

Economic Growth II: The Solow Model and Beyond

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the Solow model and beyond

- The last lecture introduced the Solow model without technical progress, in which growth of output per worker falls to zero eventually, although total output grows at the rate of population growth in the long-run.
- This lecture extends the Solow model to include exogenous technical progress.
- It also considers two further simple models – the augmented Solow model with human capital and the AK model of broad capital.

technological progress

- Exogenous vs endogenous; disembodied vs embodied; neutral vs factor biased.
- Technical progress is Hicks-neutral if the ratio of MPK/MPL is constant for a given K/L ratio (as if isoquants were being renumbered):

$$Y = F(L, K, T(t)) = T(t).F(L, K)$$

- Technical progress is Harrod-neutral if it is labour-augmenting (relative factor shares constant at any capital-output ratio) and Solow-neutral if it is capital-augmenting (relative factor shares constant at any labour-output ratio):

$$Y = F(A(t).L, K)$$

$$Y = F(L, B(t).K)$$

- The Cobb-Douglas function has all three properties.

exogenous technical progress

- Consider the labour-augmenting production function

$$(2.1) \quad Y = F(K, AL)$$

- Technical progress occurs when A rises over time, with labour becoming more productive when the level of technology is higher.
- When technical progress proceeds at a constant rate g and doesn't depend on any other variables, it is called

exogenous

$$(2.2) \quad \frac{\dot{A}}{A} = g \Leftrightarrow A_t = A_0 e^{gt}$$

Solow model with technical progress

- The Cobb-Douglas production function with technology is

$$(2.3) \quad Y = AK^\alpha L^{1-\alpha}$$

- When we have technical progress at the rate g , we can think of output as being per technology-adjusted worker

- $(2.4) \quad Y / AL = y / A = \tilde{y} = \tilde{k}^\alpha$

- In equilibrium the amount of capital per technology-adjusted worker must be constant

$$(2.5) \quad \dot{\tilde{k}} = s\tilde{k}^\alpha - (n + d + g)\tilde{k}$$

steady-state income and growth

- Setting equation (2.5) to zero gives

$$(2.6) \quad \tilde{k}^* = \left(\frac{s}{n + d + g} \right)^{1/(1-\alpha)}$$

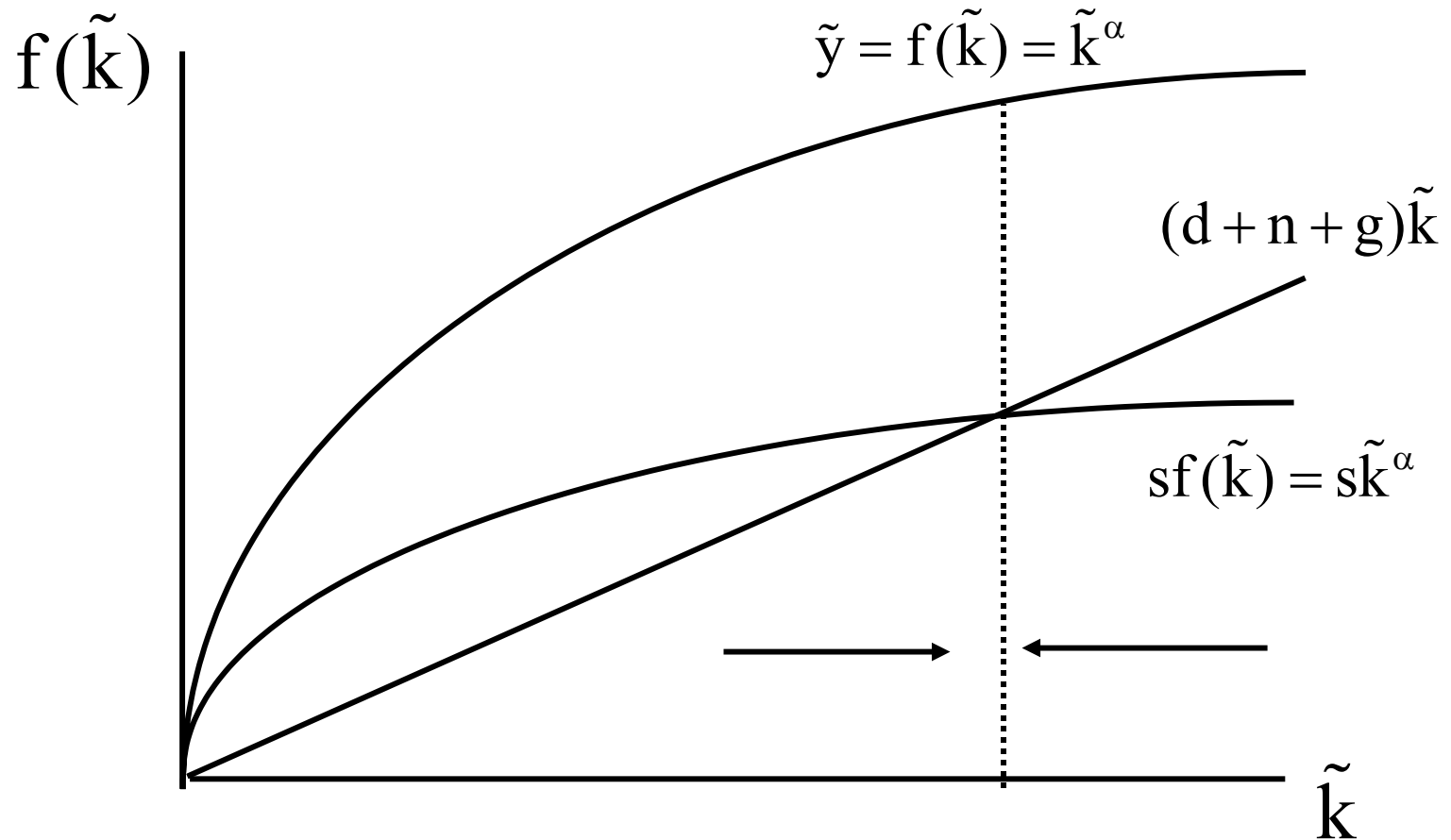
- Substituting this into the production function gives

$$(2.7) \quad \tilde{y}^* = \left(\frac{s}{n + d + g} \right)^{\alpha/(1-\alpha)}$$

- Or, in terms of output per worker

$$(2.8) \quad y^*(t) = A(t) \left(\frac{s}{n + d + g} \right)^{\alpha/(1-\alpha)}$$

the Solow diagram with technical progress



transition dynamics

- Recall the capital accumulation equation (2.5)

$$(2.9) \quad \dot{\tilde{k}} = s\tilde{y} - (n + g + d)\tilde{k}$$

- This can be re-written as

$$(2.10) \quad \frac{\dot{\tilde{k}}}{\tilde{k}} = s \frac{\tilde{y}}{\tilde{k}} - (n + g + d)$$

- Since we know that $\tilde{y} = \tilde{k}^\alpha$, this can be re-written as

$$(2.11) \quad \frac{\dot{\tilde{k}}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (n + g + d)$$

transition speed

- Barro and Sala-I-Martin (1995) show that this implies a growth rate of output, in the region of the steady-state, to be

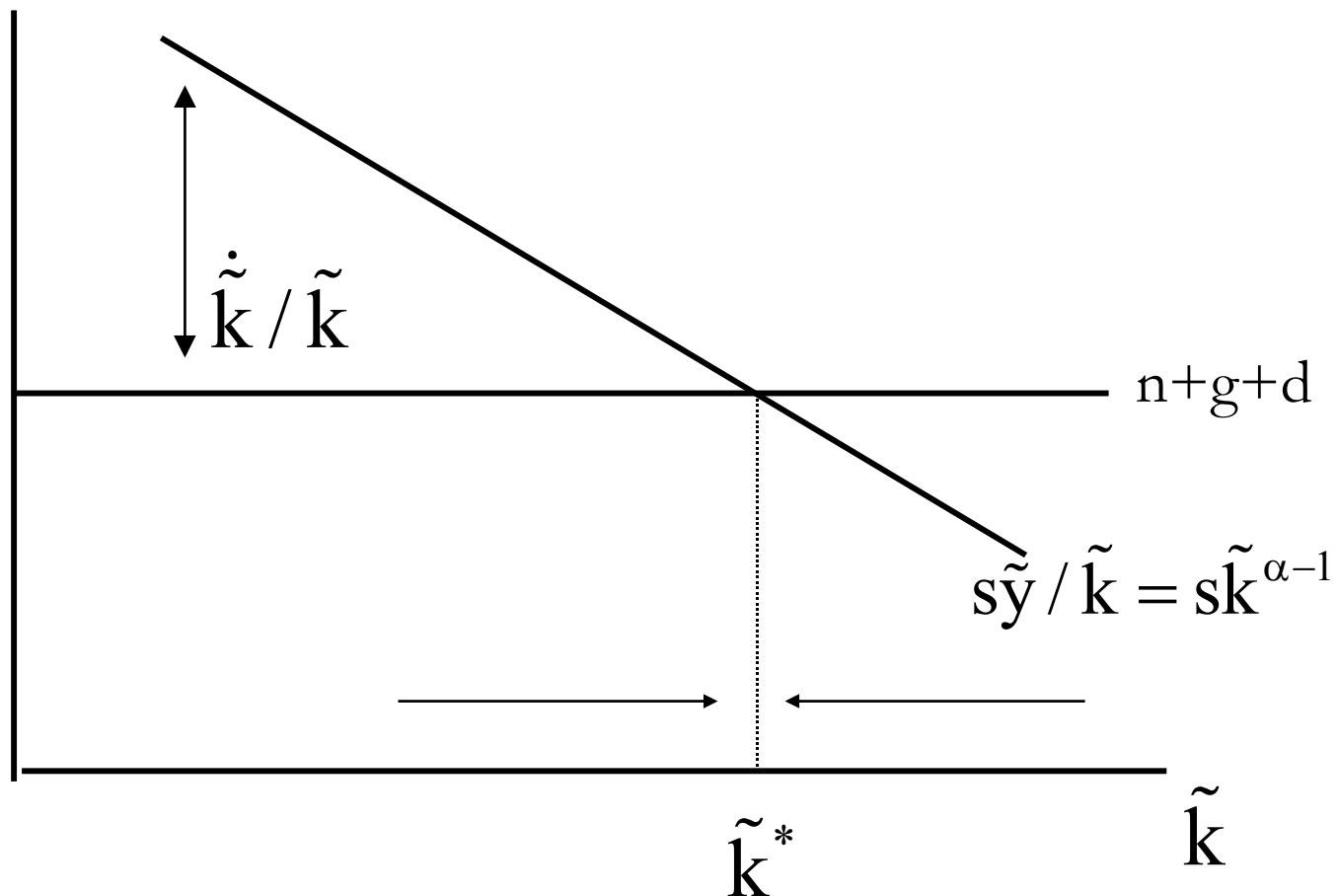
$$(2.12) \quad g_{\tilde{y}} \cong -(1 - \alpha)(n + g + d) \log(\tilde{y} / \tilde{y}^*)$$

- Where y^* is the steady-state level of output per technology-adjusted worker. We can re-write this as

$$(2.13) \quad g_{\tilde{y}} \cong -\beta \log(\tilde{y} / \tilde{y}^*)$$

- Where $\beta = (1 - \alpha)(n + g + d)$ indicates how quickly output per technology-adjusted worker approaches its steady-state value. If $b=0.05$ then 5 percent of the gap disappears each year and this is independent of the saving rate and the level of technology.
- What would be a reasonable value of β ? With $\alpha=0.3$, $d=0.05$, $n=0.01$ and $g=0.02$ we would expect $\beta=0.056$ but we tend to find $\beta=0.02$, which Mankiw (1995) argues implies a value of α of 0.75.

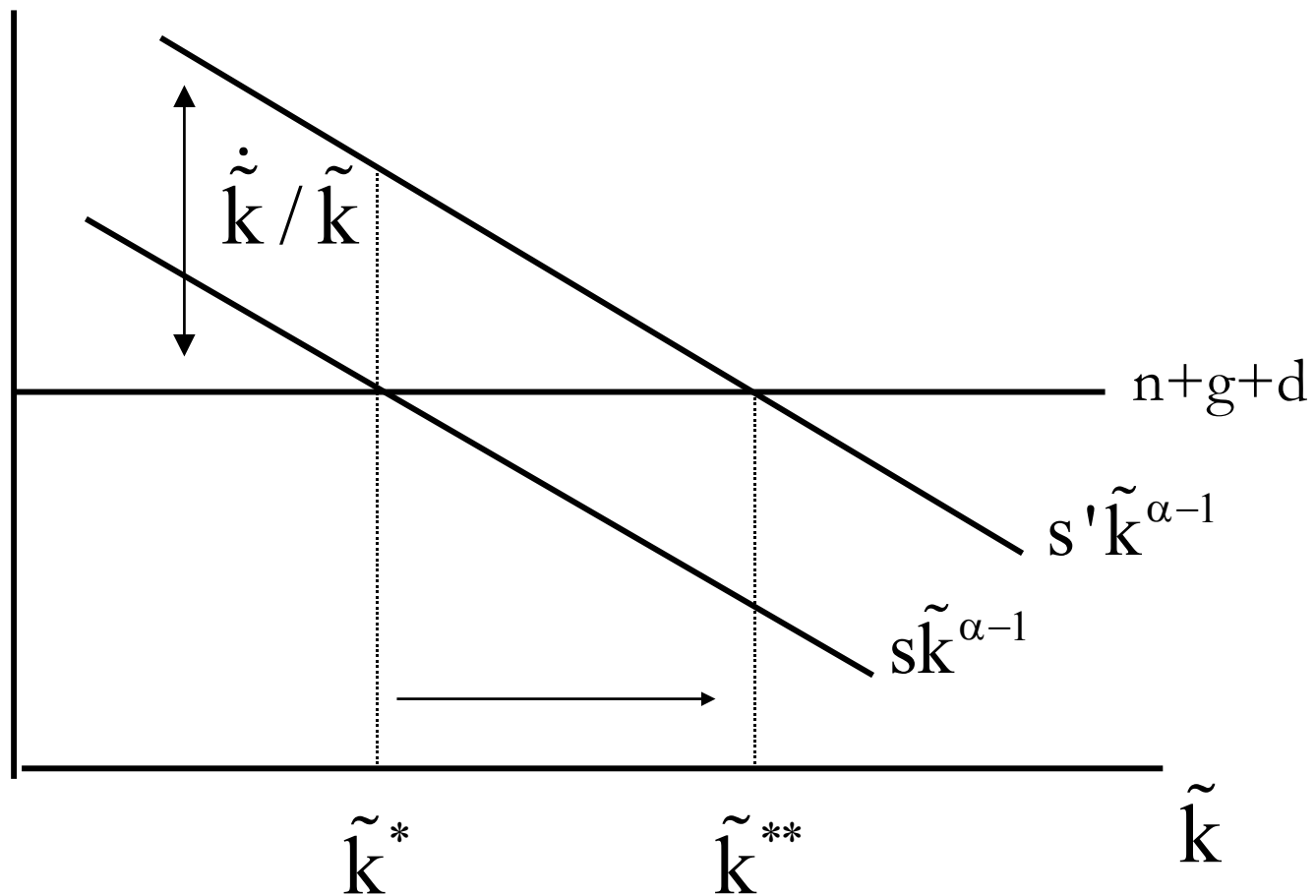
transition dynamics



a rise in the saving rate

- Suppose an economy begins in steady-state with investment rate s and then permanently increases this rate to s' (for example, because of an investment subsidy).
- At the initial level of capital per technology-adjusted worker, investment exceeds the amount needed to keep the level of capital per technology-adjusted worker constant, so it begins to rise.
- The increase in investment raises the growth rate temporarily as the economy moves to a new steady-state. But once the new higher steady-state level of income is reached, the growth rate returns to its previous level.

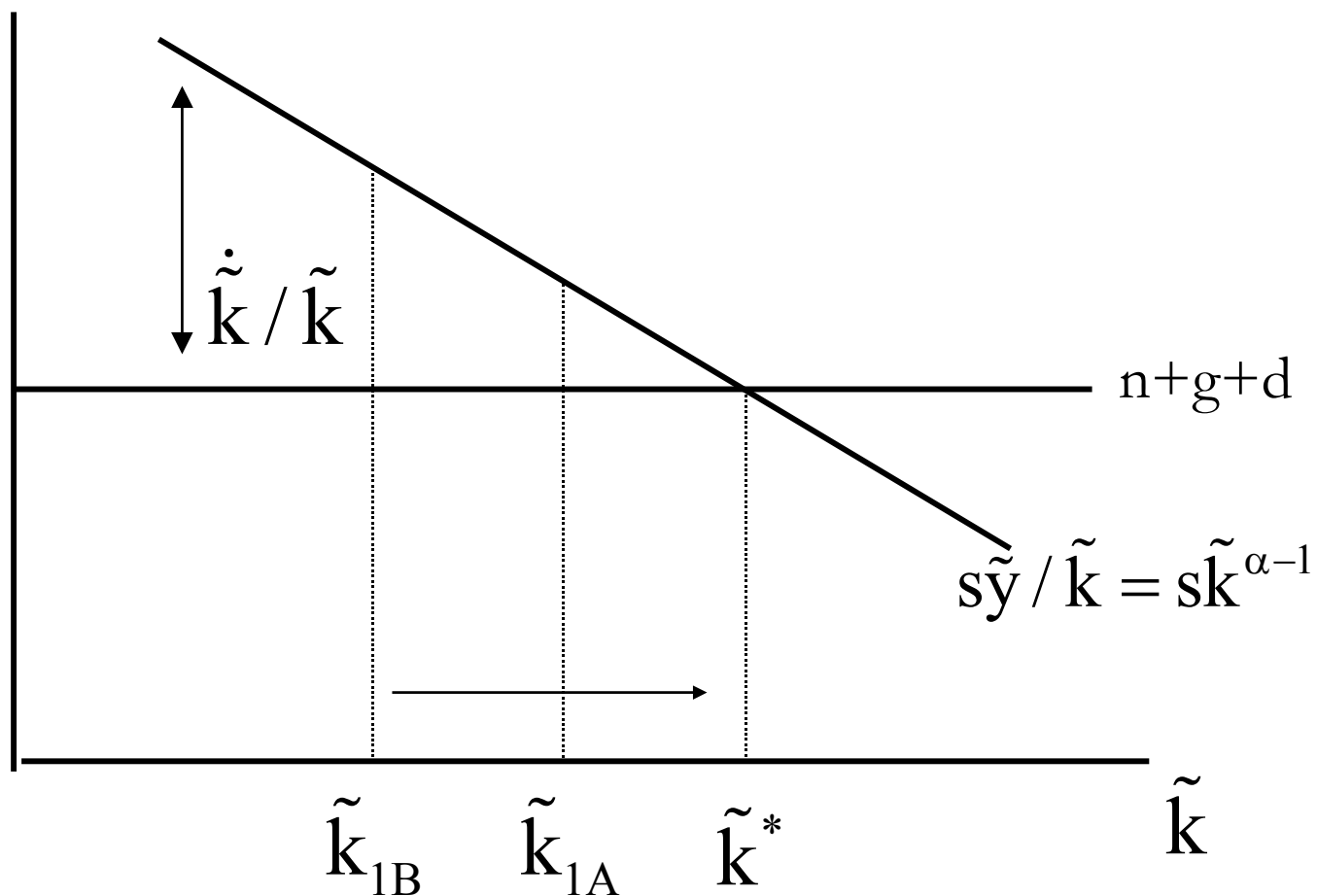
a rise in the saving rate



the rate of convergence

- Suppose one economy starts with a lower initial level of capital per technology-adjusted worker than another but with the same steady-state level.
- The capital accumulation equation says that the country with the lower initial level should accumulate capital faster and so the output per technology-adjusted worker gap between the two countries will narrow over time as both economies approach the same steady-state.
- Therefore, the Solow model predicts that ‘Among countries with the same steady-state, poor countries should grow faster on average than rich countries’.

the convergence hypothesis



the augmented Solow model

- Lucas (1988) argues that the Solow model should be extended to include human capital.
- Suppose the production function is
(2.14) $Y = K^{\alpha} (AH)^{1-\alpha}$
- People accumulate human capital by spending time learning new skills instead of working. Let $(1-u)$ denote the fraction of time devoted to learning and L the total amount of raw labour used in production.
- Unskilled labour learning skills for time $(1-u)$ generates skilled labour H :
(2.15) $H = e^{\psi(1-u)} L$

human capital

- Notice that if $(1-u)=0$, then $H=L$, that is, all labour is unskilled. An increase in $(1-u)$ leads to an increase in the effective units of skilled labour H :

$$(2.16) \quad \frac{d \log H}{d(1-u)} = \psi$$

- This implies that a small increase in $(1-u)$ raises H by the percentage ψ .
- If physical capital is accumulated as in the standard Solow model then output per worker equals

$$(2.17) \quad y^*(t) = \left(\frac{s_K}{n + d + g} \right)^{\alpha/(1-\alpha)} hA(t)$$

implications for relative incomes

- The extended Solow model suggests that some countries are richer than others because they have high investment rates in physical capital, spend a large fraction of time on education, have low population growth rates and high levels of technology.

- If we define relative national income as

$$(2.18) \quad \hat{y}^* = y^* / y_{US}^*$$

- then relative incomes are given by

$$(2.19) \quad \hat{y}^* = \left(\frac{\hat{s}_K}{\hat{n} + \hat{g} + \hat{d}} \right)^{\alpha/(1-\alpha)} \hat{h}\hat{A}$$

- Notice that this model does not explain how $(1-u)$ is chosen (i.e. it is treated as being exogenous).

broad capital

- In the Solow model, firms are able to capture all of the returns to investment.
- However, it seems reasonable that there might be externalities in capital formation so that the social return might be higher than the private rate of return.
- These externalities could arise because workers move between firms taking their knowledge of the production process with them (learning by doing).
- In an extreme case this might lead to there being constant returns to capital.

the AK model

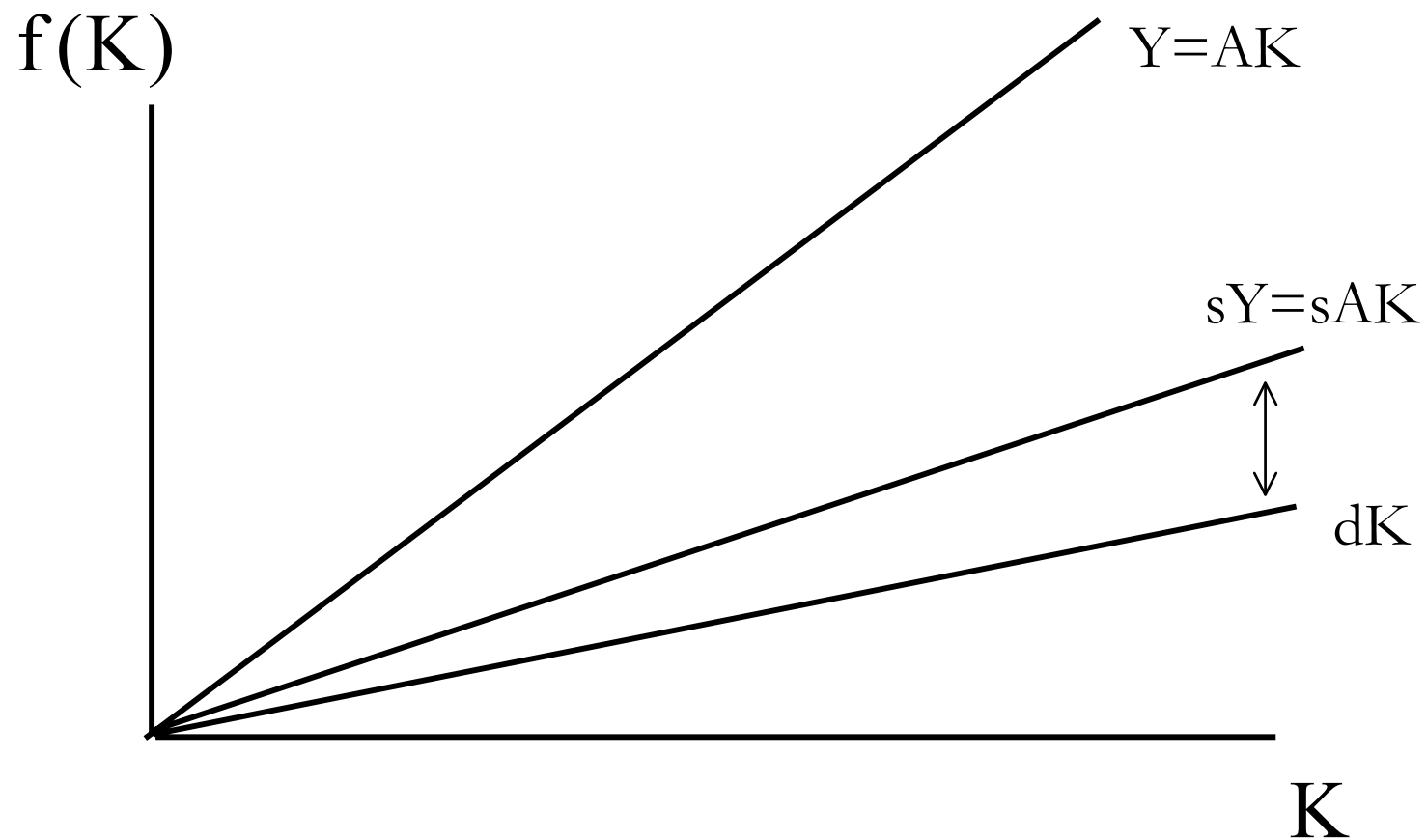
- One very simple model that allows for endogenous growth is the AK model. It has the following production function:

$$(2.20) \quad Y = AK$$

- Capital is accumulated from saving such that gross investment is $I = sY = sAK$
- Capital depreciates at a constant proportional rate, d .
- Consequently capital grows at the following rate:

$$(2.21) \quad \dot{K} = sY - dK$$

the AK model



growth in the AK model

- If we re-write the capital accumulation equation by dividing both sides by K

$$(2.22) \quad \frac{\dot{K}}{K} = s \frac{Y}{K} - d$$

- And we know from the production function that $Y/K=A$

$$(2.23) \quad \frac{\dot{K}}{K} = sA - d$$

- Taking logs and derivatives of the production function we see that the growth rate of output is equal to the growth rate of capital and therefore

$$(2.24) \quad g_Y = \frac{\dot{Y}}{Y} = sA - d$$

AK model implications

- The growth rate of an AK economy is an increasing function of the saving rate, so a government policy to raise the saving rate will raise the growth rate.
- The growth rate of an AK economy does not depend upon its initial capital stock, so there is no convergence between economies with different initial capital stocks even if they have the same saving rates, levels of technology and depreciation rates.
- Technological progress and population growth are not necessary to generate per capita growth.

the Solow model and beyond

- The Solow model (both with, and without, technical progress) model has two main predictions:
 - For countries with the same steady-state, poor countries should grow faster than rich ones.
 - An increase in investment raises the growth rate temporarily as the economy moves to a new steady-state. But once the new higher steady-state level of income is reached, the growth rate returns to its previous level.
- This is also true of the augmented Solow model with human capital presented here.
- However, the AK model yields the opposite predictions – there is no convergence, and policy changes can have permanent effects.