int_{\mathcal{S}}\int_{\mathbf{R}_+} \left\{1 - e^{-\theta f(\xi)x}\right\} U(\mathrm{d}x;\xi)\pi(\mathrm{d}\xi) \tag{3.15}$$

and, for simplicity letting $a' = 0$, this can be given the symbolic form

$$\bar{\mathrm{K}}\{\theta \ddagger f \bullet z\} = \int_{\mathcal{S}} \bar{\mathrm{K}}\{\theta f(\xi) \ddagger y|\xi\}\pi(\mathrm{d}\xi) \tag{3.16}$$

where y and ξ are interpreted as a pair of random variables with ξ having law determined by π and such that y given ξ is a positive and infinitely divisible random variable with characteristic triplet $(0, 0, U(\cdot;\xi))$. For a positive infinitely divisible random variable y formula (3.15) specializes to

$$\bar{\mathrm{K}}\{\theta \ddagger y\} = -a'\theta - \int_{\mathbf{R}_+} \left\{1 - e^{-\theta x}\right\} U(\mathrm{d}x). \tag{3.17}$$

Extension to multivariate Lévy fields $z = (z_1, ..., z_m)$ is immediate. Lévy-Khintchine's infinitesimal representation takes the form

$$\mathrm{C}\{\zeta \ddagger z(\mathrm{d}\omega)\} = i\langle a(\mathrm{d}\omega), \zeta\rangle - \frac{1}{2}\langle \zeta b(\mathrm{d}\omega), \zeta\rangle + \int \left\{e^{i\langle\zeta,x\rangle} - 1 - i\zeta\tau(x)\right\} \nu(\mathrm{d}x; \mathrm{d}\omega) \tag{3.18}$$

where now $\zeta = (\zeta_1, ..., \zeta_m)$, a is an m-dimensional measure, b an $m \times m$ matrix valued measure and (3.2) and (3.4) apply with $|x|$ interpreted as Euclidean distance.

In extension of formula (3.17) we have that an m-dimensional random variate $y = (y_1, ..., y_m)$ all of whose coordinates are positive is infinitely divisible if and only if the cumulant function $\bar{\mathrm{K}}$ is of the form

$$\bar{\mathrm{K}}\{\theta \ddagger y\} = -\langle a', \theta\rangle - \int_{\mathbf{R}_+^m} (1 - e^{-\langle\theta,x\rangle})U(\mathrm{d}x) \tag{3.19}$$

where a' is an m-dimensional vector with all coordinates nonnegative and U is a Lévy measure satisfying $U(\mathbf{R}^m \backslash \mathbf{R}_+^m) = 0$ and

$$\int_{|x|\leq 1} |x|U(\mathrm{d}x) < \infty. \tag{3.20}$$

For a proof, see Skorohod (1991, pp. 156-157).

4. Selfdecomposability and Lévy processes

4.1. Selfdecomposability

A probability measure P on $\mathbf{R}$ is said to be *selfdecomposable* or to belong to Lévy's *class L*, if for each $\lambda > 0$ there exists a probability measure Q_λ on $\mathbf{R}$ such that

$$\phi(\zeta) = \phi(e^{-\lambda}\zeta)\phi_\lambda(\zeta) \tag{4.1}$$

where ϕ and ϕ_λ denote the characteristic functions of P and Q_λ, respectively. A random variable x with law in L is also called selfdecomposable, and it is infinitely divisible. The concept of selfdecomposability is closely related to that of stationary linear autoregressive time series of order 1, i.e. $AR(1)$ processes. Indeed, for such a process

$$x_n = \rho x_{n-1} + u_n,$$

with i.i.d. innovations $\{u_n\}$, we have

$$\mathrm{C}\{\zeta \ddagger x_n\} = \mathrm{C}\{\rho\zeta \ddagger x_{n-1}\} + \mathrm{C}\{\zeta \ddagger u_n\} \tag{4.2}$$

and since, by stationarity, $\mathrm{C}\{\zeta \ddagger x_n\} = \mathrm{C}\{\zeta \ddagger x_{n-1}\}$ the relation (4.2) is a cumulant version of (4.1). Essentially, then, the only possible $AR(1)$ processes are those for which the one-dimensional marginal law is selfdecomposable. And similarly for the OU processes, i.e. "AR(1) processes in continuous time", to be discussed in Section 6.

The class L is also characterized as the class of possible limit laws for normalized sequences of the form

$$b_n^{-1}(x_1 + x_2 + \cdots + x_n) - a_n$$

where $x_1, x_2, ..., x_n, ...$ is a sequence of independent random variables satisfying the uniform asymptotic negligibility condition (see, for example, Loève (1955, pp. 319–326)).

Further important characterizations of class L as a subclass of the set of all infinitely divisible laws is given by the following theorem

Theorem 4.1 Let $U(\mathrm{d}x)$ denote the Lévy measure of an infinitely divisible probability measure P on $\mathbf{R}$. Then the following three statements are equivalent:

(i) P is selfdecomposable

(ii) The functions on $\mathbf{R}_+$ given by $U^+(e^s)$ and $U^-(-e^s)$ are both convex

(iii) U is of the form $U(\mathrm{d}x) = u(x)\mathrm{d}x$ with

$$\bar{u}(x) = |x|\, u(x) \tag{4.3}$$

increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

If u is differentiable then the necessary and sufficient condition (ii) may be reexpressed as

$$u(x) + xu'(x) \leq 0\,, \tag{4.4}$$

for $x \neq 0$.

Suppose P is concentrated on $\mathbf{R}_+$, with 0 the lower bound of the support of P. Then P is selfdecomposable if and only if P is absolutely continuous with a density p for which there exists a monotonically decreasing function $\bar{u}$ such that

$$xp(x) = \int_0^x p(x-y)\bar{u}(y)\mathrm{d}y$$

for (almost all) $x > 0$. □

The equivalence of (i), (ii) and (iii) is due to Lévy (1937). A proof may be found also in Bar-Lev, Bshouty, and Letac (1992); see also Sato (1999). The final result in the theorem was established in the form given here by Steutel (1970), cf. Sato (1999, p. 385 and 426). Here we restrict discussion to a proof that (iii) implies (i). Let (a, b, U) be the characteristic triplet of P and assume that (iii) is satisfied. Verifying (i) amounts to showing that, for any $c \in (0, 1)$, the function

$$\psi_c(\zeta) = \log\phi(\zeta) - \log\phi(c\zeta)$$

can be written in the Lévy-Khintchine form (3.1). Now

$$\begin{aligned}\psi_c(\zeta) &= -\frac{1}{2}b(1-c^2)\zeta^2 + a(1-c)i\zeta \\ &\quad + \int_{\mathbf{R}\backslash\{0\}} \{e^{i\zeta x} - 1 - i\zeta\tau(x)\}u(x)\mathrm{d}x \\ &\quad - \int_{\mathbf{R}\backslash\{0\}} \{e^{ic\zeta x} - 1 - ic\zeta\tau(x)\}u(x)\mathrm{d}x\end{aligned}$$

and the last term may be rewritten as

$$\int_{\mathbf{R}\backslash\{0\}} \{e^{ic\zeta x} - 1 - ic\zeta\tau(x)\}u(x)\mathrm{d}x = \int_{\mathbf{R}\backslash\{0\}} \{e^{i\zeta x} - 1 - i\zeta c\tau(c^{-1}x)\}c^{-1}u(c^{-1}x)\mathrm{d}x.$$

Furthermore, letting

$$\delta_c(x) = \tau(x) - c\tau(c^{-1}x)$$

we have

$$\delta_c(x) = \begin{cases} 0 & \text{for} \quad |x| \le c \\ x - c\,\mathrm{sign}x & \text{for} \quad c < |x| \le 1 \\ (1-c)\,\mathrm{sign}x & \text{for} \quad |x| > c. \end{cases}$$

It follows that

$$\psi_c(\zeta) = a_c i\zeta - \frac{1}{2}b_c\zeta^2 + \int_{\mathbf{R}} \{e^{ic\zeta x} - 1 - i\zeta\tau(x)\}u_c(x)\mathrm{d}x$$

where the drift of the innovations is

$$a_c = a(1-c)\int_{\mathbf{R}\{0\}} \delta_c(x)u_c(x)\mathrm{d}x,$$

while the variance of the Wiener component to the innovations is

$$b_c = b(1-c^2).$$

Finally the Lévy density of the innovations is

$$u_c(x) = u(x) - c^{-1}u(c^{-1}x).$$

To see this note from (4.3)

$$u_c(x) = |x|^{-1}\{\bar{u}(x) - \bar{u}(c^{-1}x)\}$$

by the monotonicity property of $\bar{u}(x)$ it follows that $u_c(x)$ is a Lévy density. □

Yet another key characterization of selfdecomposability will be given in Theorem 4.3 below.

Example 4.1 *GIG laws.* All the generalized inverse Gaussian laws are selfdecomposable, a result due to Halgreen (1979). See further in Subsection 5.1 below. □

4.2. Multivariate case

An m-dimensional infinitely divisible random variate x and its distribution are called selfdecomposable and are said to be of class L if for any $c \in (0,1)$ there exists an m-dimensional random variate x_c, independent of x, such that

$$\mathrm{C}\{\zeta \ddagger x\} = \mathrm{C}\{\zeta \ddagger cx\} + \mathrm{C}\{\zeta \ddagger x_c\}$$

i.e.

$$x \stackrel{\mathcal{D}}{=} cx + x_c \tag{4.5}$$

where $\stackrel{\mathcal{D}}{=}$ means identity in law. It can be shown that then x_c is necessarily infinitely divisible, see Sato (1999, Proposition 15.5). Furthermore, we have

Theorem 4.2 Let x be an m-dimensional infinitely divisible random variate. Then x is selfdecomposable if and only if its Lévy measure U is of the form

$$U(B \times E) = \int_B \pi(\mathrm{d}\sigma) \int_0^\infty \mathbf{1}_E(r) k(r,\sigma) r^{-1} \mathrm{d}r \tag{4.6}$$

for $B \in \mathcal{B}(\mathbf{S}^{m-1})$, $E \in \mathcal{B}(\mathbf{R}_+)$, and where π is a probability measure on $\mathbf{S}^{m-1}$ and the function $k(r,\sigma)$ is nonnegative and Borel measurable in σ, nonincreasing in r, and

$$\int_0^\infty k(r,\sigma) r (1+r^2)^{-1} \mathrm{d}r = K < \infty \tag{4.7}$$

with K independent of σ ($\pi-$almost everywhere). □

Theorem 4.2 is given in Sato (1980); see also Urbanik (1969) and Wolfe (1982) for alternative representations.

Extension of the concept of multivariate selfdecomposability to operator selfdecomposability, where c in (4.5) is changed to a matrix, is discussed extensively in Jurek and Mason (1993).

Remark 4.1 Unless U is concentrated on an $(m-1)$-dimensional hyperplane of $\mathbf{R}^m$ it possesses a density u with respect to Lebesgue measure (Sato (1982) and Sato (1999)). □

Cases where the function k does not depend on σ, so that U is a product measure, often occur in practice.

Example 4.2 *Stable laws.* For all the multivariate infinite variance α-stable distributions ($0 < \alpha < 2$), U is of product measure form with $k(r) = r^{-\alpha}$ (cf. Samorodnitsky and Taqqu (1994, p. 66)). □

Example 4.3 *(IG,NIG) Lévy motion.* Consider a pair of independent Brownian motions (b_1, b_2) with drift vector $(\gamma, 0)$ where $\gamma \geq 0$. Let $x(t)$ denote the first passage time to level δt of b_1 and define $y(t)$ as $b_2(x(t))$. Then $z(t) = (x(t), y(t))$ is a bivariate Lévy process and $z(1)$ has probability density function

$$p(x,y;\delta,\gamma) = (2\pi)^{-1} \delta e^{\delta\gamma} x^{-2} \exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x + x^{-1} y^2)\right\} \tag{4.8}$$

and cumulant function

$$\mathrm{C}\{v, \eta \ddagger u, y\} = \delta\gamma[1 - \{1 - (2iv + \eta^2)/\gamma^2\}^{1/2}]. \tag{4.9}$$

The marginal processes $x(t)$ and $y(t)$ are, respectively, IG (inverse Gaussian) and NIG (normal inverse Gaussian) Lévy motions. Hence the name (IG, NIG) Lévy motion for $z(t)$.

The characteristic triplet of (4.8) is $((\delta\gamma, 0), 0, U)$ where the bivariate Lévy measure U has density

$$u(x, y) = (2\pi)^{-1}\delta x^{-2} \exp\left\{-\frac{1}{2}(\gamma^2 x + x^{-1}y^2)\right\} \tag{4.10}$$

with respect to Lebesgue measure. (A derivation of the formula for the Lévy density is given in Barndorff-Nielsen (2000b)).

Reexpressing $u(x, y)$ in polar coordinates $r > 0$, $\phi \in (-\pi/2, \pi/2)$ we have

$$u(\phi, r) = (2\pi)^{-1} r^{-1} g(\phi, r) \tag{4.11}$$

where

$$g(\phi, r) = \delta(\cos\phi)^{-2} \exp\left\{-\frac{1}{2} r(\cos\phi)^{-1}(\gamma^2\cos^2\phi + \sin^2\phi)\right\}.$$

As a function of r, $g(\phi, r)$ is decreasing and hence the law of $z(1)$ is selfdecomposable; in other words, the process $z(t)$ is of class L.

Some applications of the (IG, NIG) Lévy motion and certain generalizations of this in the context of finance are discussed in Barndorff-Nielsen (2000b). □

4.3. Relation between selfdecomposability and Lévy processes.

The following theorem is due to Jurek and Vervaat (1983) (cf. also Jurek and Mason (1993)).

Theorem 4.3 A random variable x has law in L if and only if x has a representation of the form

$$x = \int_0^\infty e^{-t}\mathrm{d}z(t) \tag{4.12}$$

where $z(t)$ is a Lévy process.

In this case the Lévy measures U and W of x and $z(1)$ are related by

$$U(\mathrm{d}x) = \int_0^\infty W(e^t\mathrm{d}x)\mathrm{d}t. \tag{4.13}$$

□

The process $z = \{z(t)\}_{t\geq 0}$ is termed the *background driving Lévy process* or the BDLP corresponding to x.

Remark 4.2 If $z(t)$ is a Lévy process then the integral

$$\int_0^\infty e^{-t}\mathrm{d}z(t)$$

exists, as the limit of $\int_0^T e^{-t}\mathrm{d}z(t)$ for $T \to \infty$, if and only if

$$\int_{|u|>1} \log(1+|u|)W(du) < \infty.$$

This holds irrespectively of whether the limiting procedure is taken to be convergence in law or almost surely. For a proof, see Jurek and Mason (1993, Thm. 3.6.6). □

Remark 4.3 The cumulant transforms of x and $z(1)$ are related by

$$\mathrm{C}\{\zeta \ddagger x\} = \int_0^\infty \mathrm{C}\{e^{-s}\zeta \ddagger z(1)\}\mathrm{d}s = \int_0^\zeta \mathrm{C}\{\xi \ddagger z(1)\}\xi^{-1}\mathrm{d}\xi \tag{4.14}$$

and

$$\mathrm{C}\{\zeta \ddagger z(1)\} = \zeta\frac{\partial \mathrm{C}\{\zeta \ddagger x\}}{\partial\zeta} \tag{4.15}$$

as follows directly from (4.12) and (3.10). □

From (4.13) we find (recall the notation introduced by (3.6)) that for $x > 0$

$$\begin{aligned} U^+(x) &= \int_0^\infty W(e^t[x,\infty))\mathrm{d}t \\ &= \int_1^\infty s^{-1}W([sx,\infty))\mathrm{d}s \end{aligned}$$

or, equivalently,

$$U^+(x) = \int_x^\infty s^{-1}W^+(s)\mathrm{d}s \tag{4.16}$$

with a similar expression for $U^-(x)$. It follows that we have the important relations

$$u(x) = \begin{cases} x^{-1}W^+(x) & \text{for} \quad x > 0 \\ |x|^{-1}W^-(x) & \text{for} \quad x < 0. \end{cases} \tag{4.17}$$

Proposition 4.1 Suppose that the Lévy density u is differentiable. Then the Lévy measure W has a density w, and u and w are related by

$$w(x) = -u(x) - xu'(x). \tag{4.18}$$

□

PROOF. Straightforward, by differentiation of formula (4.17). □

Example 4.4 *Inverse Gaussian law.* The inverse Gaussian distribution with parameters δ and γ is denoted $IG(\delta,\gamma)$. It is concentrated on $\mathbf{R}_+$ and has probability density

$$p(x) = (2\pi)^{-1/2}\delta e^{-\delta\gamma}x^{-3/2}\exp\left\{-(\delta^2x^{-1}+\gamma^2x)/2\right\} \tag{4.19}$$

and Lévy density

$$u(x) = (2\pi)^{-1/2}\delta x^{-3/2}\exp\left(-\gamma^2x/2\right). \tag{4.20}$$

It follows immediately from this expression that $IG(\delta,\gamma)$ is selfdecomposable (a special case of the Halgreen (1979) result) and that the Lévy density of the corresponding BDLP is

$$w(x) = (2\pi)^{-1/2}\frac{\delta}{2}(x^{-1}+\gamma^2)x^{-1/2}e^{-\gamma^2x/2}. \tag{4.21}$$

□

Note 4.1 *Relation to exponential tilting.* Let u be the Lévy density of a selfdecomposable probability measure P and suppose, for simplicity, that P is concentrated on $\mathbf{R}_+$ and that u is differentiable. Furthermore, let P_θ denote the negative exponential tilt of P, i..e.

$$\frac{\mathrm{d}P_\theta}{\mathrm{d}P}(x) = a(\theta)e^{-\theta x}$$

with $\theta > 0$. The Lévy measure u_θ of P_θ then has a density u_θ satisfying

$$u_\theta(x) = u(x)e^{-\theta x}.$$

Denoting the Levy densities of the BDLPs corresponding to P and P_θ respectively by w and w_θ, we find, from formula (4.18),

$$w_\theta(x) = w(x)e^{-\theta x} + \theta x u_\theta(x). \tag{4.22}$$

It follows that the BDLP corresponding to P_θ is a sum of two independent Levy processes, the first being the BDLP of P and the second, with Levy density $\bar{u}_\theta(x) = \theta x u_\theta(x)$, being a compound Poisson process. The latter result is a consequence $xu_\theta(x)$ being integrable on $\mathbf{R}_+$ (cf. (3.20)). □

Example 4.5 $\frac{1}{2}$*-stable law.* The Lévy density of the positive stable law with index $\frac{1}{2}$ is

$$u(x) = (2\pi)^{-1/2}x^{-3/2}$$

and the exponential tilt of that law is the $IG(1,\sqrt{-2\theta})$ distribution. By Proposition 4.1 we have

$$w(x) = \frac{1}{2}u(x)$$

and hence from Note 4.1 we obtain that the Lévy density of the BDLP of $IG(1,\gamma)$ is

$$w(x) = (2\pi)^{-1/2}\frac{1}{2}(x^{-1}+\gamma^2)x^{-1/2}e^{-\gamma^2x/2} \tag{4.23}$$

in agreement with formula (4.21). □

Theorem 4.4 Let $x = (x_1, ..., x_m)$ be an m-dimensional selfdecomposable random variate. Then there exists an m-dimensional Lévy process $z = (z_1, ..., z_m)$, unique up to identity in law, such that x is representable as

$$x = \int_0^\infty e^{-s} \mathrm{d}z(s). \tag{4.24}$$

□

Theorem 4.4 is due to Jurek and Vervaat (1983), see also Sato and Yamazato (1984).

5. GIG and GH laws

Having discussed the basic probability theory associated with Lévy fields and the self-decomposability of probability measures we are now in a position to become more specific in our modelling framework. We will be interested in constructing non-negative processes in order to model changing volatility. A rather general and flexible modelling setup for distributions are the GIG (generalized inverse Gaussian) laws and the GH (generalized hyperbolic) laws, the latter determined as normal variance-mean mixtures[5] using GIG's as mixing distributions. These distributions are all selfdecomposable (Halgreen (1979)) and may serve as building blocks in the various dynamic models discussed in this paper.

A review of the definitions and properties of the classes GIG and GH is given in Eberlein (2000). Hence, in this section, we review just a few facts, mainly concerning the special cases of the IG (inverse Gaussian) and the NIG (normal inverse Gaussian) distributions. In the present paper these cases are used for illustrative purposes but we emphasise that many other distributional patterns are comprised by the GIG and GH laws (including the gamma, the reciprocal gamma, the reciprocal inverse Gaussian, the hyperbolic, and the Student distributions).

5.1. Generalized inverse Gaussian distributions

The generalized inverse Gaussian distribution $GIG(\lambda, \delta, \gamma)$ is the distribution on $\mathbf{R}_+$ given in terms of its density

$$p(x) = p(x; \lambda, \delta, \gamma) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} x^{\lambda-1} \exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right\}. \tag{5.1}$$

The parameters λ, γ and δ are such that $\lambda \in \mathbf{R}$ while γ and δ are both nonnegative and not simultaneously 0. Furthermore K_λ is the modified Bessel function of the third kind and with index λ. For $\delta\gamma = 0$ the expression (5.1) should be interpreted in the limiting sense, using the formula

$$K_\lambda(x) \sim \begin{cases} \Gamma(\lambda) 2^{\lambda-1} x^{-\lambda} & \text{for} \quad \lambda > 0 \\ \\ -\log x & \text{for} \quad \lambda = 0 \end{cases} \tag{5.2}$$

[5] A random variable x is said to be of variance-mean mixture type if x can be represented in law as $x = \mu + \beta\sigma^2 + \sigma u$ where u and σ are independent random variables with u having the standard normal distribution.

valid for $x \downarrow 0$. We recall that $K_\lambda(x) = K_{-\lambda}(x)$.

It follows from the exponential form of the representation (5.1) that if $x \sim GIG(\lambda, \delta, \gamma)$ then[6]

$$\bar{\mathrm{K}}\{\theta; x\} = \lambda \log\{1 + 2\theta/\gamma^2\}^{1/2} - \log K_\lambda(\delta\gamma) + \log K_\lambda \left\{\delta\gamma \left(1 + 2\theta/\gamma^2\right)^{1/2}\right\} \tag{5.3}$$

As already mentioned, the random variable x with law (5.1) is selfdecomposable. The Lévy-Khintchine representation of the cumulant generating function of x has Lévy density

$$u(x) = x^{-1} \left[\delta^2 \int_0^\infty e^{-x\xi} g_\lambda(2\delta^2\xi) \mathrm{d}\xi + \max\{0, \lambda\}\right] \exp\left(-\gamma^2 x/2\right) \tag{5.4}$$

where

$$g_\lambda(x) = \left[(\pi^2/2)x \left\{J^2_{|\lambda|}(\sqrt{x}) + N^2_{|\lambda|}(\sqrt{x})\right\}\right]^{-1} \tag{5.5}$$

and J_ν and N_ν are Bessel functions. A derivation of the fact that the characteristic function of x has the stated Lévy-Khintchine representation may be found in Barndorff-Nielsen (2000). Once this result has been established the selfdecomposability of x follows directly from Theorem 4.1.[7]

The special case of GIG corresponding to $\lambda = -\frac{1}{2}$ is the $IG(\delta, \gamma)$ distribution. Other special cases of the GIG laws are the *reciprocal inverse Gaussian distribution* that corresponds to $\lambda = \frac{1}{2}$ and is denoted $RIG(\delta, \gamma)$, the *gamma distribution* $\Gamma(\nu, \alpha)$ that obtains for $\delta = 0$ and with $\nu > 0$, $\lambda = \nu$ and $\alpha = \gamma^2/2$, and the *reciprocal gamma distribution* $\Gamma^{-1}(\nu, \alpha)$ which occurs for $\gamma = 0$ and with $\nu > 0$, $\lambda = -\nu$ and $\alpha = \delta^2/2$. Note that if $x \sim IG(\delta, \gamma)$ then $x^{-1} \sim RIG(\gamma, \delta)$, and if $x \sim \Gamma(\nu, \alpha)$ then $x^{-1} \sim \Gamma^{-1}(\nu, \alpha)$.

5.2. Normal inverse Gaussian distribution

If we take a $\sigma^2 \sim IG(\delta, \gamma)$ and independently draw an $\varepsilon \sim N(0, 1)$, then $x = \mu + \beta\sigma^2 + \sigma\varepsilon$ has a normal inverse Gaussian distribution. It has parameters $\alpha = \sqrt{\beta^2 + \gamma^2}, \beta, \mu$ and δ is denoted $NIG(\alpha, \beta, \mu, \delta)$. The random variable exists on $\mathbf{R}$ having the density function

$$g(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q\left(\frac{x - \mu}{\delta}\right)^{-1} K_1 \left\{\delta\alpha q\left(\frac{x - \mu}{\delta}\right)\right\} e^{\beta x} \tag{5.6}$$

where $q(x) = \sqrt{1 + x^2}$ and

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1} \alpha \exp\left\{\delta\sqrt{\alpha^2 - \beta^2} - \beta\mu\right\} \tag{5.7}$$

and where K_1 is the modified Bessel function of the third kind and index 1. The domain of variation of the parameters is given by $\mu \in \mathbf{R}$, $\delta \in \mathbf{R}_+$, and $0 \leq \beta < \alpha$.

It follows immediately from (5.6) and (5.7) that the cumulant generating function of the normal inverse Gaussian distribution is

$$\mathrm{K}(u; \alpha, \beta, \mu, \delta) = \delta\left\{\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2}\right\} + \mu u. \tag{5.8}$$

[6] We use here the fact that for an exponential model with densities $\exp\{\langle\theta, x\rangle - h(x) - k(\theta)\}$ (with respect to some σ-finite measure on $\mathbf{R}^m$) the cumulant function is of the form $\overline{\mathrm{K}}\{\cdot \ddagger x\} = k(\cdot + \theta) - k(\theta)$.

[7] In this connection, see also Pitman and Yor (1981, p. 346) where a connection to Bessel processes is established for GIG laws with $\lambda \leq 0$.

Thus, in particular, if $x_1, ..., x_m$ are independent normal inverse Gaussian random variables with common parameters α and β but having individual location-scale parameters μ_i and δ_i $(i = 1, ..., m)$ then $x_+ = x_1 + ... + x_m$ is again distributed according to a normal inverse Gaussian law, with parameters $(\alpha, \beta, \mu_+, \delta_+)$.

It is often of interest to consider alternative parametrisations of the normal inverse Gaussian laws. In particular, letting $\bar{\alpha} = \delta\alpha$ and $\bar{\beta} = \delta\beta$, we have that $\bar{\alpha}$ and $\bar{\beta}$ are invariant under location—scale changes, and when $\bar{\alpha}, \bar{\beta}, \mu, \delta$ constitute the parametrisation of interest we shall write $NIG[\bar{\alpha}, \bar{\beta}, \mu, \delta]$ instead of $NIG(\alpha, \beta, \mu, \delta)$. In terms of this alternative parametrisation the first four cumulants of $NIG[\bar{\alpha}, \bar{\beta}, \mu, \delta]$ are

$$\kappa_1 = \mu + \frac{\delta\rho}{\sqrt{1-\rho^2}}, \quad \kappa_2 = \frac{\delta^2}{\bar{\alpha}(1-\rho^2)^{3/2}} \tag{5.9}$$

and

$$\kappa_3 = \frac{3\delta^3\rho}{\bar{\alpha}^2(1-\rho^2)^{5/2}}, \quad \kappa_4 = \frac{3\delta^4(1+4\rho^2)}{\bar{\alpha}^3(1-\rho^2)^{7/2}}, \tag{5.10}$$

where $\rho = \beta/\alpha$, which is invariant since $\beta/\alpha = \bar{\beta}/\bar{\alpha}$. Further, the standardised third and fourth cumulants are

$$\frac{\kappa_3}{\kappa_2^{3/2}} = 3\frac{\rho}{\{\bar{\alpha}(1-\rho^2)^{1/2}\}^{1/2}} \quad \text{and} \quad \frac{\kappa_4}{\kappa_2^2} = 3\frac{1+4\rho^2}{\bar{\alpha}(1-\rho^2)^{1/2}}. \tag{5.11}$$

Thus

$$\frac{\kappa_3^2}{\kappa_4} = 3\frac{\rho^2}{1+4\rho^2} \tag{5.12}$$

is a function of ρ only.

We note that the NIG distribution (5.6) has semiheavy tails; specifically,

$$g(x;\alpha,\beta,\mu,\delta) \sim const.\, |x|^{-3/2} \exp\left(-\alpha\,|x| + \beta x\right) \quad \text{as} \quad x \to \pm\infty\,, \tag{5.13}$$

as follows from the well known asymptotic relation for the Bessel functions $K_\nu(x)$:

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2}} x^{-1/2} e^{-x} \quad \text{as} \quad x \to \infty\,. \tag{5.14}$$

The characteristic triplet of the normal inverse Gaussian distribution is $(a, 0, U)$ where the Lévy measure U has density

$$u(x) = \pi^{-1}\delta\alpha\,|x|^{-1} K_1(\alpha\,|x|)e^{\beta x} \tag{5.15}$$

while

$$a = \mu + 2\pi^{-1}\delta\alpha \int_0^1 \sinh(\beta x) K_1(\alpha x)\mathrm{d}x\,. \tag{5.16}$$

For a derivation of these formulae, see Barndorff-Nielsen (1998b).

Remark 5.1 An important characterization of the normal inverse Gaussian law $NIG(\alpha, \beta, \mu, \delta)$ is the following. Let $b(t) = \{b_1(t), b_2(t)\}$ be a bivariate Brownian motion starting at $(\mu, 0)$ and having drift vector (β, γ) where $\beta \in \mathbf{R}$ and $\gamma \geq 0$. Furthermore, let u denote the time when b_1 first reaches level $\delta > 0$ and let $x = b_2(u)$. Then $x \sim NIG(\alpha, \beta, \mu, \delta)$ with $\alpha = \sqrt{\delta^2 + \gamma^2}$. □

6. OU processes

6.1. General setup

Financial time series models are usually specified in continuous time. In this section we will develop continuous time processes with values constrained to fall immediately on the positive half-line.

For any $t > 0$ and $\lambda > 0$ we may rewrite the representation (4.12) of a random variable x in the class L as follows

$$\begin{aligned} x &= \int_0^\infty e^{-\lambda s} \mathrm{d}z(\lambda s) \\ &= \int_t^\infty e^{-\lambda s} \mathrm{d}z(\lambda s) + \int_0^t e^{-\lambda s} \mathrm{d}z(\lambda s) \\ &= e^{-\lambda t} x_0 + u_t \end{aligned} \tag{6.1}$$

where

$$x_0 = \int_0^\infty e^{-\lambda s} \mathrm{d}z\left(\lambda(s+t)\right)$$

and

$$u_t = e^{-\lambda t} \int_0^t e^{\lambda s} \mathrm{d}z\left(\lambda(t-s)\right)$$

x_0 and u_t being independent. Note that $x_0 \overset{\mathcal{D}}{=} x$ and

$$u_t \overset{\mathcal{D}}{=} \int_0^t e^{-\lambda(t-s)} \mathrm{d}z(\lambda s). \tag{6.2}$$

In fact, a stronger statement is true, namely that for any $\lambda > 0$, the solution to the stochastic differential equation

$$\mathrm{d}x(t) = -\lambda x(t)\mathrm{d}t + \mathrm{d}z(\lambda t) \tag{6.3}$$

is a stationary process $\{x(t)\}_{t\geq 0}$ such that $x(t) \overset{\mathcal{D}}{=} x$. A stationary process $x(t)$ of this kind is said to be an *Ornstein-Uhlenbeck type process* or an *OU process*, for short. The process $z(t)$ is termed the *background driving Lévy process* (BDLP) corresponding to the process $x(t)$.

More specifically, given a one-dimensional distribution D there exists an Ornstein-Uhlenbeck type stationary process whose one-dimensional marginal law is D if and only if D is selfdecomposable. The precise statement of existence is as follows, cf. Wolfe (1982), Jurek and Vervaat (1983) and Sato and Yamazato (1983) (see also Sato (1999, Section 17) and Barndorff-Nielsen, Jensen, and Sørensen (1998)).

Theorem 6.1 Let ϕ be the characteristic function of a random variable x. If x is selfdecomposable then there exists a stationary stochastic process $\{x(t)\}_{t\geq 0}$ and a Lévy process $\{z(t)\}_{t\geq 0}$, independent of $x(0)$, such that $x(t) \overset{\mathcal{D}}{=} x$ and

$$x(t) = e^{-\lambda t} x(0) + \int_0^t e^{-\lambda(t-s)} \mathrm{d}z(\lambda s) \tag{6.4}$$

for all $\lambda > 0$.

Conversely, if $\{x(t)\}_{t\geq 0}$ is a stationary stochastic process and $\{z(t)\}_{t\geq 0}$ is a Lévy process, independent of $x(0)$, such that $\{x(t)\}$ and $\{z(t)\}$ satisfy the equation (6.3) for all $\lambda > 0$ then $x(t)$ is selfdecomposable. □

Remark 6.1 Let x be a square integrable OU process. Then x has correlation function $r(u) = \exp\{-\lambda|u|\}$. □

Remark 6.2 A necessary and sufficient condition for the stochastic differential equation (6.3) to have a stationary solution is that

$$\mathrm{E}\{\log^+|z(1)|\} < \infty \tag{6.5}$$

where $\log^+|x| = \max\{0, \log|x|\}$ (cf. Wolfe (1982) and Sato and Yamazato (1983)). □

The stationary process $\{x(t)\}_{t\geq 0}$ can be extended to a stationary process on the whole real line. To do this we introduce an independent copy of the process z but modify it to be caglad, thus obtaining a process $\bar{z}$, say.

Now, for $t < 0$ define $z(t)$ by $z(t) = \overline{z}(-t)$, and for $t \in \mathbf{R}$ let

$$x(t) = e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}\mathrm{d}z(\lambda s). \tag{6.6}$$

Then $\{z(t)\}_{t\in\mathbf{R}}$ is a (homogeneous, cadlag) Lévy process; and $\{x(t)\}_{t\in\mathbf{R}}$ is a strictly stationary process of Ornstein-Uhlenbeck type.

Note that equivalent forms of (6.6) are

$$x(t) = e^{-\lambda t}\int_{-\infty}^{\lambda t} e^{s}\mathrm{d}z(s)$$

and

$$x(t) = \int_{-\infty}^{0} e^{s}\mathrm{d}z(s+\lambda t). \tag{6.7}$$

Remark 6.3 Let x be an OU process and let z be the corresponding BDLP. Then the cumulant transforms of $x(t)$ and $z(1)$ are related by

$$\mathrm{C}\{\zeta \ddagger x(t)\} = \int_{0}^{\zeta} \mathrm{C}\{\xi \ddagger z(1)\}\xi^{-1}\mathrm{d}\xi \tag{6.8}$$

and

$$\mathrm{C}\{\zeta \ddagger z(1)\} = \zeta\frac{\partial \mathrm{C}\{\zeta \ddagger x(t)\}}{\partial\zeta}$$

as follows directly from (4.14) and (4.15). □

From the above discussion it follows, in particular, that there exists a $\{x(t)\}_{t\in\mathbf{R}}$ stationary OU process such that $x(t)\sim IG(\delta,\gamma)$ for every $t\in\mathbf{R}$, whatever the value of the autoregressive parameter λ. We refer to this process as the *inverse Gaussian OU process* or the *IG OU process*, for short. The character of this process is studied below.

Similarly, we shall consider the character of the stationary normal inverse Gaussian Ornstein-Uhlenbeck type process or *NIG OU process*, i.e. the stationary Ornstein-Uhlenbeck type process having $NIG(\alpha,\beta,\mu,\delta)$-distributed one-dimensional marginals.

However, as a further illustration, we first consider the *gamma OU process*, the OU process whose one-dimensional marginals follow the $\Gamma(\nu,\alpha)$ law.

6.2. The gamma OU process

For the $\Gamma(\nu,\alpha)$ distribution the probability density and the Lévy density are:

$$p(x)=\frac{\alpha^{\nu}}{\Gamma(\nu)}x^{\nu-1}e^{-\alpha x}, \tag{6.9}$$

$$u(x)=\nu x^{-1}e^{-\alpha x}. \tag{6.10}$$

Hence, by (4.18), the corresponding BDLP z has Lévy density for $z(1)$ given by

$$\begin{aligned} w(x) &= -u(x)-xu'(x) \\ &= -\nu x^{-1}e^{-\alpha x}+\nu x^{-1}e^{-\alpha x}+\nu\alpha e^{-\alpha x} \\ &= \alpha\nu e^{-\alpha x}. \end{aligned}$$

Except for the factor ν, this is a probability density function and it follows that $z(t)$ is the compound Poisson process

$$z(t)=\sum_{n=1}^{N(t)}x_n$$

where $N(t)$ is the Poisson process with $\mathrm{E}\{N(t)\}=\nu t$ and x_n has law $\Gamma(1,\alpha)$.

6.3. The IG OU process

The Lévy density $w(x)$ of the BDLP for the IG OU process is given by formula (4.21). This implies (Barndorff-Nielsen (1998b))

Proposition 6.1 The BDLP $z(t)$ driving the inverse Gaussian OU process with one-dimensional $IG(\delta,\gamma)$ marginals is a sum of two independent Lévy processes, $z(t)=y(t)+p(t)$, where $y(t)$ is an inverse Gaussian Lévy process with parameters $\delta/2$ and γ for $y(1)$, while $p(t)$ is of the form

$$p(t)=\gamma^{-1}\sum_{i=1}^{N_t}u_i^2 \tag{6.11}$$

with N_t a Poisson process of rate $\delta\gamma/2$ and the u_i being independent standard normal and independent of the process N_t. □

6.4. The NIG OU process

In discussing the character of the NIG OU process we assume, for simplicity that $\mu = 0$ and, since $x(t) \sim NIG(\alpha, \beta, 0, \delta\ )$ implies that $-x(t) \sim NIG(\alpha, -\beta, 0, \delta\ )$, we further restrict attention to the case $\beta \geq 0$.

We first derive the Lévy measure of the BDLP $\{z(t)\}_{t\in\mathbf{R}}$ corresponding to the NIG OU process, using the relation (4.18). Since the Bessel functions satisfy $K_1'(x) = K_0(x)$,

$$\begin{aligned} w(x) &= -u(x) \\ &\quad +\pi^{-1}\delta\alpha\left\{|x|^{-1}K_1(\alpha|x|) - \alpha\mathrm{sign}(x)K_1'(\alpha|x|) - \beta\mathrm{sign}(x)K_1(\alpha|x|)\right\}e^{\beta x} \\ &= \pi^{-1}\delta\alpha\left[\left\{|x|^{-1} - \beta\mathrm{sign}(x)\right\}K_1(\alpha|x|) + \alpha K_0(\alpha|x|)\right]e^{\beta x} \\ &= (1-\beta x)u(x) + \pi^{-1}\delta\alpha^2 K_0(\alpha|x|)e^{\beta x}. \end{aligned} \tag{6.12}$$

Proposition 6.2 The BDLP $z(t)$ for the normal inverse Gaussian OU process with parameters $(\alpha, \beta, 0, \delta)$ is, for $\beta \geq 0$, representable as the sum of three independent Lévy processes: $z(t) = y(t) + p(t) + q(t)$. The first process $y(t)$ is the normal inverse Gaussian Lévy process, with parameters $(\alpha, \beta, 0, (1-\rho)\delta)$, and the second has the form

$$p(t) = \frac{1}{2}\alpha^{-1}(1-\rho^2)^{-1/2}\sum_{i=1}^{N_t}(u_i^2 - u_i'^2) \tag{6.13}$$

where N_t denotes a Poisson process with rate $[\{(1-\rho)/(1+\rho)\}^{1/2}\delta\alpha]^{-1}$ and the u_i and u_i' $(i = 0, 1, 2, ...)$ are independent standard normally distributed and independent of the process N_t . Finally, the Laplace transform $E\exp(\theta q(t))$ of $q(t)$ is

$$\exp\left\{t\rho\delta\left[\beta\{(\alpha-\beta)/(\alpha+\beta)\}^{1/2} - (\theta+\beta)\{(\alpha-\theta-\beta)/(\alpha+\theta+\beta)\}^{1/2}\right]\right\}. \tag{6.14}$$

□

For the derivation of this result, see Barndorff-Nielsen (1998b).

7. BDLP modelling

Instead of specifying the law of the one-dimensional marginal distribution of an OU process $x(t)$ and working out the density for $z(1)$ of the BDLP, as above for the IG OU and NIG OU processes, it is possible to go the other way and construct the model through the BDLP.

Of course there are constraints on valid BDLPs which must be satisfied. More specifically we have

Proposition 7.1 Let z be a Lévy jump process, denote the Lévy measure of $z(1)$ by W, and define the function u on $\mathbf{R}$ by $u(0) = 0$ and

$$u(x) = \begin{cases} x^{-1}W^+(x) & \text{for} \quad x > 0 \\ \\ |x|^{-1}W^-(x) & \text{for} \quad x < 0. \end{cases} \tag{7.1}$$

If u satisfies

$$\int_{\mathbf{R}} \min\{1, x^2\}u(x)\mathrm{d}x \tag{7.2}$$

then u is the density of a Lévy jump process z and there exists an OU process x such that $x(t) \stackrel{\mathcal{D}}{=} z(1)$ (for all t) and such that z is the BDLP of x. □

PROOF This follows directly from the characterization of the Lévy measures of selfdecomposable distributions (Theorem 4.1) together with formula (4.17) and Theorem 6.1. □

Corollary 7.1 Let w be the density of a Lévy measure W on $\mathbf{R}_+$ and suppose that

$$\int_1^\infty \log x w(x) \mathrm{d}x < \infty \tag{7.3}$$

Then Proposition 7.1 applies. □

PROOF We only need to check that condition (7.2) is satisfied, and that is a consequence of using (7.1) and the fact that w must satisfy

$$\int_0^\infty \min\{1, x^2\} w(x) \mathrm{d}x < \infty$$

More specifically, noting that

$$u(x) = \int_1^\infty w(\tau x) \mathrm{d}\tau \tag{7.4}$$

we find

$$\begin{aligned}
\int_{0+}^\infty \min\left\{1, x^2\right\} u(x) \mathrm{d}x &= \int_1^\infty \int_{0+}^\infty \min\left\{1, x^2\right\} w(\tau x) \mathrm{d}x \mathrm{d}\tau \\
&= \int_1^\infty \int_{0+}^\infty \min\left\{1, \tau^{-2} y^2\right\} \tau^{-1} w(y) \mathrm{d}y \mathrm{d}\tau \\
&= \int_{0+}^1 y^2 w(y) \mathrm{d}y \int_1^\infty \tau^{-3} \mathrm{d}\tau \\
&\quad + \int_1^\infty w(y) \left(\int_1^y \tau^{-1} \mathrm{d}\tau + y^2 \int_y^\infty \tau^{-3} \mathrm{d}\tau \right) \mathrm{d}y \\
&= \frac{1}{2} \int_{0+}^1 y^2 w(\mathrm{d}y) + \int_1^\infty \log y w(y) \mathrm{d}y + \frac{1}{2} \int_1^\infty w(y) \mathrm{d}y.
\end{aligned}$$

In the latter expression the first and third integrals are finite since W is a Lévy measure and the second integral is finite by assumption. □

Because of the results to be discussed in Section 8 below, it may be numerically advantageous to define the Lévy measure W of the BDLP in terms of the tail mass rather than by the density w.

Example 7.1 Suppose the Lévy measure W is concentrated on $\mathbf{R}_+$ and has tail integral

$$W^+(x) = cx^{-\varepsilon}(1+x)^{-\beta} \exp\left(-\frac{1}{2}\gamma^2 x\right)$$

where c is a positive constant, $0 \le \varepsilon < 1$, $0 \le \beta$, $0 \le \gamma$ and $\max\{(\beta - 1), \gamma\} > 0$. Then

$$\begin{aligned}
w(x) &= c\{\varepsilon x^{-1} + \beta(1+x)^{-1} + \frac{1}{2}\gamma^2\} x^{-\varepsilon}(1+x)^{-\beta} \exp\left(-\frac{1}{2}\gamma^2 x\right) \\
&= c\{\varepsilon x^{-1} + \beta(1+x)^{-1} + \frac{1}{2}\gamma^2\} W^+(x)
\end{aligned} \tag{7.5}$$

and we have

$$u(x) = x^{-1}W^{+}(x) = cx^{-1-\varepsilon}(1+x)^{-\beta}\exp\left(-\frac{1}{2}\gamma^2 x\right) \tag{7.6}$$

which clearly satisfies (7.3).

Note that for $\varepsilon = \frac{1}{2}$ and $\beta = 0$, (7.6) reduces to the Lévy density of the IG law. □

8. Series representations

Lévy processes and fields and related integrals were discussed in Section 3. In the present section we briefly review some series representations of random objects of this kind, for simplicity restricting the treatment to positive fields. The representations are useful, in particular, for simulation of such processes and integrals, whereas direct simulation is not really practical due to the jump character of the processes. Bondesson (1982) was the first to discuss the type of approach we shall consider, more recent work being due to Marcus (1987), Rosinski (1991), Asmussen (1998, Sect. VIII.2), Wolpert and Ickstadt (1998) and Wolpert and Ickstadt (1999). Rosinski (2000) gives an up to date survey of series representations.

Recall the definition (3.6) of $U^{+}(x;\omega)$ and let

$$U^{-1}(a;\omega) = \inf\{x \geq 0 : U^{+}(x;\omega) \leq a\} \tag{8.1}$$

Theorem 8.1 Let z be a positive Lévy jump field on a region $\mathcal{S}$ and with cumulant functional

$$\mathrm{C}\{f \ddagger z\} = \int_{\mathbf{R}_+}\int_{\mathcal{S}}(e^{if(\omega)x} - 1)U(\mathrm{d}x;\omega)\pi(\mathrm{d}\omega) \tag{8.2}$$

for some probability measure π on $\mathcal{S}$. Furthermore, let $a_1 < a_2 < \cdots < a_i < \cdots$ be the arrival times of a Poisson process with intensity 1, let $\omega_1, \omega_2, ..., \omega_i, ...$ be an i.i.d. sequence from the law π, and suppose that the sequences $\{a_i\}$ and $\{\omega_i\}$ are independent.

For nonnegative integrable functions f we then have

$$f \bullet z \overset{\mathcal{D}}{=} \sum_{i=1}^{\infty} f(\omega_i)U^{-1}(a_i;\omega_i) \tag{8.3}$$

□

PROOF Let τ be a positive number, let n_τ be the largest integer i for which $a_i < \tau$, and write

$$\sigma = \sum_{i=1}^{\infty} f(\omega_i)U^{-1}(a_i;\omega_i)$$

and

$$\sigma_\tau = \sum_{i=1}^{n_\tau} f(\omega_i)U^{-1}(a_i;\omega_i)$$

Conditionally on n_τ the joint law of $a_1, ..., a_{n_\tau}$ is equal to the joint law of $\tau(r_{(1)}, ..., r_{(n_\tau)})$ where $r_{(1)} < ... < r_{(n_\tau)}$ are the order statistics of a sample $(r_1, ..., r_{n_\tau})$ from the uniform law on $(0, 1)$.

Hence, by conditioning on n_τ and $\{\omega_i\}$ and using product integration, we find for the Laplace transform $\bar{\mathrm{L}}\{\theta \ddagger \sigma_\tau\} = \exp \bar{\mathrm{K}}\{\theta \ddagger \sigma_\tau\}$

$$\begin{aligned}
\bar{\mathrm{L}}\{\theta \ddagger \sigma_\tau\} &= \mathrm{E}\left\{\prod_{i=1}^{n_\tau} \exp\left[\bar{\mathrm{K}}\{f(\omega_i)\theta \ddagger U^{-1}(a_i;\omega_i)\}\right]\right\} \\
&= \mathrm{E}\left\{\prod_{i=1}^{n_\tau} \exp\left[\bar{\mathrm{K}}\{f(\omega_i)\theta \ddagger U^{-1}(\tau r_{(i)};\omega_i)\}\right]\right\} \\
&= \mathrm{E}\left\{\prod_{i=1}^{n_\tau} \exp\left[\bar{\mathrm{K}}\{f(\omega_i)\theta \ddagger U^{-1}(\tau r_i;\omega_i)\}\right]\right\} \\
&= \mathrm{E}\left\{\exp\left[n_\tau \bar{\mathrm{K}}\{f(\omega_i)\theta \ddagger U^{-1}(\tau r;\omega)\}\right]\right\}
\end{aligned}$$

with ω and r distributed as the ω_i and r_j, respectively. Since n_τ follows a Poisson distribution with mean 1 we consequently have

$$\begin{aligned}
\bar{\mathrm{L}}\{\theta \ddagger \sigma_\tau\} &= \mathrm{E}\left\{e^{-\tau}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\exp\left[n\bar{\mathrm{K}}\{f(\omega)\theta \ddagger U^{-1}(\tau r;\omega)\}\right]\right\} \\
&= \exp\left[\tau\left(\bar{\mathrm{L}}\{f(\omega)\theta \ddagger U^{-1}(\tau r;\omega)\} - 1\right)\right]. \qquad (8.4)
\end{aligned}$$

A direct calculation shows that

$$\bar{\mathrm{L}}\{\theta \ddagger U^{-1}(\tau r;\omega)\} = \mathrm{E}\left\{\tau^{-1}\int_{U^{-1}(\tau;\omega)}^{\infty}(1-e^{-\theta x})U^{+}(\mathrm{d}x;\omega)\right\} + 1$$

and hence, as $\tau \to \infty$,

$$\exp\left[\tau\left(\bar{\mathrm{L}}\{\theta \ddagger U^{-1}(\tau u;\omega)\} - 1\right)\right] \to \mathrm{E}\left\{\int_{\mathbf{R}_+}(1-e^{-\theta x})U^{+}(\mathrm{d}x;\omega)\right\}. \qquad (8.5)$$

Therefore, by (8.4) and (8.5),

$$\begin{aligned}
\bar{\mathrm{K}}\{\theta \ddagger \sigma\} &= \lim_{\tau\to\infty}\bar{\mathrm{K}}\{\theta \ddagger \sigma_\tau\} \\
&= \int_{\mathcal{S}}\int_{\mathbf{R}_+}(1-e^{-f(\omega)\theta x})U^{+}(\mathrm{d}x;\omega)\pi(\mathrm{d}\omega)
\end{aligned}$$

or equivalently $\bar{\mathrm{K}}\{\theta \ddagger \sigma\} = \bar{\mathrm{K}}\{\theta f \ddagger z\} = \bar{\mathrm{K}}\{\theta \ddagger f \bullet z\}$ as was to be shown. $\square$

In essence, the method of proof of Theorem 8.1 is adopted from Marcus (1987).

Corollary 8.1 Consider a subordinator z with positive increments and Lévy measure U. Let $\{a_i\}$ and $\{r_i\}$ be two independent sequences of random variables such that $r_1, r_2, ...$ are independent copies of a uniform random variable r on $[0,1]$ while $a_1 < \cdots < a_i < \cdots$ are the arrival times of a Poisson process with intensity 1. The Lévy process z is representable in law, on the time interval $[0,1]$, as

$$\{z(s) : 0 \le s \le 1\} \stackrel{\mathcal{D}}{=} \{\tilde{z}(s) : 0 \le s \le 1\} \qquad (8.6)$$

where

$$\tilde{z}(s) = \sum_{i=1}^{\infty} U^{-1}(a_i)\mathbf{1}_{[0,s]}(r_i). \tag{8.7}$$

□

Corollary 8.2 Let the process $z(t)$ be as above and let f be a positive and integrable function on $[0,T]$. Then

$$\int_0^T f(s)\mathrm{d}z(s) \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} U^{-1}(T^{-1}a_i)f(Tr_i). \tag{8.8}$$

□

It should be noted that the convergence of the series (8.3), (8.7) and (8.8) will often be quite slow.

Example 8.1 *OU gamma ($\Gamma(\upsilon, \alpha)$ marginals) process.* We need a method to sample from $e^{-\lambda t}\int_0^{\lambda t} e^s \mathrm{d}z(s)$. In the gamma case

$$W^{-1}(x) = \max\left\{0, -\frac{1}{\alpha}\log\left(\frac{x}{\nu}\right)\right\}.$$

Thus, defining $c_1 < c_2 < \ldots$ as the arrival times of a Poisson process with intensity $\nu\lambda t$ and $N(1)$ as the corresponding number of events up until time 1, then

$$\begin{aligned}
e^{-\lambda t}\int_0^{\lambda t} e^s \mathrm{d}z(s) &\stackrel{\mathcal{L}}{=} e^{-\lambda\Delta}\sum_{i=1}^{\infty} W^{-1}(a_i/\lambda t)e^{\lambda t r_i} \\
&= -\alpha^{-1}e^{-\lambda t}\sum_{i=1}^{\infty}\mathbf{1}_{]0,\nu[}(a_i/\lambda t)\log(a_i/\nu\lambda t)e^{\lambda t r_i} \\
&= \alpha^{-1}e^{-\lambda t}\sum_{i=1}^{\infty}\mathbf{1}_{]0,1[}(c_i)\log(c_i^{-1})e^{\lambda t r_i} \\
&= \alpha^{-1}e^{-\lambda t}\sum_{i=1}^{N(1)}\log(c_i^{-1})e^{\lambda t r_i}.
\end{aligned}$$

Example 8.2 *BDLP of IG.* When the law of x is $IG(\delta, \gamma)$ the upper tail integral of the Lévy measure for the corresponding BDLP is

$$W^{+}(x) = \frac{\delta}{\sqrt{2\pi}}x^{-1/2}\exp\left(-\frac{1}{2}\gamma^2 x\right),$$

cf. formula (4.17). The inverse function W^{-1} of W^{+} satisfies

$$W^{-1}(y) \sim \frac{\delta^2}{2\pi}y^{-2} \quad \text{for} \quad y \to \infty.$$

Hence the series (8.7) and (8.8) (with W^{-1} instead of U^{-1}) will only converge slowly. □

Example 8.3 The Bessel function $K_0(x)$ is positive and decreasing, with

$$K_0(x)\begin{cases} \sim & -\log x \quad \text{as} \quad x \downarrow 0 \\ \\ \to & 0 \qquad\quad\; \text{as} \quad x \to \infty. \end{cases}$$

Thus, letting $w(x) = -K_0'(x) = K_1(x)$ we find, by Proposition 7.1 and Corollary 7.1, that w is the Lévy density of a BDLP of a positive OU process with Lévy density $u(x) = x^{-1}K_0(x)$. In this case, since $W^{-1}(y) \sim e^{-y} \quad \text{as} \quad y \to \infty$, the series (8.7) and (8.8) will converge rapidly. □

9. Superpositions of OU processes

At first sight modelling dynamic processes using OU type processes seems very restrictive as these processes are linear and Markov. However, we could think of these models as building blocks for more general processes. These richer processes can be achieved by simply adding and weighting different types of OU processes. The theory behind such superpositions of processes is considered in this section.

In the notation of Section 3, let $\mathcal{S} = \mathbf{R} \times \mathbf{R}_+$, with points $\omega = (s, \xi)$, and let z be a Lévy jump field such that $a(d\omega)$ and $b(d\omega)$ (of the Lévy-Khintchine representation (3.3)) are identically 0 while the generalized Lévy measure ν is of the form

$$\nu(\mathrm{d}x; \mathrm{d}\omega) = W(\mathrm{d}x)\mathrm{d}s\pi(\mathrm{d}\xi) \tag{9.1}$$

where W is a Lévy measure of an infinitely divisible distribution on $\mathbf{R}$ and π is a probability measure on $\mathbf{R}_+$. Then we have the following theorem, which is proved in Barndorff-Nielsen (2000a).

Theorem 9.1 Suppose that the tail masses W^- and W^+ are of the form

$$W^-(x) = |x|\, u(x) \quad \text{and} \quad W^+(x) = xu(x), \tag{9.2}$$

u being the Lévy density of a selfdecomposable distribution on $\mathbf{R}$.

Define the family $x(\cdot, \mathrm{d}\xi) = \{x(t, \mathrm{d}\xi) : t \in \mathbf{R}\}$ of random measures on $\mathbf{R}_+$ by

$$x(t, B) = \int_B e^{-\xi t} \int_{-\infty}^{\xi t} e^s z(\mathrm{d}s, \mathrm{d}\xi) \tag{9.3}$$

and let

$$x(t) = x(t, \mathbf{R}_+). \tag{9.4}$$

Then $\{x(t)\}_{t \in \mathbf{R}}$ is a well-defined, infinitely divisible and stationary process, and the cumulant transforms of the finite dimensional distributions of x are given by

$$\mathrm{C}\{\zeta_1, ..., \zeta_m \ddagger x(t_1), ..., x(t_m)\} = \int_{\mathbf{R}_+} \int_{\mathbf{R}} \kappa \left\{ \sum_{j=1}^{m} \mathbf{1}_{[0,\infty)}(t_j - s)\zeta_j e^{-\xi(t_j - s)} \right\} \xi \mathrm{d}s\pi(\mathrm{d}\xi) \tag{9.5}$$

where κ is the cumulant function corresponding to the Lévy measure W and $t_1 < \cdots < t_m$. □

Remark 9.1 Formal calculation from the formulae (9.4) and (9.3) gives

$$\mathrm{d}x(t) = \int_{\mathbf{R}_+} \{-\xi x(t, \mathrm{d}\xi)\mathrm{d}t + z(\mathrm{d}t, \mathrm{d}\xi)\} \tag{9.6}$$

showing that x is a superposition of, perhaps infinitesimally determined, Ornstein-Uhlenbeck type processes. We shall refer to any such process as a *supOU process*.

PROOF We first note that (as verified in Barndorff-Nielsen (2000a)) the random measure $x(t, \cdot)$ is well-defined in accordance with the theory of independently scattered random measures.

To derive (9.5) we write

$$\begin{aligned} \sum_{j=1}^{m} \zeta_j x(t_j) &= \int_{\mathbf{R}_+} \sum_{j=1}^{m} \zeta_j e^{-\xi t_j} \int_{-\infty}^{\xi t_j} e^s z(\mathrm{d}s, \mathrm{d}\xi) \\ &= \int_{\Omega} g(s, \xi) z(\mathrm{d}s, \mathrm{d}\xi) \end{aligned}$$

where

$$g(s, \xi) = \sum_{j=1}^{m} \zeta_j e^{-\xi t_j} \mathbf{1}_{(-\infty, \xi t_j]}(s) e^s.$$

Hence, by formula (3.10),

$$\begin{aligned} \mathrm{C}\{\zeta_1, ..., \zeta_m \ddagger x(t_1), ..., x(t_m)\} &= \int_{\mathbf{R}_+} \int_{\mathbf{R}} \kappa \left\{ \sum_{j=1}^{m} \zeta_j e^{-\xi t_j} \mathbf{1}_{(-\infty, \xi t_j]}(s) e^s \right\} \mathrm{d}s \pi(\mathrm{d}\xi) \\ &= \int_{\mathbf{R}_+} \int_{\mathbf{R}} \kappa \left\{ \sum_{j=1}^{m} \zeta_j e^{-\xi(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s) \right\} \mathrm{d}s \xi \pi(\mathrm{d}\xi). \end{aligned}$$

The stationarity and infinite divisibility of the process x follow immediately from this expression and the infinite divisibility of κ. □

Note that condition (9.2) implies that W is the Lévy measure of the BDLP corresponding to the selfdecomposable law whose Lévy density is u.

Corollary 9.1 We have $\mathrm{C}\{\zeta \ddagger x(t)\} = \acute{\kappa}(\zeta)$ where $\acute{\kappa}$ is the cumulant function of the selfdecomposable law with Lévy density u. □

PROOF Formula (9.5) implies, in particular, that

$$\mathrm{C}\{\zeta \ddagger x(t)\} = \int_0^{\infty} \kappa(\zeta e^{-s}) \mathrm{d}s$$

and the result now follows from formula (4.14). □

Corollary 9.2 Assuming that x is square integrable, the autocorrelation function r of x is given by

$$r(\tau) = \int_0^\infty e^{-\tau\xi}\pi(\mathrm{d}\xi) = \exp\bar{K}\{\tau \ddagger \xi\} \tag{9.7}$$

for $\tau \geq 0$ and where for the last conclusion we interpret ξ as a (non-negative) random variable with distribution π. □

Example 9.1 Suppose that π is the gamma law $\Gamma(2\bar{H}, 1)$ where $\bar{H} > 0$. Then

$$r(\tau) = (1+\tau)^{-2\bar{H}}. \tag{9.8}$$

In particular, then, the process $\{x(t)\}_{t\in\mathbf{R}}$ exhibits second order long range dependence if $H \in (\frac{1}{2}, 1)$ where $H = 1 - \bar{H}$. □

Note 9.1 Corollaries 9.1 and 9.2 together show that to any selfdecomposable distribution D with finite second moment and to any Laplace transform of a distribution π on $\mathbf{R}_+$ there exists a stationary process on $\mathbf{R}$ whose one-dimensional marginal law is D and whose autocorrelation function equals the given Laplace transform. □

Example 9.2 IG supOU *processes.* For $\gamma > 0$ there exists a supOU process with one-dimensional marginal distribution $IG(\delta, \gamma)$ and autocorrelation function (9.8). □

Example 9.3 NIG supOU *processes.* The normal inverse Gaussian law $NIG(\alpha, \beta, \mu, \delta)$ is selfdecomposable and hence there exists a supOU process with one-dimensional marginal distribution $NIG(\alpha, \beta, \mu, \delta, \gamma)$ and autocorrelation function (9.8). □

It should also be noted that questions of moduli of continuity and large increments of infinite sums of classical, i.e. Gaussian, Ornstein-Uhlenbeck processes have been discussed in papers by Csáki, Csörgö, Lin, and Révész (1991) and Lin (1995). See also Walsh (1981).

10. Return to financial economics

10.1. Background

Continuous time models built out of Brownian motion play a crucial role in modern mathematical finance, providing the basis of most option pricing, asset allocation and term structure theory currently being used. An example is the so called Black-Scholes or Samuelson model which models the log of an asset price by the solution to the stochastic differential equation

$$\mathrm{d}x^*(t) = \left\{\mu + \beta\sigma^2\right\}\mathrm{d}t + \sigma\mathrm{d}w(t), \quad t \in [0, S], \tag{10.1}$$

where $w(t)$ is standard Brownian motion. Here $\mu + \beta\sigma^2$ represents the drift of the log-price, while σ is the volatility. The reason the size of the drift depends upon the volatility is that investors are usually thought to require a "risk premium" for holding stochastic assets, compared to holding their wealth in a riskless interest banking account. Hence if the volatility is high, we would expect the drift also to be high. Overall, this model implies aggregate returns over intervals of length $\Delta > 0$ are

$$y_n = \int_{(n-1)\Delta}^{n\Delta} \mathrm{d}x^*(t) = x^*(n\Delta) - x^*\left((n-1)\Delta\right) \tag{10.2}$$

implying returns are normal and independently distributed with a mean of $\mu\Delta + \beta\sigma^2\Delta$ and a variance of $\Delta\sigma^2$. Unfortunately, for moderate to large values of Δ (corresponding to returns measured over 5 minute to one day intervals) returns are typically heavy-tailed, exhibit volatility clustering (in particular the $|y_n|$ are correlated) and are skew (see the discussion in, for example, Campbell, Lo, and MacKinlay (1997) and Bollerslev, Engle, and Nelson (1994)), although for higher values of Δ a central limit theorem seems to hold and so Gaussianity becomes a less poor assumption for $\{y_n\}$ in that case. This means that at this "macroscopic" time scale every single assumption underlying the Black-Scholes model is routinely rejected by the type of data usually used in practice.

One possible response to the empirical rejection of (10.1) is to replace Brownian motion by a heavier tailed Lévy process — such as the generalised hyperbolic (see, for example, Eberlein and Keller (1995) and Rydberg (1999); questions of pricing and hedging for Lévy processes is discussed by Chan (1999) and Hubalek and Krawczyk (1999)). This will allow returns to be both heavy tailed and skewed, however these returns are going to be independent and stationary, by the definition of Lévy processes. Hence these models are also easily rejected empirically, as well as missing a major concept in financial economics — that of changing volatility or risk in financial markets.

In order to improve these "macroscopic" models we can allow the volatility process to change over time according to an OU process or a superposition of such processes as was suggested by Barndorff-Nielsen and Shephard (1999). In these stochastic volatility (SV) models we write

$$\begin{aligned} \mathrm{d}x^*(t) &= \left\{\mu + \beta\sigma^2(t)\right\}\mathrm{d}t + \sigma(t)\mathrm{d}w(t) + \rho\mathrm{d}\bar{z}(\lambda t), \quad t \in [0, S], && (10.3) \\ \mathrm{d}\sigma^2(t) &= -\lambda\sigma^2(t)\mathrm{d}t + \mathrm{d}z(\lambda t) && (10.4) \end{aligned}$$

where $\bar{z}(t) = z(t) - \mathrm{E}z(t)$, the centred version of the BDLP (which is here to capture leverage, see Black (1976) and Nelson (1991)). SV models driven entirely by Brownian motions have been extensively studied in the econometrics literature, see Taylor (1994), Ghysels, Harvey, and Renault (1996) and Shephard (1996) for reviews, however the use of Lévy based OU type processes is new and powerful.

When financial economists look at returns with very small values of Δ even these SV models are not sufficiently rich. Assets are neither continuously traded in time nor, generally, in price (see, for example, Engle and Russell (1998)). Instead trades occur irregularly in time and usually at discrete prices (e.g. 1/16 of a US dollar on the New York stock exchange). In order to deal with these "microscopic" datasets Rydberg and Shephard (2000) have suggested the use of compound processes. A stylised version of this model, based on OU type processes, is

$$\begin{aligned} x^*(t) &= \sum_{k=1}^{N(t)} u_k, \\ \sigma^{2*}(t) &= \int_0^t \sigma^2(u)\mathrm{d}u, \\ \mathrm{d}\sigma^2(t) &= -\lambda\sigma^2(t)\mathrm{d}t + \mathrm{d}z(\lambda t), \end{aligned}$$

with $N(t)$ being modelled as the number of trades in $[0, t]$ of a Cox process with instantaneous intensity $\left\{\sigma^2(t)\right\}$. We will write τ_k as the time of the $k\,th$ event and so $\tau_{N(t)}$ is the time of the last recorded event when we are standing at calendar time t. Further, we let $\{u_k\}$ be some stationary sequence with conditional (on the intensity) mean

$$\begin{aligned} \mu_k &= \mu\left(\tau_k - \tau_{k-1}\right) + \beta\sigma_k^2 + \rho\bar{z}_k, \\ \sigma_k^2 &= \beta\left\{\sigma^{2*}\left(\tau_k\right) - \sigma^{2*}\left(\tau_{k-1}\right)\right\}, \\ \bar{z}_k &= \rho\left\{\bar{z}\left(\tau_k\right) - \bar{z}\left(\tau_{k-1}\right)\right\}. \end{aligned}$$

If the support of $\{u_k\}$ is discrete then the price process will move at irregularly spaced times and the prices at which trades are recorded will be discrete.

In this section we will give some of the basic properties of the macroscopic models. Due to space limitations it is not possible to give a full discussion of all of the issues (such as the non-existence of arbitrage), instead we refer the reader to Barndorff-Nielsen and Shephard (1999) and Nicolato (1999). The first of these papers discusses, in particular, the connection between microscopic and macroscopic models.

10.2. The SV model

We first discuss the distributional properties of the stochastic volatility process x^* given by (10.3). Those properties are embodied in the class of integrals of the form

$$f \bullet x^* = \int_0^\infty f(t)\mathrm{d}x^*(t) \tag{10.5}$$

where f is a deterministic real function and we interpret $f \bullet x^*$ as a stochastic integral (as defined, for instance, in Protter (1992)). We therefore proceed to determine the cumulant function of such integrals (when they exist). In particular, by suitable choice of f one obtains the cumulant functions of the multivariate marginal distributions of x^* in terms of the cumulant generating function $k(\theta)$ of $z(1)$.

Theorem 10.1 The cumulant function of $f \bullet x^*$ is expressible as

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger f \bullet x^*\} &= \lambda \int_0^\infty \left\{k(-Je^{-\lambda s}) + k(-H(s))\right\} \mathrm{d}s \\ &\quad + i\zeta(\mu - \lambda\rho\xi) \int_0^\infty f(s)\mathrm{d}s \end{aligned} \tag{10.6}$$

where

$$J = \int_0^\infty \left\{\frac{1}{2}\zeta^2 f^2(u) - i\zeta\beta f(u)\right\} e^{-\lambda u}\mathrm{d}u \tag{10.7}$$

and

$$H(s) = \int_0^\infty \left\{\frac{1}{2}\zeta^2 f^2(s+u) - i\zeta\beta f(s+u)\right\} e^{-\lambda u}\mathrm{d}u - i\zeta\rho f(s). \tag{10.8}$$

□

PROOF From (10.1) we have

$$f \bullet x^* = (f\sigma)\bullet w + \beta\int_0^\infty f(t)\sigma^2(t)\mathrm{d}t + \rho f(\lambda^{-1}t)\bullet z + (\mu - \rho\xi\lambda)\int_0^\infty f(t)\mathrm{d}t. \tag{10.9}$$

Now

$$\mathrm{E}\{\exp(i\zeta(f\sigma)\bullet w)|z(\cdot)\} = \exp\left\{-\frac{1}{2}\zeta^2\int_0^\infty f^2(t)\sigma^2(t)\mathrm{d}t\right\}$$

and hence

$$\mathrm{E}\{\exp(i\zeta f \bullet x^*)\} = \mathrm{E}\{\exp G[f]\}\exp\left\{i\zeta(\mu - \rho\xi\lambda)\int_0^\infty f(t)\mathrm{d}t\right\}$$

where

$$G[f] = \int_0^\infty \int_0^\infty \left\{ -\frac{1}{2}\zeta^2 f^2(t) + i\zeta\beta f(t) \right\} \sigma^2(t)\mathrm{d}t + i\zeta\rho \int_0^\infty f\left(\lambda^{-1}t\right)\mathrm{d}z(t).$$

Furthermore, using the representation

$$\sigma^2(t) = e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}\mathrm{d}z(\lambda s)$$

we find for an arbitrary function h

$$\int_0^\infty h(t)\sigma^2(t)\mathrm{d}t = I_0 + I_1$$

where

$$\begin{aligned} I_0 &= \int_0^\infty h(t)e^{-\lambda t}\int_{-\infty}^{0} e^{\lambda s}\mathrm{d}z(\lambda s)\mathrm{d}t \\ &= \int_0^\infty e^{-\lambda t}h(t)\mathrm{d}t \int_{-\infty}^{0} e^{s}\mathrm{d}z(s) \end{aligned}$$

and

$$\begin{aligned} I_1 &= \int_0^\infty \int_s^\infty e^{-\lambda(t-s)}h(t)\mathrm{d}t\mathrm{d}z(\lambda s) \\ &= \int_0^\infty \int_0^\infty e^{-\lambda u}h(s+u)\mathrm{d}u\mathrm{d}z(\lambda s). \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{E}\{\exp(i\zeta f \bullet x^*)\} &= \mathrm{E}\left[\exp\left\{-J\int_0^\infty e^{-s}\mathrm{d}z(s)\right\}\right] \\ &\quad\times\mathrm{E}\left[\exp\left\{-\int_0^\infty H(s)\mathrm{d}z(\lambda s)\right\}\right] \\ &\quad\times\exp\left\{i\zeta(\mu-\rho\xi\lambda)\int_0^\infty f(t)\mathrm{d}t\right\} \end{aligned} \tag{10.10}$$

where J and $H(s)$ are given by (10.7) and (10.8). Application of the key formula (3.10) now yields (10.3). □

In particular, letting $\zeta = 1$ and

$$f(t) = \zeta_1 \mathbf{1}_{[0,t_1]} + \cdots + \zeta_m \mathbf{1}_{[0,t_m]} \tag{10.11}$$

(where $0 < t_1 < \cdots < t_m$) we obtain the joint cumulant function of $x^*(t_1), \cdots, x^*(t_m)$.

Let us consider the special case of $m = 1$, $t_1 = t$ and $\mu = \beta = \rho = 0$ in more detail. For this we have

Corollary 10.1 In the case $\mu = \beta = \rho = 0$, the cumulant function of $x^*(t)$ is

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger x^*(t)\} &= \lambda\int_0^\infty k\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})e^{-\lambda s}\right\}\mathrm{d}s \\ &\quad+\lambda\int_0^t k\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda(t-s)})e^{-\lambda s}\right\}\mathrm{d}s. \end{aligned} \tag{10.12}$$

$\square$

From this formula the cumulants of $x^*(t)$ are explicitly expressible in terms of the cumulants of $z(1)$ or, alternatively, of $\sigma^2(t)$. More specifically, with

$$k(\theta) = \sum_{m=1}^{\infty} \kappa_m (-1)^m \frac{\theta^m}{m!}$$

and

$$\mathrm{C}\{\zeta \ddagger x^*(t)\} = \sum_{m=1}^{\infty} \acute{\kappa}_m \frac{(i\zeta)^m}{m!}$$

as the series representations of the cumulant functions of $z(1)$ and $x^*(t)$, we find that the even order cumulants of $x^*(t)$ are (all the odd order cumulants being 0)

$$\acute{\kappa}_{2m} = \kappa_m \frac{(2m)!}{m!} 2^{-m} \lambda^{-m+1} c_m(t;\lambda) \tag{10.13}$$

where

$$c_m(t;\lambda) = \int_0^t (1 - e^{-\lambda(t-s)})^m \mathrm{d}s + (m\lambda)^{-1}(1 - e^{-\lambda t})^m \tag{10.14}$$

$$= t + \lambda^{-1} \sum_{\nu=1}^{m} (-1)^\nu \nu^{-1}(1 - e^{-\nu\lambda t}) + (m\lambda)^{-1}(1 - e^{-\lambda t})^m. \tag{10.15}$$

In particular

$$c_1(t;\lambda) = c_2(t;\lambda) = t \tag{10.16}$$

Furthermore we find for the kurtosis of $x^*(t)$

$$\acute{\gamma}_2 = \frac{\acute{\kappa}_4}{\acute{\kappa}_2^2} = 3\gamma_2 \lambda^{-1} t^{-1}$$

where $\gamma_2 = \kappa_4/\kappa_2^2$.

In other words

$$3\acute{\gamma}_2^{-1} = \gamma_2^{-1}\lambda t.$$

Example 10.1 Suppose $\sigma^2(t) \sim IG(\delta,\gamma)$. Then $\bar{\mathrm{K}}\{\theta \ddagger \sigma^2(t)\} = \delta\gamma\{1 - (1 + 2\theta/\gamma^2)^{1/2}\}$ and so, by formula (4.15),

$$k(\theta) = \frac{\delta\theta}{\gamma}(1 + 2\theta/\gamma^2)^{-1/2}.$$

Consequently

$$\kappa_1 = \delta/\gamma \quad \text{and} \quad \kappa_m = m\, 1\, 3 \cdots (2m-3)(\delta/\gamma)\gamma^{-2(m-1)}.$$

In particular, we find, by (10.3) and (10.16)

$$\acute{\kappa}_2 = (\delta/\gamma)t \quad \text{and} \quad \acute{\kappa}_4 = 6(\delta/\gamma^3)\lambda^{-1}t.$$

As a result the kurtosis is

$$\acute{\gamma}_2 = 6\gamma^{-1}\lambda^{-1}t^{-1} \quad \text{while} \quad 3\acute{\gamma}_2^{-1} = 2\gamma^{-1}\lambda t.$$

□

10.3. The SSV model

In this subsection we shall indicate how a stationary process of the form (1.2) lies imbedded in a stochastic volatility model for the term structure of interest rates. The model, which is discussed in detail in Nicolato (1999), is an extension of the Heath-Jarrow-Morton (HJM) model, with the Hull-White factorisation of the volatility.

As in the HJM setup, we assume that the forward rate

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}$$

satisfies a stochastic differential equation of the form

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}w(t)$$

where w is Brownian motion and the drift $\alpha(t,T)$ and volatility $\sigma(t,T)$ are restricted by the relation

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)\mathrm{d}v - \sigma(t,T)g(t)$$

for some function $g(t)$ which has the interpretation of the market price of risk. The Hull-White specification of the volatility is

$$\sigma(t,T) = \sigma(t)e^{-\phi(T-t)}$$

ϕ being a positive parameter.

A widely adopted rough approximation specifies that the short rate $s(t) = f(t,t)$ follows a mean-reverting dynamics

$$\mathrm{d}s(t) = \{\Phi(t) - \phi s(t)\}\mathrm{d}t + \sigma(t)\mathrm{d}t$$

for some function or process $\Phi(t)$. If, in particular,[8]

$$\Phi(t) = \mu + \beta\sigma^2(t)$$

and if $\sigma^2(t)$ is now assumed to be a stationary stochastic process (rather than a deterministic function) then $s(t)$ obeys an equation of the SSV type (1.2). A preliminary statistical analysis of LIBOR data has indicated that, under certain regimes, this type of dynamics for the short rate, with the volatility process $\sigma^2(t)$ defined as a superposition of IG-OU processes, provides a fairly realistic model. Moreover, in that setting, the cumulant functional of the process $s(t)$ is available in a form similar to that for integrated processes x^* given in Theorem 10.1.

[8]this corresponds to assuming that the market price of risk satisfies the equation

$$\sigma(t)g(t) = \gamma^*(t) + \int_0^t \sigma^2(u)e^{-2a(t-u)}\mathrm{d}u - \Phi(t)$$

where $\gamma^*(t) = \frac{\mathrm{d}f^*(t)}{\mathrm{d}t} + af^*(t)$ and f^* denotes the initially observed forward rate curve, i.e. $f^*(t) = f(0,t)$.

11. Discussion: Further issues and future work

- We have here focussed on GH laws as a flexible class of distributions for describing asset returns in finance and velocity differences in turbulence. An alternative approach is to seek to capture the observed distributional behaviour by specification of Lévy densities rather than probability densities. For some case studies see Novikov (1994), Koponen (1995), Cont, Potters, and Bouchaud (1997), Mantegna and Stanley (2000, Sect. 8). Levendorskii (2000) develops this approach considerably, including applications to option pricing.

- Pricing of derivatives under the type of stochastic volatility models given by (1.1) with $\sigma^2(t)$ of OU type is discussed by Nicolato and Prause (2000).

- In view of the roles of the concepts of selfsimilarity and scaling in finance and turbulence on the one hand and the applicability of the generalized hyperbolic distributions on the other it seems natural to ask whether for any given GH law D, say, there exists a selfsimilar process y with stationary increments such that the law of $y(1)$ is D and all the one-dimensional marginals of y belong to the class GH. In particular, one may ask whether there is such a y whose one-dimensional marginals are normal inverse Gaussian. This is still an open problem. However, by relaxing the requirements somewhat, either asking only for second order stationary increments or only for second order selfsimilarity, it is possible to make some headway, see Barndorff-Nielsen and Pérez-Abreu (1999) and Barndorff-Nielsen (2000a), respectively.

- A simple q-dimensional version of the SV model for log-prices sets $x^*(t) = \left\{x_1^*(t), ..., x_q^*(t)\right\}$ with

$$\mathrm{d}x^*(t) = \{\mu + \beta\Sigma(t)\}\,\mathrm{d}t + \Sigma(t)^{1/2}\mathrm{d}w(t),$$

where $\Sigma(t)$ is a time varying stochastic covariance matrix and β is a vector of risk premiums. We can estimate $\Sigma^*(t)$ using quadratic variation

$$\begin{aligned}
[x^*](t) &= \text{p-}\lim \sum \{x^*(t_{i+1}^r) - x^*(t_i^r)\}'\{x^*(t_{i+1}^r) - x^*(t_i^r)\} \\
&= \Sigma^*(t) = \int_0^t \Sigma(u)\mathrm{d}u
\end{aligned}$$

as $x^*(t)$ is a continuous $q-$dimensional local martingale. An important problem is to specify a model for $\Sigma^*(t)$. One approach is to do this indirectly via a factor structure

$$\Sigma(u) = diag(\sigma_1^2(u), ..., \sigma_q^2(u)) + \phi'\phi\sigma_{q+1}^2(u).$$

Here $\phi = (\phi_1, ..., \phi_q)$ are unknown parameters and the $\sigma_1, \sigma_2, ..., \sigma_{q+1}$ are mutually independent OU processes which are square integrable and stationary — a model first suggested and analysed by Barndorff-Nielsen and Shephard (1999). The process $x^*(t)$ has a common, scaled stochastic volatility component. In addition, each element of the vector process is hit by an independent stochastic volatility process. It generalizes straightforwardly to allow for two or more factors. This style of model is in keeping with the latent factor models of Diebold and Nerlove (1989), King, Sentana, and Wadhwani (1994) and Pitt and Shephard (1999). Its motivation is that in financial assets it is often the case that returns move together, with a few common driving mechanisms. The common factors allow us to pick this up in a straightforward and parsimonious way. This model could be generalised by allowing the volatilities to be dependent using the multivariate OU type processes discussed in Barndorff-Nielsen and Shephard (1999).

- Extensions of the type of stochastic volatility models discussed in the present paper to settings where t instead of time denotes a multi-dimensional index, such as position in the plane or in space-time, are of considerable interest. In that context the focus of formulation is on Lévy random fields rather than Lévy processes. Some relevant references are Brix (1998) and Wolpert and Ickstadt (1998).

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