

SCORING METHODS AND BANKS STABILITY: NATURE VERSUS THE AGENDA SETTER

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ABSTRACT. In this paper we introduce new scoring methods for use in social choice situations. The new methods incorporate information relevant to the Banks set. The new scoring methods are compared, as well as traditional scoring methods to investigate what information (if any) they convey about the robustness of alternatives in the Banks set. We then investigate the usefulness of such scores to agenda setters when they are uncertain about the set of alternatives to be chosen from.

1. INTRODUCTION

This paper considers uncertainty in collective choice situations and examines scoring methods potentially useful to an agenda setter in the presence of uncertainty. Banks [1985] and subsequent work examines the power an agenda setter – one able to order alternatives to be voted on in an amendment agenda – may exercise in collective choice settings with perfect information.¹ However, such information may not always be available. Uncertainty in agenda-setting environments may come from multiple sources: (1) uncertainty about the alternatives to be voted on, (2) uncertainty about the preferences of voters, or (3) uncertainty about the length of the agenda (i.e. how long voting will take place).² While all of the above mentioned are interesting and worthy of investigation, we focus on uncertainty of the first sort.

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¹Though Ordeshook and Palfrey [1988] are a notable exception, investigating how uncertainty about others' preferences effect strategic voting outcomes.

²A possible fourth source of uncertainty may be the how rational voters are. We take voters to be forward looking, strategic agents.

The environment considered here is one in which an agenda setter must order alternatives (from a finite set) to be voted on in an amendment agenda prior to knowing which alternatives will *actually* be voted on. Specifically, the way we incorporate uncertainty about the choice set is by allowing nature to remove some alternatives from consideration prior to voting, but after the agenda has been set.

To facilitate the investigation, new scoring methods are introduced that incorporate information relevant to the Banks set. The Banks set [Banks, 1985] is a useful solution concept identifying the set of alternatives that can be chosen when voters vote strategically under an amendment agenda. An amendment agenda is a method of voting on many alternatives as follows: alternatives are ordered by an agenda setter and the first alternative is voted against the second alternative, the winner of which is pitted against the third, and so on. The alternative that wins after the last alternative has been considered is adopted. The Banks set is equal to those alternatives that may be reached via *some* ordering, when an amendment agenda is used, when voters vote strategically. In essence the Banks set shows the agenda setters' power – when an agenda setter may choose the order of alternatives to be voted on and voters vote strategically, the Banks set is the set of possible outcomes. However, in this paper we ask “when an agenda setter is uncertain about which alternatives will be available for consideration, what should she do? Further, what information is sufficient to answer this question?” Specifically, we look at several simple scoring methods (established ones as well as new methods) that one might suspect capture an alternatives' relative “strength” in a collective choice situation.

The paper is organized as follows. Background and definitions are given in section 2, some existing scoring methods are reviewed and new ones introduced in section 3. Section 4 empirically explores one aspect of scoring methods and future work is discussed in section 5.

2. BACKGROUND

This section closely follows that of Moser et al. [2006], with the exception of section 2.1. Consider a collective choice situation (hereafter CCS) in which an outcome x must be chosen from some finite set of $n \geq 3$ alternatives, X . Denote by $T \subset X^2$ the binary majority

preference relation and assume that T is a complete and antisymmetric binary relation on X (such a relation is referred to as a *tournament*). If alternative x is majority preferred to alternative y , write xTy . The set of alternatives that are majority preferred to an alternative x is denoted by $T(x)$ and the set of alternatives that are defeated (under majority rule) by x are denoted by $T^{-1}(x)$. For any set X and any subset $Y \subset X$, let $X_{-Y} \equiv X \setminus Y$ and, for any tournament T on X , let $T|_Y$ denote the tournament induced by T on Y .³ An *agenda* is any (finite) ordered subset of X . Agendas are denoted by the \sim superscript (for example, \tilde{a} , \tilde{x} , etc.), and the set of all agendas is denoted S_X .

Tournaments are frequently (and equivalently) represented as complete, asymmetric directed graphs, with X being the set of vertices, and two vertices $x, y \in X$ being connected by an edge only if $(x, y) \in T$. Tournaments have been studied at least since the 1960's, in connection with a wide range of topics from majoritarian voting to ranking of participants in competitions [Moon, 1968]. They are an important tool for studying collective choice situations because when the number of alternatives is finite and majority preference is strict, constructing a social choice prediction is equivalent to the problem of selecting a set of “winners” from a tournament. In such applications, it is generally assumed that each vertex of the tournament represents a feasible alternative.

2.1. Maximality and Chains. Given a tournament T , a subset $Y \subseteq X$ is

- (1) *maximal* with respect to a collection of subsets $\mathcal{H} \subseteq 2^X$, if there is no $Y' \in \mathcal{H}$ such that $Y \subset Y'$, and
- (2) a *chain* if $T|_Y$ is transitive.

Given a tournament T on a set X the set of all chains is denoted by $H(X, T)$. Further, $Y \subseteq X$ is a *maximal chain* if Y is both a chain and maximal with respect to the set of all chains. That is, Y is a maximal chain if $\nexists Y' \subseteq H(X, T)$ with $Y \subset Y'$. The set of all maximal chains is denoted by $MC(X, T)$. Further, let $H(x)$ be the set of all chains for which x is top-ranked and let $MC(x)$ be the set of all maximal chains for which x is top-ranked.

³In graph theory, $T|_Y$ is referred to as a subtournament. Formally, for any $x, y \in X$, $(x, y) \in T|_Y$ if and only if $(x, y) \in Y \cap T$.

2.2. The Banks and Uncovered Sets. Extending the work of Miller [1980], Shepsle and Weingast [1984] and others, Banks [1985] demonstrated that outcomes of sophisticated voting in amendment agendas can be identified with maximal elements of externally stable chains.⁴ For any tournament T and pair of alternatives $x, y \in X$, write $x C_T y$ (read “ x covers y with respect to T ”) if $x T y$ and $T(x) \cap T(y) = T(x)$ (that is, x beats y and any alternative z that beats x beats y as well).

Given X and T , define the *uncovered set* [Miller, 1980], $UC(X, T)$, as follows:

$$UC(X, T) = \{x \in X : y T x \Rightarrow \exists z \text{ s.t. } x T z T y\} = \{x \in X : \{y \in X : y C_T x\} = \emptyset\}.$$

Additionally, define the *Banks set* (Banks [1985]), $B(X, T)$, as follows:

$$B(X, T) = \{x \in X : MC(x) \neq \emptyset\}.$$

The following important fact is well-known.

Theorem 2.1 (Banks [1985]).

$$B(X, T) \subseteq UC(X, T).$$

2.3. Uncertainty of Choice Set in Collective Choice Settings. In this section we operationalize an agenda setter’s uncertainty of the choice set. Specifically, the sequence of events is as follows: an agenda setter orders all alternatives in some finite set X (call this ordering \tilde{a}); nature removes some subset $Z \subset X$; voters vote strategically on the remaining alternatives $Y = X \setminus Z$ according to the ordering \tilde{a} .

In brief, if x is a feasible outcome before nature acts (that is, if $x \in B(X, T)$), how likely is it to be a feasible outcome after removal of some subset of alternatives (how likely is it that $x \in B(Y, T|_Y)$?) This leads to the following definition.

⁴See Penn [2006] for an interesting treatment of Banks points and maximal chains when chains may be infinitely long.

Definition 2.1. *The k -probabilistic Banks score of a Banks point $x \in B(X, T)$, written $pB_k(x)$, is the likelihood that x will be in the banks set of $X \setminus S$ with $|S| \leq k$.*

$$pB_k(x) = \frac{|\{S \in \mathcal{S} : x \in B(X \setminus S)\}|}{|\mathcal{S}|}$$

where \mathcal{S} is the set of all subsets of X of cardinality at most k .

While one may explicitly calculate $pB_k(x)$ for given k, T , and x , doing so is extremely computationally intensive. For example, if $|X| = n$, one must calculate the Banks set of $n + 1$ tournaments to determine $pB_1(x)$. Further, as Woeginger [2003] shows, finding Banks points is NP-complete, loosely meaning that the problem gets much much more difficult to solve as the size of X increases. It is this observation that prompts our attention to turn to tournament statistics that may contain some information relevant to $pB_k(x)$.

3. SCORING METHODS

Here we consider scoring alternatives for the purpose of comparison. The notion of assigning numbers to alternatives in a tournament that represent their relative ‘strength’ is far from novel, though systematic work exploring scoring methods seem to have begun with Wei [1952] and Kendall [1955]. Moon [1968] and Laslier [1997, ch. 3] also discuss scoring methods as they relate to tournament solution concepts.

Perhaps the most intuitive score to give an alternative is simply the number of other alternatives it beats. This gives rise to the Copland score, $COP(x)$, which is simply the number of alternatives x beats. Specifically, $COP(x) = |T^{-1}(x)|$. An alternative is called a *Copland winner* if it has highest Copland score.⁵ In one regard, the Copland score captures the “first-order strength” of an alternative: $COP(x)$ is computed without consideration of who is beaten by points that x beats – that is, the Copland score only takes first order “strength” of alternatives into consideration.

A related idea, but one that takes the entire tournament structure into consideration is the *Markov score* of an alternative, $MARK(x)$. The Markov score may be motivated as follows:

⁵It is easily shown that every Copland winner is uncovered, though the converse is not necessarily true.

suppose two distinct alternatives, $x(1)$ and $y(1)$, are selected at random at time $t = 1$ and for all $t > 1$, the winner of the previous pair-wise contest (call this $x(t)$) is pitted against a randomly chosen (distinct) alternative (call this $y(t) \in X \setminus x(t)$). This process defines a Markov chain on the state space X . Now let $p(t)$ be a $n \times 1$ vector of probabilities for the random variable $x(t)$ to be $x \in X$. One then has [Laslier, 1997, pg.57]:

$$(1) \quad p(t+1) = \frac{1}{n-1}(T+S)p(t)$$

where S is the diagonal matrix of Copland scores and T (in an abuse of notation) is the $n \times n$ matrix representation of the tournament. Hence, equation 1 gives the transition matrix for the Markov process. One may investigate the steady state of this process, \bar{p} .⁶ This gives rise to the Markov score, formally defined as follows.

Definition The vector \bar{p} , called the *Markov scores*, is the eigenvector of the transition matrix associated to the eigenvalue 1 such that $\sum_{x \in X} \bar{p}(x) = 1$.

Hence, Markov scores may be thought of as the likelihood of an alternative surviving, after indefinitely long pair-wise contests between alternatives.⁷

Alternatively, one may score points based on the “ease” with which they are adopted as the outcome of strategic voting. Just as the Copland score captures the first order strength of an alternative (via the number of alternatives it beats), we may similarly be interested in the number of alternatives used to ensure an alternative is in the Banks set. This gives rise to two different scores, one based on the number of externally stable transitive chains an alternative is top-ranked in, and another based on the number of alternatives in all of one’s externally stable chains. To this end, define the *Union score* of an alternative, $UN(x)$, as the total number of distinct alternatives that appear in all of the maximal chains x is top-ranked in:

⁶That such a limit exists, and that $\bar{p}(x) > 0$ iff x is in the top-cycle are the results of elementary Markov theory. See Laslier [1997, ch. 3].

⁷Properties of Markov winners – points with highest Markov score – are investigated in Laslier [1997, ch. 3.3].

$$UN(x) = \left| \bigcup_{H \in MC(x)} H \right|.$$

Trivially, $UN(x) \leq COP(x)$.

Additionally, define the *Number-of-chains* score, $NUMCH(x)$ as the total number of chains x is top-ranked in:

$$NUMCH(x) = |MC(x)|.$$

These two scores capture different aspects of the “breadth” of support for x to be a Banks point.

Alternatively, one may wish to know how “fragile” a Banks point, x , is: if alternatives are removed at random, how many must be removed (in expectation) before x is no longer a Banks point? In a related issue, what is the smallest set of alternatives S such that x is ensured *not* to be in $B(X \setminus S)$, for any S ? Considering an alternative’s “immunity” from removal of alternatives gives rise to considering the length of the shortest chain $H \in MC(x)$:

$$SHCH(x) = \min\{|H| : H \in MC(x)\}$$

with $SHCH(x) = \infty$ if $x \notin B(x)$. The shortest chain score of x is equal to the minimal number of alternatives need to place x in the Banks set.

These are five different ways of giving alternatives a score that may capture some element of “robustness”. Alternatives with a high *COP* score beat lots of things, and hence do not lose to many things. Markov winners survive with high probability in a long-run pair-wise random matching amendment agenda. An alternative x with a high *UN* score has a large “net” form which an agenda setter may use to schedule agendas that results in x being chosen. Likewise, *NUMCH* score captures the number of maximal chains that lead to the adoption of a given alternative. This may be thought of as a proxy for the ease with which an alternative is chosen. The *shortest chain score*, *SHCH*, captures ideas of “strength”: if x is

top of a “short” maximal chain, then random removal of alternatives is unlikely to prevent x from being selected.⁸ As we will see in the next section, these scores contain varying degrees of information regarding an alternative’s robustness.

3.1. Relation to Probabilistic Banks score. To aid in our investigation, we employ computer simulations to examine the relationship between probabilistic Banks scores and other scores. In particular, we generate a sequence of random tournaments $\{T_m\}$, calculate the Banks set $B(T_m)$, remove a set Z of k alternatives at random (so that $Y = X \setminus Z$ remains), and find the Banks set of the remaining tournament $B(T_{m|Y})$. For each tournament, we repeat this for all $Z \subset X$ such that $|Z| \leq k$. In this manner, $pB_k(x)$ is calculated for all $x \in X$. Notice that even if x is a Condorcet winner, $pB_k(x) < 1$ because some subset contains x , and hence is removed itself. Additionally, each of the five scoring methods discussed above are calculated for all $x \in X$.

3.2. pB_1 Results. To begin we look at removals of size one. That is, we compare $pB_1(x)$ to $COP(x)$, $UN(x)$, $NUMCH(x)$, $SHCH(x)$ and $MARK(x)$. In figure one, results of generating 100 random tournaments on 10 nodes is presented. For each plot, the ordinate axis shows the average 1-probabilistic Banks score for alternatives with score equal to the abscissa.⁹

[FIGURE 1]

⁸Alternatively, alternatives may be ranked by the number of different amendment agendas resulting in a particular alternative. To this end one could define the *all-agendas score* (or *ALLAG*) of an alternative as the percentage of amendment agendas leading to x being selected:

$$ALLAG(x) = \frac{|\{\tilde{a} \in S_X : x \text{ is the sophisticated voting outcome of } \tilde{a}\}|}{n!},$$

recalling that $|X| = n$ and S_X is the set of all permutations of X . For example, if the amendment agenda to be voted on is chosen at random, alternatives with high *all agendas score* are likely to be selected by the group. Like the k -probabilistic Banks score, this score suffers from extremely high computational costs to calculate, and is left for future study.

⁹For the interested reader, the frequency of each score is given in the appendix.

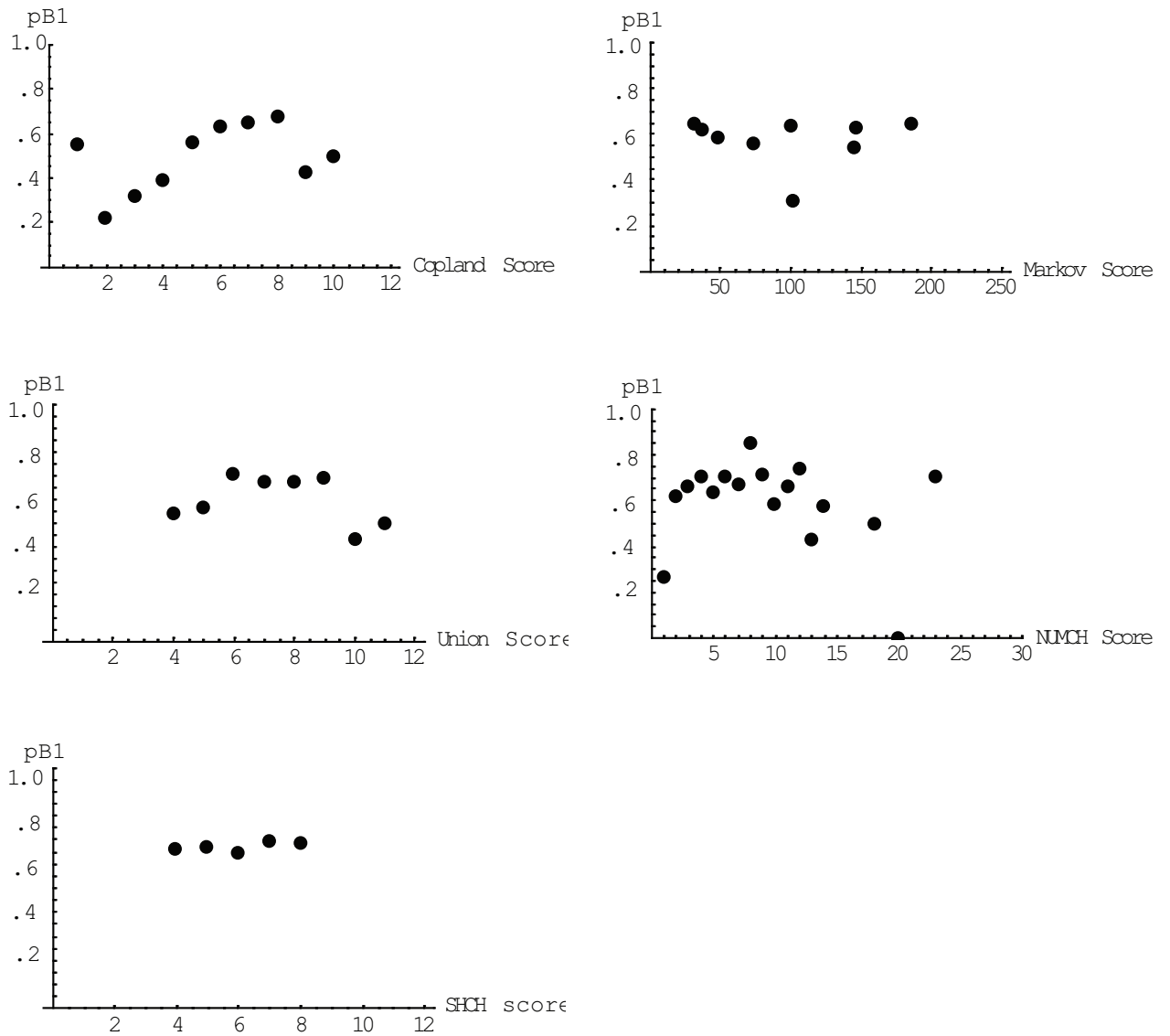


FIGURE 1. Numerical results for 100 random tournaments on 10 alternatives. The abscissa lists the each of the five scores discussed, respectively. The ordinate shows the average 1-probabilistic Banks score for alternatives with score equal to the abscissa.

Disappointingly, none of the scores seem to convey much information regarding 1-probabilistic Banks scores.

Figure 1 shows the empirical relation (or lack thereof) of the absolute level of various scores to $pB_1(x)$, with no clear pattern observed. Alternatively, one may suspect that the relative rank is more informative than absolute levels. However, as the following example demonstrates, alternatives that score the highest on pB_1 score can be different than alternatives scoring highest *NUMCH* or *UN* scores.

Example Consider the following collective choice situation, (X, T) :

- $X = \{a, b, c, d, e, f, g\}$,
- aTb, aTf ,
- bTd, bTe, bTg ,
- cTa, cTb, cTd, cTf, cTg ,
- dTa ,
- eTa, eTc, eTd, eTf ,
- fTb, fTd, fTg ,
- gTa, gTd, gTe .

While tedious, scores may be computed directly and are show in the next table.

	<i>MARK</i>	<i>NUMCH</i>	<i>UN</i>	<i>SHCH</i>	pB_1
<i>a</i>	.232	0	0	∞	-
<i>b</i>	.061	1	4	4	5
<i>c</i>	.175	3	6	4	4
<i>d</i>	.242	0	0	∞	-
<i>e</i>	.075	3	5	4	3
<i>f</i>	.198	0	0	∞	-
<i>g</i>	.015	1	4	4	0

TABLE 1

As Table 1 shows, alternative b survives the most 1-alterative removals (a total of 5) from the choice set. However, alternatives scoring highest on the *number-of-chains* score are $\{c, d\}$, while c is the alterative with the highest *union* score. Note also that the Markov winner (the alterative with the highest Markov score), d , is not even in the Banks set of X .

The next example demonstrates that alternatives that score highest on pB_1 score may be different than those alternatives scoring lowest on shortest-chain score.

Example Consider the following collective choice situation, (X, T) :

- $X = \{a, b, c, d, e, f\}$,
- aTb ,
- bTd, bTe, bTf ,
- cTa, cTc, cTd ,
- dTa ,
- eTa, eTc, eTd, eTf ,
- fTa, fTc, fTd .

Again, direct computation shows the following.

	<i>MARK</i>	<i>NUMCH</i>	<i>UN</i>	<i>SHCH</i>	pB_1
<i>a</i>	.065	0	0	∞	-
<i>b</i>	.162	1	4	4	0
<i>c</i>	.157	2	4	3	0
<i>d</i>	.419	0	0	∞	-
<i>e</i>	.089	1	5	5	0
<i>f</i>	.106	0	0	∞	-

TABLE 2

As Table 2 shows, alternatives b, c and e are all in the Bank set of X , but are not in the Banks set when any one alternative is removed. However, alternatives b, c and e all have different shortest-chain scores.

4. CONCLUSIONS

In this paper we introduced three new scoring methods for ranking alternatives in tournaments. Development of the scores was motivated by the problem of agenda setting in face of uncertainty about the choice set. Scores were constructed using principals relevant to the Banks set. Empirically, the scores were not seen to have any relation with “Banks stability” – the likelihood that an alternative remains in the Banks set after alternatives are removed.

We end with some possible future directions for this line of research:

- (1) What are the properties of UN , $NUMCH$ and $SHCH$? What is the relation between their maximal elements and established properties such as monotonicity, independence of losers, strong superset property, etc.?
- (2) How are various scoring methods related to each other – are they measuring some common attribute of alternatives?

Finally, this work can be seen as an attempt to find low-dimensional sufficient statistics of a tournament to convey information about an alternative's "robustness" to the removal of other alternatives. It simply might be the case that such statistics are inherently of a high dimension.

5. APPENDIX

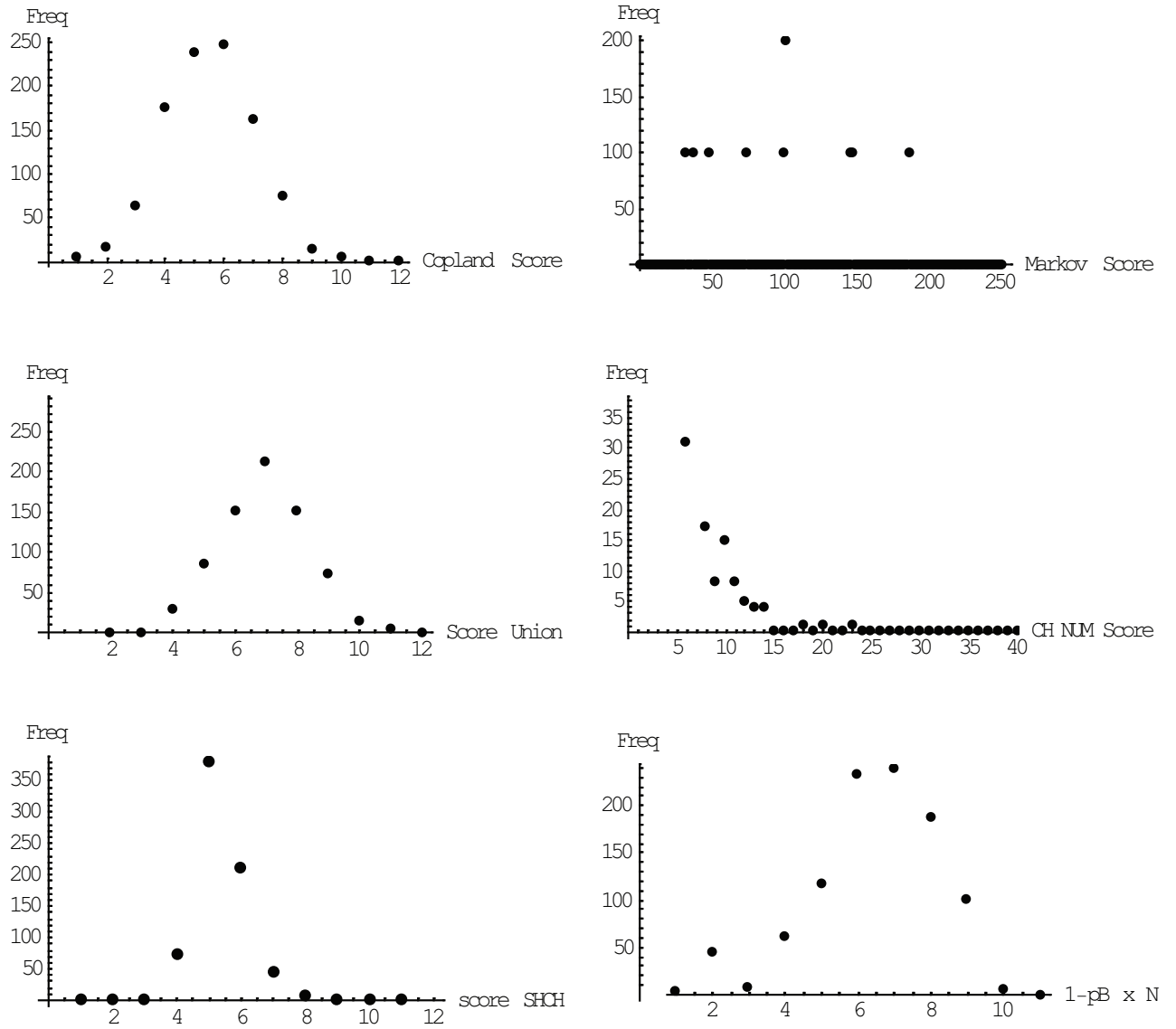


FIGURE 2. Frequency of scores.

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