Supplementary Material for

Increasing Interdependence of Multivariate Distributions

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This document contains supplementary material for the proof of Proposition 7 in Appendix D.

1 Details of the derivations of equations (27) and (28)

Given $\delta \equiv g - f$ satisfying top-to-bottom symmetry, we want to decompose δ into a sum of elementary transformations (ETs). On the hypercube $L = \{0, 1\}^4$, there are 24 possible ETs of the form defined in (4) in the text, where in principle the sign of each ET can be either positive or negative. Given our restriction to distributions satisfying top-to-bottom symmetry, these 24 can be grouped into 12 distinct pairs. In Appendix D, with reference to Figure 2 there, we defined the two ETs involving the nodes a, b_i, b_j, and c_{ij} (there are two such ETs because of top-to-bottom symmetry) to have size $\beta_{ij} = \beta_{ji}$. Similarly, we defined the two ETs involving the nodes b_i, c_{ik}, c_{il}, b_j (once again, there are two because of top-to-bottom symmetry) to have size $\alpha_{ij} = \alpha_{ji}$. There are 6 distinct values of β_{ij} and 6 distinct values of α_{ij}. We now verify that for the 24 ETs so defined to sum to δ, it is necessary and sufficient that β_{ij} and α_{ij} satisfy equations (27) in Appendix D.

To do so, we proceed in two steps. In Step 1, we compute, for each node in $L = \{0, 1\}^4$, the net effect of the 6 distinct pairs of ETs whose sizes are represented by $\{\beta_{ij}\}_{i<j}$; in Step 2, we compute, for each node, the net effect of the 6 distinct pairs of ETs whose sizes are represented by $\{\alpha_{ij}\}_{i<j}$. We then show that, after all 12 pairs of ETs have been performed, i.e. after both Step 1 and Step 2, the overall change in the probability assigned to each node in $L = \{0, 1\}^4$ equals the value of δ at that node as defined in Figure 2.

To track the effect of the ETs on each node in $L = \{0, 1\}^4$, it is helpful to label the nodes of $L = \{0, 1\}^4$ as shown in Figure S1 below. For any node (a_1, a_2, a_3, a_4), we use the same numeral to label node $(1 - a_1, 1 - a_2, 1 - a_3, 1 - a_4)$ as to label (a_1, a_2, a_3, a_4), since any distribution satisfying top-to-bottom symmetry assigns the same probability to these two nodes. Thus, there are only 8 distinct labels in Figure S1.

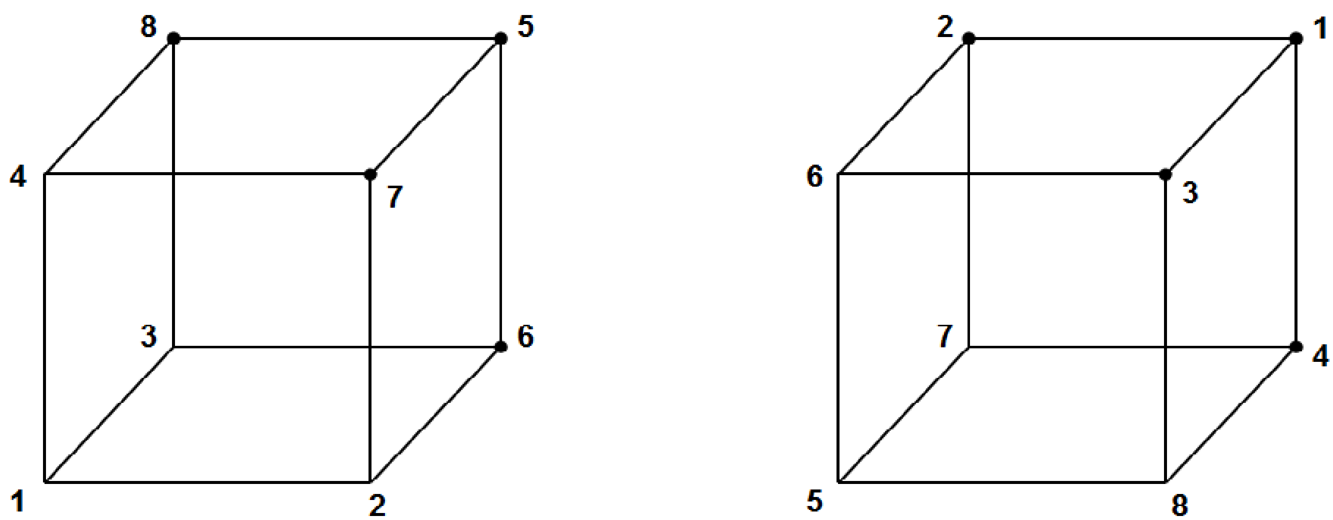

Figure S1: Labels for the nodes of $L = \{0,1\}^4$, given top-to-bottom symmetry. In the left cube, node 1 corresponds to $(0,0,0,0)$; in the right cube, node 1 corresponds to $(1,1,1,1)$.

In Table S1 below, the first column lists the labels for the 8 distinct pairs of nodes in Figure S1. The second column lists the values of δ (from Figure 2 in Appendix D) corresponding to each of the labels in Figure S1. The third column lists, for each of the 8 pairs of nodes in Figure S1, the net effect of Step 1 as described above, and the fourth column lists the net effect of both Step 1 and Step 2. The condition that the 6 pairs of ETs of size $\{\beta_{ij}\}_{i<j}$, coupled with the 6 pairs of ETs of size $\{\alpha_{ij}\}_{i<j}$, sum to δ corresponds to the condition that the fourth column of Table S1 matches the second column of Table S1. In particular, the equality of the first entries in these two columns corresponds to the first condition in (27), the equality of the second through fifth entries in these two columns corresponds to the second condition in (27), and the equality of the sixth through eighth entries corresponds to the third condition in (27).

Conditions (27) can be rearranged to yield conditions (28) as follows. Add the following three equations from (27),

$$\begin{aligned} b_i + \beta_{ij} + \beta_{ik} + \beta_{il} &= \alpha_{ij} + \alpha_{ik} + \alpha_{il} \\ b_j + \beta_{ij} + \beta_{jk} + \beta_{jl} &= \alpha_{ij} + \alpha_{jk} + \alpha_{jl} \\ c_{ij} - \beta_{ij} - \beta_{kl} &= -\alpha_{ik} - \alpha_{il} - \alpha_{jk} - \alpha_{jl}, \end{aligned}$$

Label in Figure S1	Value of δ (see Fig. 2 in Apx. D)	Net effect of Step 1	Net effect of Steps 1 and 2
1	a	$\sum_{i<j} \beta_{ij}$	$\sum_{i<j} \beta_{ij}$
2	b_1	$-\beta_{12}-\beta_{13}-\beta_{14}$	$\alpha_{12}+\alpha_{13}+\alpha_{14}-\beta_{12}-\beta_{13}-\beta_{14}$
3	b_2	$-\beta_{12}-\beta_{23}-\beta_{24}$	$\alpha_{12}+\alpha_{23}+\alpha_{24}-\beta_{12}-\beta_{23}-\beta_{24}$
4	b_3	$-\beta_{13}-\beta_{23}-\beta_{34}$	$\alpha_{13}+\alpha_{23}+\alpha_{34}-\beta_{13}-\beta_{23}-\beta_{34}$
5	b_4	$-\beta_{14}-\beta_{24}-\beta_{34}$	$\alpha_{14}+\alpha_{24}+\alpha_{34}-\beta_{14}-\beta_{24}-\beta_{34}$
6	$c_{12}=c_{34}$	$\beta_{12}+\beta_{34}$	$\beta_{12}+\beta_{34}-\alpha_{13}-\alpha_{14}-\alpha_{23}-\alpha_{24}$
7	$c_{13}=c_{24}$	$\beta_{13}+\beta_{24}$	$\beta_{13}+\beta_{24}-\alpha_{12}-\alpha_{14}-\alpha_{23}-\alpha_{34}$
8	$c_{23}=c_{14}$	$\beta_{23}+\beta_{14}$	$\beta_{23}+\beta_{14}-\alpha_{12}-\alpha_{13}-\alpha_{24}-\alpha_{34}$

Table S1

cancel terms, and simplify using the first condition in (27) to get

$$a + b_i + b_j + c_{ij} = 2\alpha_{ij} + 2\beta_{kl},$$

which is the third condition in (28). (The first two conditions in (28) match the first two in (27).)

2 Proof that the values of β_{kl} and α_{ij} defined in equations (29) satisfy equations (28)

Given the values of $\{\beta_{kl}\}_{k<l}$ defined in (29), we have

$$\sum_{i<j} \beta_{ij} = \frac{a}{(4a + \sum_{h=1}^{4} b_h)} \left[6a + 3\sum_{h=1}^{4} b_h + \sum_{i<j} c_{ij} \right]. \tag{1}$$

Now, doubling equation (26) and recalling that $c_{ij} = c_{kl}$ because of top-to-bottom symmetry yields

$$2a + 2\sum_{h=1}^{4} b_h + \sum_{i<j} c_{ij} = 0. \tag{2}$$

Using (2) to cancel terms on the right-hand side of (1) yields

$$\sum_{i<j} \beta_{ij} = \frac{a}{(4a + \sum_{h=1}^{4} b_h)} \left[4a + \sum_{h=1}^{4} b_h \right] = a,$$

so the first condition in (28) is satisfied.

Now substitute the values of $\{\beta_{kl}\}_{k<l}$ defined in (29) into the left-hand side of the second condition in (28) to get

$$
\begin{aligned}
b_i + \beta_{ij} &+ \beta_{ik} + \beta_{il} \\
&= b_i + \frac{a}{(4a + \sum_{h=1}^4 b_h)} \left[(a + b_k + b_l + c_{kl}) + (a + b_j + b_l + c_{jl}) + (a + b_j + b_k + c_{jk})\right] \\
&= \frac{1}{(4a + \sum_{h=1}^4 b_h)} \left[4ab_i + b_i(\sum_{h=1}^4 b_h) + 3a^2 + 2a(b_j + b_k + b_l) + a(c_{kl} + c_{jl} + c_{jk})\right] \\
&= \frac{1}{(4a + \sum_{h=1}^4 b_h)} \left[2ab_i + 3a^2 + (2a + b_i)(\sum_{h=1}^4 b_h) + a(c_{kl} + c_{jl} + c_{jk})\right]. \qquad (3)
\end{aligned}
$$

Substitute the values of $\{\alpha_{ij}\}_{i<j}$ defined in (29) into the right-hand side of the second condition in (28) to get

$$
\begin{aligned}
\alpha_{ij} + \alpha_{ik} + \alpha_{il} &= \frac{(2a + \sum_{h=1}^4 b_h)}{2(4a + \sum_{h=1}^4 b_h)} \left[3a + 3b_i + b_j + b_k + b_l + c_{ij} + c_{ik} + c_{il}\right] \\
&= \frac{1}{(4a + \sum_{h=1}^4 b_h)} \Bigg[3a^2 + 2ab_i + a(\sum_{h=1}^4 b_h) + a(c_{ij} + c_{ik} + c_{il}) + (a + b_i)(\sum_{h=1}^4 b_h) \\
&\quad + \frac{1}{2}(\sum_{h=1}^4 b_h)(a + b_i + b_j + b_k + b_l + c_{ij} + c_{ik} + c_{il})\Bigg] \\
&= \frac{1}{(4a + \sum_{h=1}^4 b_h)} \left[3a^2 + (2a + b_i)(\sum_{h=1}^4 b_h) + 2ab_i + a(c_{ij} + c_{ik} + c_{il})\right], \qquad (4)
\end{aligned}
$$

where we use equation (26) to derive the final equality. Since top-to-bottom symmetry implies that $c_{ij} = c_{kl}$, $c_{ik} = c_{jl}$, and $c_{il} = c_{jk}$, the right-hand sides of (3) and (4) are equal, and hence the second condition in (28) is satisfied.

Finally, given the values of $\{\beta_{kl}\}_{k<l}$ and $\{\alpha_{ij}\}_{i<j}$ defined in (29), we have

$$
\begin{aligned}
2\alpha_{ij} + 2\beta_{kl} &= \frac{(2a + \sum_{h=1}^4 b_h)(a + b_i + b_j + c_{ij})}{4a + \sum_{h=1}^4 b_h} + \frac{2a(a + b_i + b_j + c_{ij})}{4a + \sum_{h=1}^4 b_h} \\
&= \frac{(4a + \sum_{h=1}^4 b_h)(a + b_i + b_j + c_{ij})}{4a + \sum_{h=1}^4 b_h} = a + b_i + b_j + c_{ij},
\end{aligned}
$$

so the third condition in (28) is satisfied.