INSTANT EXIT FROM THE WAR OF ATTRITION

DAVID P. MYATT
NUFFIELD COLLEGE, UNIVERSITY OF OXFORD

Abstract. This paper takes a new look at the classic concession game. It argues that exit from an asymmetric war of attrition is likely to be *instant*. Selecting a unique equilibrium using a "craziness" perturbation device, it finds a notion of *stochastic strength* determines the outcome, with a stochastically weaker player giving up immediately.

"It's not true that nice guys finish last. Nice
guys are winners before the game even starts."
Addison Walker

1. Introduction

When should a player exit from a war of attrition? If the player is perceived to be weaker *a priori*, then she should exit immediately. Can this be? The standard model of the war of attrition suggests that players fight for some period of time before exiting. Indeed, in the symmetric case, the highest valuation player wins the war. This chapter argues that in fact the two-player war of attrition should reach a prompt conclusion.

1.1. **The War of Attrition.** The *war of attrition* is a common modelling device in economics. Consider the simplest two-player version. It formalises a *concession game*. The participants incur costs of delay prior to a concession by one of the players. Following any eventual concession, the remaining player receives a prize.[1] The model suffers from some flaws, however. In particular, the model exhibits multiple equilibria. The textbook examples — for instance, Tirole (1988) and Fudenberg and Tirole (1991) — focus on the symmetric case. Naturally, any device that picks a unique equilibrium will predict symmetry given a symmetric specification. But what of the asymmetric case?

Date: This version: June 1999.

[1]Initial investigation of this scenario in a biological context began with Maynard Smith (1974) and Bishop, Cannings and Maynard Smith (1978). The dragon slaying and ballroom dancing interests of Bliss and Nalebuff (1984) introduce the model to the economic literature. In a recent paper, Bulow and Klemperer (1997) extend the war of attrition to "many player" case, where the number of participants exceeds the number of prizes by more than one.

Drawing upon earlier work, this paper argues that exit from the asymmetric war of attrition is likely to be *instant*. The model of Fudenberg and Tirole (1986) is reformulated, with the introduction of a "craziness" perturbation. This enables a characterisation of the unique equilibrium for vanishing "craziness". When one player is "stochastically stronger" at the outset, the device posited here selects an equilibrium in which the weaker player exits *immediately*.

1.2. **The Complete Information War of Attrition.** Suppose that two players fight for a single prize, where the valuations and fighting costs are commonly known. This is a complete information war of attrition. At each point in time, a player can choose to remain in the fight or (irreversibly) exit. If Player 1 is the first to exit at time t, then Player 2 is the solo recipient of a prize, valued at v_2. Both players pay fighting costs, totalling $c_1 t$ and $c_2 t$ respectively — it is without loss of generality to normalise $c_1 = c_2 = 1$, since (for positive fighting costs and prize valuations) it is only the ratio v_i/c_i that is of importance.

There are two stationary pure strategy equilibria to this game. They involve *instant exit* — one player exits immediately at the start of the game, and the other fights forever. The latter player wins immediately with zero fighting costs. Alternatively, there is a stationary mixed strategy equilibrium, where players exit according to a constant hazard Poisson process. The hazard rates satisfy $\lambda_1 = 1/v_2$ and $\lambda_2 = 1/v_2$. Indeed, in the symmetric case, this yields a symmetric equilibrium with $\lambda_1 = \lambda_2 = 1/v$. At each point in time, a player considers waiting a fraction longer prior to exit. Costs are incurred at rate 1, whereas the expected gain is $\lambda_j v_i$. The exit rates of players are chosen so that $\lambda_j v_i = 1$ and hence a player is indifferent between exit and continuation at each point in time.

The comparative statics of the mixed equilibrium make it unattractive. Notice that if $v_2 > v_1$, then $\lambda_2 > \lambda_1$ — the stronger player, with the higher valuation, exits at a *faster* rate. Indeed, as $v_1 \to 0$ for fixed v_2, the weaker player wins with probability approaching 1. In the asymmetric case, the equilibrium in which the weaker player exits immediately may seem more reasonable.

The desire to select among a multiplicity motivated Kornhauser, Rubinstein and Wilson (1989). They consider a discrete time, complete information war of attrition. Drawing upon the suggestions of Selten (1975), Myerson (1978), Kohlberg and Mertens (1986) and others, they choose to perturb this game in particular way. With some small probability, a player is restricted to fight forever. In the asymmetric case ($v_2 > v_1$), this selects an equilibrium in which the weaker player (v_1) exits *immediately*.

1.3. **The Incomplete Information War of Attrition.** This paper aims to incorporate this approach into the incomplete information war of attrition. With incomplete information, a player's valuation is unknown to her opponent. Bayesian equilibrium strategies are increasing in a player's valuation — stronger players fight for longer. As Riley (1980) notes, there are a range of equilibria. Essentially, first order conditions determine a pair of differential equations that must be satisfied by any solution. A full solution requires two boundary conditions. Equilibrium arguments determine only one of these, and hence there are a range of possible equilibria.

Fudenberg and Tirole (1986, FT) provide a solution to this problem. They study a model in which fighting costs are uncertain. The support of these fighting costs extends below zero, and hence there is positive probability that a player has a dominant strategy to remain in the game forever. They develop an asymptotic boundary condition that must be satisfied as $t \to \infty$, tying down a unique solution. The regular boundary condition, however, is tied at a valuation of zero.[2] It follows that initial boundary conditions do not change as the parameters move, and "instant exit" comparative statics are not available.

1.4. **Instant Exit from the War of Attrition.** This paper argues that instant exit by a weaker player is a more reasonable prediction in the incomplete information war of attrition. The analysis considers a model with unknown valuations. These are bounded away from zero, so that the Lipschitz continuity problem addressed by Fudenberg and Tirole (1986) is not an issue. A multiplicity remains, however. The optimality restrictions imposed by a Bayesian Nash equilibrium yield a pair of differential equations to be satisfied by players' stopping time functions. A range of boundary conditions is available, however, necessitating an equilibrium selection device.

To tie down a unique equilibrium, the possibility of "craziness" is introduced, and is a direct reinterpretation of the FT (1986) negative fighting costs. With some small probability, a player is restricted to fight forever. This ties down the boundary conditions for the stopping time functions, and yields a unique equilibrium.

The basic model is thus a reformulation of the FT (1986) device, moving the incomplete information away from fighting costs and onto valuations. Crucially, however, the support of valuations is bounded away from zero, which yields to the selection of initial conditions. This enables a new set of comparative statics to emerge. These are as follows.

First, if the valuation distribution of one player first order stochastically dominates that of her opponent, then the weaker player exits *immediately* with positive probability. Secondly, a notion of "stochastic strength" is introduced. This couples first order stochastic dominance with

[2] In fact, there model is equivalent to one in which there is positive probability that a player has a fighting cost of $+\infty$. Such a player exits immediately at $t = 0$. This results in an equilibrium in which no finite-valued player exits immediately. Furthermore, it results in a lack of Lipschitz continuity at this point — this is the source of multiple equilibria in their model.

an asymptotic hazard rate dominance condition. If the "craziness" perturbations are allowed to vanish, then the stochastically weaker player exits immediately with probability approaching 1. Viewing this as a selection device for the unperturbed war of attrition, the analysis predicts instant exit for a stochastically weaker player. The war of attrition ends immediately.

1.5. **Guide to the Paper.** Section 2 describes the war of attrition model, including the perturbation device, and defines a notion of "stochastic strength". The analysis is split into two sections. Section 3 deduces the basic properties of any equilibrium. The results in that section will be familiar to readers of earlier work. The main contribution of this paper is the analysis of Section 4. Concluding remarks are made in Section 5.

2. The Model

The model is a straightforward two player war of attrition augmented by a perturbation device.

2.1. **The Standard War of Attrition.** Begin with a standard war of attrition. There are two players, $i \in \{1, 2\}$, each choosing a stopping time $t_i \in [0, +\infty) \cup \{\infty\}$, which may be revised at any time $t \leq t_i$. Each player has a fighting cost c_i, which is normalised to $c_i = 1$ without loss of generality. The players have valuation u_i for a prize, where $u_i \in (\underline{u}_i, \infty)$. Realised payoffs are:

$$\pi_i(t_i, t_j) = u_i \mathbb{I}\{t_i > t_j\} - \min\{t_i, t_j\}$$

where $\mathbb{I}$ is the indicator function. Notice that if two players exit simultaneously ($t_i = t_j$) then the prize is lost. It is assumed that $\underline{u}_2 \geq \underline{u}_1$.

A player's valuation is unknown to her opponent — this is a game of incomplete information. The payoff u_i is drawn from the distribution $F_i(u)$ with density $f_i(u)$. Standard regularity conditions are assumed. In particular, $f_i(u)$ is bounded above and continuously differentiable, the associated hazard $f_i(u)/(1 - F_i(u))$ is increasing, and first and second moments exist.

2.2. **The Perturbed War of Attrition.** A perturbation is introduced. With probability $\xi_i > 0$, a player is "crazy" and fights forever irrespective of valuation — such a player is restricted to play $t_i = \infty$. Informally, this will be regarded as a payoff of ∞, thus extending the valuation support to $u_i \in (\underline{u}_i, \infty) \cup \{\infty\}$. The density over valuations becomes $g_i(u) = (1 - \xi_i) f_i(u)$ with associated cumulative distribution function $G_i(u) = (1 - \xi_i) F_i(u)$.

This device is equivalent to that introduced by FT (1986). In their model, valuations are fixed, with the fighting cost of a player unknown to her opponent. Justifying their assumptions with an application to exit from a declining industry, the authors extend the support of fighting costs to below zero. This implies that, with positive probability, a player has a strictly dominant strategy to fight forever. The model presented here reformulates this idea, and replaces the incomplete information over costs with incomplete information over valuations.

There is a key difference between this model and its predecessor, however. A direct transformation of the FT (1986) model yields a valuation support that extends below zero. Hence there is also a positive probability that a player will have a dominant strategy to exit at zero. This ties down the boundary conditions for stopping time functions at this point. A multiplicity still arises, however. The appropriate first order conditions are not Lipschitz continuous at zero, and hence there are multiple solutions to the associated differential equations. The "craziness" device yields a terminal boundary condition at $t \to \infty$, and ties down a unique equilibrium.

Here, however, the valuation supports are bounded away from zero. It is thus possible for one of the players to exit at time $t = 0$ with positive probability, despite possessing a positive valuation. Indeed, this initial condition is also determined by the terminal boundary condition. Of course, this starting point varies as the "craziness" perturbation is allowed to vanish, and hence enables the equilibrium selection.

2.3. **Stochastic Strength.** This paper considers the outcome from the perturbed war of attrition when Player 2 is in some sense "stronger" than Player 1. This comparison is aided by the following definitions.

Definition 1. *Player 2* first order stochastically dominates *Player 1 in the perturbed game if $G_2(u) < G_1(u)$ for all $u > \underline{u}_1$. This will be denote $G_2 \succ_{FSD} G_1$. Similarly, in the unperturbed game, $F_2 \succ_{FSD} F_1$ if this inequality holds for $\{F_i\}$.*

Definition 2. *Player 2* hazard rate dominates *Player 1 in the perturbed game if:*

$$\frac{g_1(u)}{1 - G_1(u)} > \frac{g_2(u)}{1 - G_2(u)} \quad \text{for all} \quad u > \underline{u}_2$$

This will be denoted $G_2 \succ_{HRD} G_1$. Similarly, $F_2 \succ_{HRD} F_1$ if this holds for $\{F_i\}$.

Definition 3. *Player 2* asymptotically hazard rate dominates *Player 1 in the unperturbed game if her hazard rate is boundedly lower for high valuations. Specifically, for some $\overline{u} > \underline{u}_2$ and some $\hat{\lambda} > 0$:*

$$\frac{f_1(u)}{1 - F_1(u)} - \frac{f_2(u)}{1 - F_2(u)} \geq \hat{\lambda} \quad \text{for all} \quad u > \overline{u}$$

This will be denoted $F_2 \succ_{AHRD} F_1$.

Combining Definitions 1 and 3 yields a definition of stochastic strength.

Definition 4. *Player 2 is* stochastically stronger *if $G_2 \succ_{FSD} G_1$ and $F_2 \succ_{AHRD} F_1$.*

3. Preliminary Analysis

This section establishes the basic properties of any equilibrium in the perturbed war of attrition. All of the results presented here are mild modifications of those presented by FT (1986) and readers of that paper may wish to move forward to the main analysis of Section 4. Their results are modified here to apply to the case of unknown valuations.

The analysis begins by first showing that stopping times in the war of attrition are increasing in a player's valuation. The next step is to show that they are continuous, strictly increasing and differentiable. Finally, the asymptotic behaviour of stopping rules is considered.

A strategy profile in the war of attrition is a pair of stopping rules $t_1(u)$ and $t_2(u)$. These carry a player's type realisation to the extended real line:

$$t_i(u) : (\underline{u}_i, \infty) \mapsto \mathbb{R}^+ \cup \{\infty\}$$

For a Bayesian Nash equilibrium profile, these stopping rules need to be mutually optimal, and hence must satisfy the following functional equations:

$$t_i(u_i) \in \arg\max_t \mathrm{E}_{u_j}\left[u_i \mathbb{I}\{t > t_j(u_j)\} - \min\{t, t_j(u_j)\}\right] \quad \text{for} \quad i, j \in \{1,2\}\ i \neq j$$

where $\mathbb{I}$ is the indicator function. Recall that $\xi_i > 0$ for $i \in \{1,2\}$ and hence all information sets are reached with positive probability. It follows that a Bayesian Nash equilibrium profile will yield a Kreps-Wilson (1982) sequential equilibrium.

The first step is to establish the monotonicity of stopping rules.

Lemma 1. *Stopping times are weakly increasing in a player's valuation.*

Proof. Suppose that $t_H = t_1(u_L) > t_1(u_H) = t_L$ for $u_L < u_H$. This implies that:

$$u_L \mathrm{E}_{u_2}\left[\mathbb{I}\{t_H > t_2(u_2)\} - \mathbb{I}\{t_L > t_2(u_2)\}\right] \geq \mathrm{E}_{u_2}\left[\min\{t_H, t_2(u_2)\} - \min\{t_L, t_2(u_2)\}\right] \tag{1}$$

Observe that the right hand side of this inequality is strictly positive:

$$\mathrm{E}_{u_2}\left[\min\{t_H, t_2(u_2)\} - \min\{t_L, t_2(u_2)\}\right] \geq \xi_2(t_H - t_L) > 0$$

This follows since there is positive probability ($\xi_2 > 0$) that Player 2 remains in the game forever. Combining this with the optimality condition, the left hand side of Equation (1) must be strictly positive, so that:

$$\mathrm{E}_{u_2}\left[\mathbb{I}\{t_H > t_2(u_2)\} - \mathbb{I}\{t_L > t_2(u_2)\}\right] > 0$$

Hence, from $u_H > u_L$, conclude that:

$$u_H \mathrm{E}_{u_2}\left[\mathbb{I}\{t_H > t_2(u_2)\} - \mathbb{I}\{t_L > t_2(u_2)\}\right] > u_L \mathrm{E}_{u_2}\left[\mathbb{I}\{t_H > t_2(u_2)\} - \mathbb{I}\{t_L > t_2(u_2)\}\right]$$

This implies that a player with a valuation u_H strictly prefers stopping time t_H to t_L. Conclude that stopping times must be weakly increasing in valuations. Perform a symmetric operation to establish the same result for $t_2(u)$. □

Since stopping times are weakly monotonic, it is immediate that they must be continuous almost everywhere. In fact, as the following lemma reveals, they are continuous everywhere.

Lemma 2. *Stopping times are continuous in players' valuations.*

Proof. Take a valuation u_2^* such that $\lim_{u \uparrow u_2^*} t_2(u) = t_L < t_H = \lim_{u \downarrow u_2^*} t_2(u)$, yielding a discontinuity at u_2^*. These limits are well defined, since $t_2(u_2)$ is weakly increasing. Player 1 will not quit in the interval $(t_L, t_H]$, since there is no chance of winning in this interval, and hence fighting costs would be saved by exiting at t_L. For arbitrarily small ϵ, there is some $u_2 \geq u_2^*$ satisfying $t_H \leq t_2(u_2) \leq t_H + \epsilon$. Such a player u_2 pays a fighting cost of at least $t_H - t_L$ to arrive at this point. This is a non-negligible fighting cost, and hence there must be a non-negligible probability of Player 1 exiting just after t_H. Taking the limit as $\epsilon \downarrow 0$ reveals that there must be an atom at t_H in player 1's exit strategy. But this is a contradiction. □

The next step establishes strict monotonicity. Notice that this applies only for valuations mapping to $t > 0$, and hence the stopping function may be constant for valuations mapping to $t = 0$ or $t = \infty$.

Lemma 3. *Stopping times are strictly increasing in players' valuations, for $t > 0$.*

Proof. Suppose not, so that $t_1(u_L) = t_1(u_H) = t > 0$ for $u_L < u_H$. Then a positive mass $G_1(u_H) - G_1(u_L)$ of players exits at time t. Thus for small ϵ, dropping out by Player 2 in the interval $(t - \epsilon, t]$ is dominated by staying in until just after t. But this implies a discontinuity in $t_2(u)$. This proof assumes that drop out times for Player 2 extend as far as t, so that $\lim_{u \to \infty} t_2(u) \geq t$. But this cannot be, since otherwise Player 1 would have exited before t. □

Lemma 3 is of relevance only when a player does in fact fight for a positive length of time with some probability. The next lemma establishes that this occurs.

Lemma 4. *Both players fight for some positive length of time with positive probability.*

Proof. Suppose that Player 1 exits immediately for all valuations, so that $t_1(u) = 0$ for all $\underline{u}_1 < u < \infty$. Player 2 will not exit at $t = 0$, since by waiting for a short time she can benefit from the near-certain (probability $1-\xi_1$) exit of Player 1. She will not wait any length of time beyond $t = 0$, since any lack of exit by Player 1 will reveal craziness. It follows that she wishes to choose $\min\{t > 0\}$. Of course, this minimum does not exist, since $\{t > 0\}$ is an open set. Suppose, however, that Player 2 did wait an arbitrarily small period of time before exiting. It would no longer be optimal for Player 1 to exit at time $t = 0$ — she could wait a little longer, in anticipation of Player 2's forthcoming exit. □

The exit strategy of the lowest type player is now investigated.

Lemma 5. *The lowest valuation players exit immediately:* $\lim_{u\downarrow \underline{u}_i} t_i(u) = 0$

Proof. Suppose that $\lim_{u\downarrow \underline{u}_1} t_1(u) = t_L > 0$. It follows that Player 2 will never choose to exit in the interval $(0, t_L]$. Now, all types of Player 1 fight beyond t_L, hence incurring a fighting cost of at least t_L. There must be a non-neglible probability of Player 2 exiting just after t_L. But this implies that there is a non-neglible probability of Player 2 exiting at t_L, a contradiction. □

Since the stopping rules are strictly increasing and continuous, their inverses are well defined for $t \in (0, \infty)$. These inverses will be denoted as $v(t)$ and $w(t)$ respectively, corresponding to Players 1 and 2.

Lemma 6. *At points of differentiability, inverse stopping rules satisfy:*

$$v'(t) = \frac{1 - G_1(v(t))}{g_1(v(t))w(t)} \quad \textit{and} \quad w'(t) = \frac{1 - G_2(w(t))}{g_2(w(t))v(t)} \tag{2}$$

Proof. Consider Player 2, with valuation $w(t)$. This player considers the distribution over stopping times for Player 1. This satisfies $Pr[t_1 \leq t = G_1(v(t))]$, and hence has hazard:

$$\frac{g_1(v(t))v'(t)}{1 - G_1(v(t))}$$

Fighting costs are incurred at rate 1. The prize is $w(t)$. This yields the first order condition:

$$1 = \frac{g_1(v(t))v'(t)}{1 - G_1(v(t))} w(t)$$

Re-arrange to obtain Equations 2. □

Lemma 7. *The inverse stopping rules are differentiable for* $t > 0$.

Proof. The aim is to show the Lipschitz continuity of $G_1(v(t))$ on $(0, \infty)$. Take a compact subset C of this set with members $t_L < t_H$ and consider $G_1(v(t_H)) - G_1(v(t_L))$. Consider a Player 2 of type $w(t_L)$ waiting until t_H. The cost is bounded above by $t_H - t_L$. The probability of winning is bounded below by $G_1(v(t_H)) - G_1(v(t_L))$. It follows that:

$$G_1(v(t_H)) - G_1(v(t_L)) \leq \frac{t_H - t_L}{w(t_L)} \leq \frac{t_H - t_L}{\underline{u}_2}$$

This uses the optimality of t_L as a stopping time for $w(t_L)$. Taking absolute values, it follows that $G_1(v(t))$ is Lipschitz continuous. Next:

$$v(t_H) - v(t_L) = G_1^{-1}(G_1(v(t_H))) - G_1^{-1}(G_1(v(t_L)))$$
$$\leq \max_{v(t):t\in C} \left\{ \frac{1}{g_1(v)} \right\} \times [G_1(v(t_H)) - G_1(v(t_L))]$$

The inverse density term is bounded above. This is because the density is bounded below on a compact set, and $v(t)$ is continuous — hence mapping the compact subset of the stopping times into a compact subset of valuations. It follows that $v(t)$ is also Lipschitz continuous. Hence $v(t)$, and similarly $w(t)$, is differentiable almost everywhere. Where this derivative exists, it satisfies the appropriate first order conditions. It follows that the inverse stopping time functions are integrals of Lipschitz continuous functions of t, and hence differentiable everywhere by the Fundamental Theorem of Calculus. □

Assembling the results so far, a number of properties are available. Any equilibrium to the perturbed war of attrition must involve weakly increasing strategies (Lemma 1). For $t \in (0, \infty)$, stopping times are strictly increasing (Lemma 3). Behaviour for low valuations has been partially established (Lemma 5). Analysis of behaviour at this point is now expanded. There are two possibilities. Focus on Player 1, without loss of generality. The first possibility is that $t_1(u) > 0$ for all $u > \underline{u}_1$, so that $\lim_{t\downarrow 0} v(t) = v(0) = \underline{u}_1$. The second possibility is that $\lim_{t\downarrow 0} v(t) = v(0) > \underline{u}_1$. In this case, $t_1(u) = 0$ for all $\underline{u}_1 < u \leq v(0)$, so that a positive mass of players exits at the beginning of the game. Of course, it is impossible for a positive mass of both players to exit at the beginning of the game.

Lemma 8. *Either $v(0) = \underline{u}_1$ or $w(0) = \underline{u}_2$.*

Proof. Suppose not, so that $v(0) > \underline{u}_1$ and $w(0) > \underline{u}_2$. A positive mass of both players exit at $t = 0$. But then it cannot be optimal for any player to exit at the beginning — by waiting a little longer, they win the prize with non-neglible probability. □

It remains to consider the properties of equilibrium stopping time functions for larger valuations.

Lemma 9. *Selection ends at the same time:* $\lim_{u\to\infty} t_1(u) = \lim_{u\to\infty} t_1(u) = \overline{T}$

Proof. Suppose that $\lim_{u\to\infty} t_1(u) = \overline{T}_1 < \overline{T}_2 = \lim_{u\to\infty} t_2(u)$. For $\overline{T}_2 > t > \overline{T}_1$ Player 2 knows that she is facing a type that will fight forever, and hence should exit earlier. Hence no Player 2 will wait beyond $\overline{T}_1$. □

Lemma 10. *There is no infinite delay: $t_i(u) < \infty$ for all $u < \infty$.*

Proof. Suppose that $\lim_{t\to\overline{T}} v(t) = \overline{v} < \infty$. Now, $1/w(t)$ is decreasing in t and bounded below. It follows that:

$$\frac{d}{dt}\left(\frac{1}{w(t)}\right) = -\frac{1}{w(t)^2}w'(t) \longrightarrow 0 \quad \text{as} \quad t \to \infty$$

Recall the first order condition:

$$\frac{w'(t)}{w(t)^2} = \frac{1 - G_2(w(t))}{g_2(w(t))w(t)^2 v(t)}$$

Taking the limit of this expression, $v(t) \to \overline{v}$, a finite value. Furthermore, $1-G_2(w(t)) \geq \xi_2 > 0$. It must be the case that $(1-\xi_2)f_2(w(t))w(t)^2 \to \infty$. But this would eliminate the existence of second moments for $f_2(u)$. Conclude that there is never infinite delay. □

Corollary 1. *Allowing t to grow large:* $\lim_{t\to\overline{T}} v(t) = \lim_{t\to\overline{T}} w(t) = \infty$

This completes the basic characterisation of necessary properties for the stopping time functions. The next lemma establishes that, given an equilibrium profile, second order conditions hold.

Lemma 11. *The first order conditions yield globally optimal stopping times.*

Proof. Consider the payoff of Player 1 with valuation u_1 waiting until time t:

$$\begin{aligned} \pi_1(u_1, t) &= \mathrm{E}_{u_2}[u_1 \mathbb{I}\{t > t_2(u_2)\} - \min\{t, t_2(u_2)\}] \\ &= u_1 G_2(w(t)) - \int_{\underline{u}_2}^{w(t)} t_2(x) g_2(x)\, dx - t[1 - G_2(w(t))] \end{aligned}$$

This expression is differentiable for $t > 0$, yielding:

$$\begin{aligned} \frac{d\pi_1}{dt} &= u_1 g_2(w(t))w'(t) - t_2(w(t))g_2(w(t))w'(t) - [1 - G_2(w(t))] + t g_2(w(t))w'(t) \\ &= u_1 g_2(w(t))w'(t) - [1 - G_2(w(t))] \end{aligned}$$

It follows that:

$$\frac{d\pi_1}{dt} \gtreqless 0 \quad \Leftrightarrow \quad \log u_1 + \log w'(t) \gtreqless \log \frac{1 - G_2(w(t))}{g_2(w(t))}$$

From the (necessary) first order condition for Player 2:

$$\log w'(t) = \log \frac{1 - G_2(w(t))}{g_2(w(t))} - \log v(t)$$

The last term $-\log v(t)$ is decreasing in t. Hence if the first order condition is satisfied for $t_1(u_1)$, then:

$$\frac{d\pi_1}{dt} \gtreqless 0 \quad \Leftrightarrow \quad t \lesseqgtr t_1(u_1)$$

It follows that first order conditions yield a global optimum. □

4. Analysis

This section presents the main results of the paper. First, the existence of a unique equilibrium is established. Behaviour of the inverse stopping time functions $v(t)$ and $w(t)$ is then characterised, focusing on behaviour at $t = 0$. The perturbations are then allowed to vanish, yielding the appropriate selection results.

4.1. **Uniqueness of Equilibrium.**

Lemma 12. *Take two times* $0 < t_L < t_H < \overline{T}$*. The inverse stopping time functions satisfy:*

$$\int_{v(t_L)}^{v(t_H)} \frac{g_1(x)}{x(1-G_1(x))}\,dx = \int_{w(t_L)}^{w(t_H)} \frac{g_2(x)}{x(1-G_2(x))}\,dx \tag{3}$$

Proof. To show this, differentiate the left hand side with respect to t_H to obtain:

$$\frac{d}{dt_H}\left\{\int_{v(t_L)}^{v(t_H)} \frac{g_1(x)}{x(1-G_1(x))}\,dx\right\} = \frac{g_1(v(t_H))v'(t_H)}{v(t_H)(1-G_1(v(t_H)))} = \frac{1}{v(t_H)w(t_H)}$$

Differentiating the right hand side yields an identical expression. □

Next, take limits as $t_H \to \infty$. Taking the first of expression in Equation (3), the integral diverges as $v(t) \to \infty$ for $\xi_1 = 0$. To see this, notice that $\xi_1 = 0$ yields $G_1 = F_1$. The hazard is increasing, and hence:

$$\int_{v(t_L)}^{v(t_H)} \frac{f_1(x)}{x(1-F_1(x))}\,dx \leq \frac{f_1(v(t_L))}{1-F_1(v(t_L))}\int_{v(t_L)}^{v(t_H)} \frac{1}{x}\,dx \longrightarrow \infty \quad \text{as} \quad v(t_H) \to \infty$$

With $\xi_1 > 0$ this does *not* occur:

$$\int_{v(t_L)}^{v(t_H)} \frac{g_1(x)}{x(1-G_1(x))}\,dx \leq \frac{1-\xi_1}{v(t_L)\xi_1}\int_{v(t_L)}^{v(t_H)} f_1(x)\,dx \longrightarrow \frac{1-\xi_1}{v(t_L)\xi_1}[1-F(v(t_L))] < \infty$$

This observation ensures that any equilibrium is unique.

Proposition 1. *There is a unique equilibrium to the perturbed war of attrition.*

Proof. Corollary 1 shows that $v(t_H) \to \infty$ and $w(t_H) \to \infty$ as $t \to \overline{T}$. The integrals of Equation (3) are well-defined for these valuations, and hence:

$$\int_{v(t_L)}^{\infty} \frac{g_1(x)}{x(1-G_1(x))}\,dx = \int_{w(t_L)}^{\infty} \frac{g_2(x)}{x(1-G_2(x))}\,dx$$

Next take limits as $t_L \to 0$. The hazard rate is bounded above, and x is bounded below, since $\underline{u}_i > 0$. It follows that the following integrals are well defined:

$$\int_{v(0)}^{\infty} \frac{g_1(x)}{x(1-G_1(x))}\,dx = \int_{w(0)}^{\infty} \frac{g_2(x)}{x(1-G_2(x))}\,dx \tag{4}$$

This places a single equation restriction on $v(0)$ and $w(0)$, and this is a strictly increasing and continuous relationship. Recall from Lemma 8 that either $v(0) = \underline{u}_1$, or $w(0) = \underline{u}_2$. There are two cases to consider:

1. One possible equilibrium is for $v(0) = \underline{u}_1$. Equation (4) then determines $w(0)$. If there is such a $w(0)$ satisfying $w(0) \geq \underline{u}_2$, then this is an equilibrium. Furthermore, there cannot be another equilibrium. Raising $v(0)$ implies a higher $w(0)$, yielding $w(0) > \underline{u}_2$ and $v(0) > \underline{u}_1$, in contradiction of Lemma 8.
2. Suppose that the first possibility fails. Then $v(0)$ may be raised until $w(0) = \underline{u}_2$ satisfies Equation (4).

It remains to show that this procedure does in fact yield an equilibrum. Focus on the first case, with $v(0) = \underline{u}_1$, finding $w(0) \geq \underline{u}_2$ to satisfy Equation (4). Write:

$$H_1(v) = \int_{v(0)}^{v} \frac{g_1(x)}{x(1 - G_1(x))}\, dx \quad \text{and} \quad H_2(w) = \int_{w(0)}^{w} \frac{g_2(x)}{x(1 - G_2(x))}\, dx$$

These functions are well defined, strictly increasing and differentiable. This implicitly defines $w = w(v) = H_2^{-1}(H_1(v))$, which is again continuously differentiable. Next, recall that:

$$v'(t) = \frac{1 - G_1(v(t))}{g_1(v(t))w(t)} \quad \Rightarrow \quad t_1'(v) = \frac{g_1(v)w(v)}{1 - G_1(v)}$$

Integrate this to obtain:

$$t_1(v) = \int_{v(0)}^{v} \frac{g_1(x)w(x)}{1 - G_1(x)}\, dx$$

which yields the stopping time function $t_1(v)$, and implicitly (via $w(v)$) the stopping time function $t_2(w)$. This solution satisfies all appropriate conditions, and hence yields the unique equilibrium. □

This result corresponds directly to the uniqueness result of Fudenberg and Tirole (1986). In their paper, the authors set boundary conditions at zero. This prevents the Lipschitz continuity of the differential equations about zero, and hence leads to a multiplicity of solutions. Here the support of valuations is bounded away from zero. The selection device thus highlights *which* lower boundary conditions are used, based on the terminal condition generated by the "craziness" perturbation. This reformulation allows a new set of comparative statics, which form the main contribution of this paper and are explored in the next section.

It is also worth noting that, despite the tedium of results, the analysis is somewhat simpler than that of FT (1986). Concentrating on the relationship between v and w yields an immediate equilibrium characterisation, without resort to the FT (1986) "relative toughness" condition.

4.2. **Instant Exit.** The first step draws upon Definition 1, establishing that instant exit occurs.

Proposition 2. *If $G_2 \succ_{FSD} G_1$, then Player 1 exits at $t = 0$ with positive probability.*

Proof. Suppose not. Then $v(0) = \underline{v}_1$. Consider Equation (4). Using integration by parts:

$$\begin{aligned}
\int_{w(0)}^{\infty} \frac{g_2(x)}{x(1 - G_2(x))}\, dx &= -\int_{w(0)}^{\infty} \frac{1}{x}\, d\log[1 - G_2(x)] \\
&= -\left[\frac{\log[1 - G_2(x)]}{x}\right]_{w(0)}^{\infty} + \int_{w(0)}^{\infty} \log[1 - G_2(x)]\, d(1/x) \\
&= \frac{\log[1 - G_2(w(0))]}{w(0)} - \int_{w(0)}^{\infty} \frac{\log[1 - G_2(x)]}{x^2}\, dx
\end{aligned}$$

A symmetric expression holds for the left hand side. Note, however, that since $v(0) = \underline{v}_1$ it follows that $G_1(v(0)) = 0$ and hence $\log[1 - G_1(v(0))] = 0$. Thus:

$$\int_{v(0)}^{\infty} \frac{g_1(x)}{x(1 - G_1(x))}\, dx = -\int_{v(0)}^{\infty} \frac{\log[1 - G_1(x)]}{x^2}\, dx$$

Recall that $w(0) \geq \underline{u}_2 \geq \underline{u}_1 = v(0)$, an inequality that is in fact implied by $G_2 \succ_{FSD} G_1$. Using this observation, equate the previous expressions to obtain:

$$\underbrace{\frac{\log[1 - G_2(w(0))]}{w(0)}}_{\text{negative}} = \underbrace{\int_{w(0)}^{\infty} \frac{\log[1 - G_2(x)] - \log[1 - G_1(x)]}{x^2}\, dx}_{\text{strictly positive}} \underbrace{- \int_{v(0)}^{w(0)} \frac{\log[1 - G_1(x)]}{x^2}\, dx}_{\text{positive}}$$

This equation cannot hold, since the left hand side is weakly negative and the right hand side strictly positive. To see this, note first that $1 - G_i(x) \leq 1$ and hence $\log[1 - G_i(x)] \leq 0$. Secondly:

$$G_2(x) < G_1(x) \quad \Rightarrow \quad \log[1 - G_2(x)] > \log[1 - G_1(x)]$$

so that the first term on the right hand side is strictly positive. This contradicts the original supposition that $v(0) = \underline{v}_1$. It follows that $v(0) > \underline{v}_1$, and Player 1 exits with positive probability at $t = 0$. □

Corollary 2. *If $G_2 \succ_{HRD} G_1$, then Player 1 exits at $t = 0$ with positive probability.*

Proposition 2 establishes that instant exit occurs with *some* probability. Employing the notion of stochastic strength (Definition 4), this result can be sharpened.

Proposition 3. *Suppose that Player 2 is stochastically stronger than Player 1. Allowing $\max\{\xi_1, \xi_2\} \to 0$, and ensuring that $G_2 \succ_{FSD} G_1$ throughout this sequence, the unique equilibrium entails the instant exit of Player 1 with probability arbibtrarily close to 1.*

Hence, in the limit, the stochastically weaker player always exits immediately.

Proof. Begin by picking $\overline{u}$ such that $\overline{u} \geq \underline{u}_2 = w(0)$ and, for all $u \geq \overline{u}$:

$$\frac{f_1(u)}{1-F_1(u)} - \frac{f_2(u)}{1-F_2(u)} \geq \hat{\lambda} > 0$$

This is possible, since the stochastic strength of Player 2 implies that $F_2 \succ_{AHRD} F_1$. Re-write Equation (4) as:

$$\begin{aligned} (5) \quad & \int_{\max\{v(0),\overline{u}\}}^{\infty} \frac{1}{x}\left[\frac{g_1(x)}{x(1-G_1(x))} - \frac{g_2(x)}{x(1-G_2(x))}\right] dx \\ & \qquad = \int_{w(0)}^{\max\{v(0),\overline{u}\}} \frac{g_2(x)}{x(1-G_2(x))}\, dx - \int_{v(0)}^{\max\{v(0),\overline{u}\}} \frac{g_1(x)}{x(1-G_1(x))}\, dx \end{aligned}$$

Bound the right hand side of Equation (5) by:

$$\begin{aligned} \int_{w(0)}^{\max\{v(0),\overline{u}\}} \frac{g_2(x)}{x(1-G_2(x))}\, dx &\leq \int_{w(0)}^{\max\{v(0),\overline{u}\}} \frac{f_2(x)}{x(1-F_2(x))}\, dx \\ &\leq \frac{f_2(\max\{v(0),\overline{u}\})}{(1-F_2(\max\{v(0),\overline{u}\}))} \log\left(\frac{\max\{v(0),\overline{u}\}}{w(0)}\right) \end{aligned}$$

Attention now turns to the left hand side of Equation (5). For arbitrary $U > \max\{v(0), \overline{u}\}$, formulate the lower bound:

$$\inf_{\max\{v(0),\overline{u}\} \leq x \leq U} \left\{\frac{g_1(x)}{1-G_1(x)} - \frac{g_2(x)}{1-G_2(x)}\right\} \log\left(\frac{\log U}{\max\{v(0),\overline{u}\}}\right)$$

The infimum is taken over a compact set. It follows that for sufficiently small ξ_1 and ξ_2, the difference in hazards becomes arbitrarily close to the difference in unperturbed hazards. Hence, for $\max\{\xi_1, \xi_2\}$ sufficiently small:

$$\frac{g_1(x)}{1-G_1(x)} - \frac{g_2(x)}{1-G_2(x)} \geq \frac{\hat{\lambda}}{2}$$

Combining these inequalities, and using $G_2 \succ_{FSD} G_1$ to set $w(0) = \underline{u}_2$, obtain:

$$\frac{\hat{\lambda}}{2} \log\left(\frac{\log U}{\max\{v(0),\overline{u}\}}\right) \leq \frac{f_2(\max\{v(0),\overline{u}\})}{(1-F_2(\max\{v(0),\overline{u}\}))} \log\left(\frac{\max\{v(0),\overline{u}\}}{\underline{u}_2}\right)$$

which holds for sufficiently small $\{\xi_i\}$. For bounded $v(0)$, the left hand side of this equality is arbitrarily large for suitable choice of U, breaking the inequality. It follows that, for sufficiently small $\{\xi_i\}$, $v(0)$ must grow arbitrarily large. This implies that Player 1 exits at $t = 0$ with arbitrarily high probability for sufficiently small $\{\xi_i\}$ — the desired result. $\square$

4.3. **Instant Exit with Truncated Valuations.** Truncated valuations are also of interest: Players have identical distributions, but the valuation of one player is truncated below. In companion work, Dworak, Johnson and Myatt (1999) consider a three-player war of attrition with team effects. The truncated specification arises as a continuation game of that model, due to asymmetric rates of exit in an initial stage.

Proposition 4. *Consider the following specification. The valuations of both players are drawn from a common distribution F with lower bound $\underline{u}$. The perturbations are balanced, so that $\xi_1 = \xi_2 = \xi$. The valuation of Player 2 is then truncated below at $\underline{u}_2 > \underline{u} = \underline{u}_1$. In this model, the unique equilibrium entails $w(t) = v(t)$, satisfying $w(0) = v(0) = \underline{u}_2$. This solution is symmetric from $\underline{u}_2$ upwards, and Player 1 exits immediately if she has a valuation $\underline{u} < u \leq \underline{u}_2$.*

Proof. The proof checks that the posited solution is in fact an equilibrium. If Player 1 drops out for valuations as stated, the remaining subgame is symmetric, yielding the symmetric solution as described. Hence this is an equilibrium. □

4.4. **Illustration.** When will Proposition 3 apply? The following example presents an illustration. Suppose that the basic valuations of Players 1 and 2 are drawn from a common distribution F. The valuation of Player 2 is then multiplied by some constant $\alpha > 1$. Hence $F_1(u) = F(u)$ and $F_2(u) = F(u/\alpha)$. The perturbations are common to both players, so that $\xi_1 = \xi_2$.

This simple example is sufficient for the conditions of Proposition 3 to apply. To see this, first check the first order stochastic dominance property. Notice that:

$$F_2(u) = F(u/\alpha) < F(u) = F_1(u)$$

and hence the first condition holds for the distributions F_i. Since the craziness perturbations are equal, this also holds for G_i, so that $G_2 \succ_{FSD} G_1$. Next, check the hazard dominance property. Differentiating:

$$f_2(u) = \frac{d}{du} F\left(\frac{u}{\alpha}\right) = \frac{1}{\alpha} f\left(\frac{u}{\alpha}\right)$$

This yields the hazard:

$$\frac{f_2(u)}{1 - F_2(u)} = \frac{1}{\alpha} \frac{f(u/\alpha)}{1 - F(u/\alpha)}$$

By inspection, this is bounded away from $f(u)/1 - F(u)$ for large u, since $\alpha > 1$ and the hazard is increasing. The conditions of Proposition 3 hold, and hence Player 1 exits immediately for vanishing ξ_i.

5. Conclusion

The analysis of this paper suggests that wars of attrition should end quickly: a stochastically weaker player may exit immediately. Of course, this is a consequence of a particular equilibrium selection device. How reasonable is this?

In the absence of the "craziness" perturbation, there are many equilibria. There is no direct reason to focus on a particular one. Certainly, under a symmetric specification the symmetric equilibrium may be focal. This requires *exact* symmetry, however. With any asymmetry between the players, there is no symmetric equilibrium on which to focus. Adopting the FT (1986) perspective, the "craziness" perturbation is a reasonable one. It reflects the possibility of negative fighting costs. Fudenberg and Tirole's example is one of exit from a duopoly: A duopolist cannot cover her fixed costs, whereas a monopolist can, yielding a war of attrition to determine exit. Of course, it is perfectly reasonable to suppose that a duopolist *may* cover her fixed costs with *some* probability. The small probability of a dominant "always fight" strategy is then reasonable.

Suppose the argument of this paper is accepted. With vanishing craziness, we should see no fighting. So might explain the lack of exit in an observed war of attrition? A possible answer is the lack of *learning.* Of course, during a war of attrition, each player learns about the valuation of her opponent. This learning, however, is not *direct.* Rather, it occurs by *revelation*: The continued presence of a player allows her opponent to restrict the range of possible valuations.

What might happen with the introduction of direct learning? First notice that in the model presented here, there is great advantage in *a priori* strength; the "stochastically stronger" player wins the war at minimal cost. Consider a pre-game stage. Players may wait for the revelation of a public signal about their respective valuations. Should a player wait? The signal may yield a public posterior that switches or maintains that stochastic ordering of the valuation distributions, prior to the start of the "real" war of attrition. A player has an incentive to wait for such a signal: A favourable realisation will yield an instant win in the subsequent war.

Formal investigation of this idea is the topic of ongoing research. Motivated by the results of this paper, the objective is build a model of combined learning and revelation. Until that research is complete, however, it may well be wise to follow the advice of Alfred Hitchcock:

> "There's nothing to winning, really. That is, if you happen to be blessed with a keen eye, an agile mind, and no scruples whatsoever."

References

Bishop, D. T., C. Cannings, and J. Maynard Smith (1978): "The War of Attrition with Random Rewards," *Journal of Theoretical Biology*, 3, 377–388.

Bliss, C., and B. J. Nalebuff (1984): "Dragon-slaying and Ballroom Dancing: The Private Supply of a Public Good," *Journal of Public Economics*, 25, 1–12.

Dworak, J. A., J. P. Johnson, and D. P. Myatt (1999): "Team Play in the War of Attrition," mimeo, Nuffield College, Oxford and MIT Department of Economics.

Fudenberg, D., and J. Tirole (1986): "A Theory of Exit in Duopoly," *Econometrica*, 54(4), 943–960.

——— (1991): *Game Theory*. MIT Press, Cambridge, MA.

Klemperer, P. D., and J. Bulow (1997): "The Generalized War of Attrition," Nuffield College Discussion Paper, forthcoming AER.

Kohlberg, E., and J. F. Mertens (1986): "On the Strategic Stability of Equilibria," *Econometrica*, 54, 1003–1038.

Kornhauser, L., A. Rubinstein, and C. Wilson (1989): "Reputation and Patience in the War of Attrition," *Economica*, 56, 15–24.

Kreps, D. M., and R. Wilson (1982): "Sequential Equilibria," *Econometrica*, 50(4), 863–894.

Maynard Smith, J. (1974): "The Theory of Games and the Evolution of Animal Conflicts," *Journal of Theoretical Biology*, 47.

Myerson, R. (1978): "Refinements of the Nash Equilibrium Concept," *International Journal of Game Theory*, 4, 73–80.

Riley, J. G. (1980): "Strong Evolutionary Equilibrium and the War of Attrition," *Journal of Theoretical Biology*, 83.

Selten, R. (1975): "A Re-Examination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4, 25–55.

Tirole, J. (1988): *The Theory of Industrial Organization*. MIT Press, Cambridge MA.