SUMMARY - Weak Axiomatic Demand Theory

This paper gives a unified and simple treatment of three related questions in the demand theory of the weak axiom: (i) Is there an elementary, i.e., non-fixed point theoretic, proof of equilibrium existence when the excess demand function of an economy satisfies the weak axiom? (ii) What conditions are sufficient for a non-transitive preference to generate a continuous demand function? Note that such a demand must satisfy the weak, though not necessarily the strong, axiom. This motivates the next question. (iii) Given a function that satisfies the weak axiom, can we find a (not necessarily transitive) preference that generates it? To answer the first question, we give a proof using the separating hyperplane theorem. With the help of this result, we identify a class of non-transitive preferences which generate continuous demand functions, and within which any demand function satisfying the weak axiom can be rationalized.

KEYWORDS: weak axiom, representative agent, transitive preferences, rationalizability, demand function.

AUTHOR: John K.-H. Quah

ADDRESS: St Hugh's College, Oxford, OX2 6LE. United Kingdom

NOTE: The paper makes reference to three diagrams that are present in the hard copy of this paper, but not in the electronic version. But the diagrams are not crucial and could be quite easily imagined!

WEAK AXIOMATIC DEMAND THEORY

By John K.-H. Quah[1]

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1. Introduction

There is a rich theory of the consumer demand arising from the utility maximization hypothesis, something which every economist would have come across in their training. Economists are interested in demand of the individual consumer principally because they are interested in market demand. Unfortunately for the theory, the properties of individual demand are not always robust to aggregation; in particular, while the utility maximizing consumer must have a negative semi-definite and symmetric Slutsky matrix, the Slutsky matrix constructed from market demand need not have either property. Put another way, market demand need not behave as though it arises from a utility maximizing representative agent (equivalently, that it need not satisfy the strong axiom); indeed it may even violate a weaker property like the weak axiom.

For market demand to have some structure, additional properties, different from or in addition to utility maximization, have to be found. If one is interested in a market having a utility-maximizing representative agent, such a search is still likely to lead to disappointment, because the conditions needed for this to hold are very strong. However, if one is interested in a weaker property, in particular the weak axiom, then there might be reason for some satisfaction, because there is a fairly large set of conditions, in the context of either general or partial equilibrium models, under which aggregate demand can be shown to satisfy the weak axiom (see, for example, Hildenbrand (1983, 1995), Grandmont (1992), Marhuenda (1995), Jerison (1999) and Quah (1997a, 1997b, 1999)). These models typically involve restrictions on utility functions, or the distribution of preference, income or endowment characteristics; in at least some of these models, the assumptions appear to be plausible. In short, the theoretical justification for market demand satisfying the strong axiom is weak, while the justification for the weak axiom is considerably stronger. Empirical tests on aggregate data tend to mirror this observation: there is certainly support for the negative semi-definiteness of the Slutsky matrix, but considerably less for its symmetry (see, for example, Blundell, Pashardes and Weber (1993)).[2]

Since consumer theory is more often applied to aggregate rather than individual demand, it seems right to investigate the properties of demand that satisfy the weak, but not necessarily the strong, axiom - which is what this paper sets out to do. We raise three questions:

(i) Is there an elementary, i.e., non-fixed point theoretic, proof of equilibrium existence in the case when the excess demand of an economy satisfies the weak axiom?

(ii) What conditions are sufficient for a non-transitive preference to generate a continuous demand function? Note that such a demand must satisfy the weak, though not necessarily the strong, axiom. This motivates the next question.

(iii) Given a function that satisfies the weak axiom, can we find a (not necessarily transitive) preference that generates it?

At least the last two questions have been raised and answered before. Motivated perhaps by concerns different from those expressed here, Sonnenschein (1971) was the first to raise and answer question (ii); this was followed by a fairly substantial literature that is partially surveyed in Kim and Richter (1986). Question (iii) was raised as a conjecture by Kihlstrom, Mas-Colell, Sonnenschein and Shafer (1976), where they also required the rationalizing preference to have a number of desirable properties. Kim and Richter (1986) was the first attempt at answering the conjecture; Al-Najjar (1993) also dealt with this issue.

Our objective in raising these issues again is not to try to achieve the most general results, though as a bonus, our conclusions turn out not to be weaker than what has already been done; instead, we wish to subject these three related issues to a treatment that is at once synthetic and simple.

Non-Transitive Preferences

We identify in this paper a class of preferences which are sufficiently well behaved that it generates a continuous demand function. We also show that every demand function satisfying the weak axiom can be rationalized by a preference in this class. The essential properties of this class of preferences are the following: the preference is strongly monotone, at every commodity bundle x, the set of bundles weakly preferred to it, which we call *the preferred set of* x, is closed and convex; lastly, the *strictly* preferred set of x, i.e., the set of points strictly preferred to x is *full* in the preferred set, in the sense that while the former is contained in the latter, they both admit the same supporting prices at x.

It is also worth highlighting the properties we have omitted. The preference is not necessarily continuous since we have only asumed the closedness of the preferred sets. Secondly, the preference is not strongly convex in the sense which Kihlstrom et al required in their conjecture (and which Sonnenschein (1971) assumed in his proof of the existence of demand), i.e., if x and y are weakly preferred to z, then any non-degenerate convex combination of x and y is strictly preferred to z. In particular, the strictly preferred sets in our class of preferences need not be convex. Nevertheless, the conditions we impose on this class are sufficient to guarantee that the prices supporting the preferred sets at each commodity bundle form a well behaved correspondence. Demand can be found by searching for the commodity bundle on the budget plane with a supporting price that is collinear with the prevailing price.

Within this class of preferences, the rationalizability problem can also be settled in a straightforward way. Given a function f satisfying the weak axiom, let us assume that, at each commodity bundle, there is a non-empty set of prices at which demand equals that bundle when income is 1. Denote this set by $Q(x)$. In Section 5 of the paper, we shall show why $Q(x)$ is indeed non-empty provided f satisfies certain mild conditions; for now we just make this assumption, and also assume that $Q(x)$ is unique; hence Q is a function. With Q, we can construct the function $S(x,y) = y \cdot Q(x) - x \cdot Q(y)$, where x and y are any two commodity bundles. The function S satisfies $S(x,y) = -S(y,x)$ and, as pointed out by Shafer (1974), functions of this sort could be thought of as a generalized utility function generating a non-transitive preference: we simply say that y is weakly preferred to x if $S(x,y) \geq 0$. It is not difficult to see that S (or rather the preference it generates) generates the demand function f. Suppose that the bundle x is not equal to $f(p,1)$, i.e., the demand at price p and income 1, and satisfies $p \cdot x \leq 1$. By the weak axiom $Q(x) \cdot f(p,1) > 1$, so $S(x, f(p,1)) = [f(p,1) \cdot Q(x) - 1] + [1 - x \cdot Q(f(p,1))]$ must be strictly positive since $Q(f(p,1)) = p$. We conclude that $f(p,1)$ is strictly preferred to x; equivalently the preferred set of $f(p,1)$ lies beyond the budget set of $f(p,1)$ (see Figure 1). Since S is continuous, the preferred set of $f(p,1)$ is closed, but it will not, in general, be convex. The straightforward way to get round this problem is to expand the preferred set by convexifying and monotonizing it. It turns out that this can be done with the preferred set remaining closed, and without it intruding into the budget set. Furthermore, if Q is indeed a function, we show that the preference is also continuous; dis-continuity could only occur when Q is a true correspondence.

It should come as no surprise, given the simple procedure described, that while the preferred sets are convex, the preference is not strongly convex in the sense of Kihlstrom et al. This begs the question of whether a cleverer procedure will succeed in producing a rationalizing preference with strong convexity. We show in the conclusion (Section 6) of this paper, that the answer is - probably - no. There is a reason for this enigmatic qualification, which will be plain to the reader who reads that section. Of course, neither the rationalizing procedures of Kim and Richter (1986), nor that of Al-Najjar (1990), leads to a strongly convex preference. Kim and Richter's preferred sets had convexity properties, but they appear to be neither stronger nor weaker than ours (again, see the concluding section for the details).

An Elementary Equilibrium Existence Theorem

The substantive part of this paper begins in Section 2 with an equilibrium existence theorem. The classic proofs of the existence of competitive equilibria employ Kakutani's fixed point theorem (see Debreu (1982) and its references). It is also known that this is the right theorem to use, if one is not willing to assume that excess demand has any other structural property besides Walras' Law. This is because, as observed by Uzawa (1962) one could *prove* Kakutani's fixed point theorem by assuming a fundamental lemma used in equilibrium existence. (A proof of this result could also be found in Debreu (1982).)

On the other hand, if we assume that excess demand has additional structural properties, then a proof without using fixed point theorems may be available. For example, there is a simple proof of equilibrium existence when excess demand satisfies gross substitubility (see, for example, Hildenbrand and Kirman (1988)). In Section 2, we give a proof of equilibrium existence using the separating hyperplane theorem, under the added assumption that excess demand satisfies the weak axiom.

The proof we give has the virtue that it separates very sharply the function of the geometric and continuity properties of excess demand. Provided an excess demand function satisfies the weak axiom, there will be some price vector with the following almost-equilibrating property: holding all other prices fixed, raising the price of good i leads to excess supply, and lowering it leads to excess demand. To guarantee the existence of such a price vector, we do not need the continuity of excess demand, but it is quite easy to see that this price vector *is* an equilibrium price, i.e., has an excess demand of zero, provided the excess demand function is continuous. Indeed, continuity is needed at this stage of the proof, and nowhere else.

In addition to being interesting in its own right, this result has another purpose: it could be employed to establish the existence of a continuous demand function for the class of non-transitive preferences identified above. In fact, this problem is the dual analogue of the equilibrium existence problem; finding an optimal bundle in a budget set with a non-transitive preference is analogous to looking for an equilibrium price when the excess demand satisfies the weak axiom. So, once again, unlike virtually all other proofs of this result, we obtain a proof without using a fixed point theorem or something similar.[3]

It might be interesting to note that a similar process of simplification has also occurred in game theory. Von Neumann's (1928) original proof of the minimax theorem employed Brouwer's fixed point theorem, but the problem in fact has a nice geometrical structure which allows it to be quite intuitively solved with the separating hyperplane theorem (see Gale, Kuhn and Tucker (1950)). Similarly, Hart and Schmeidler (1989) showed that the existence of correlated equilibria could be established with linear methods, even though the standard proof (nowadays) of the existence of Nash equilibria uses Kakutani's fixed point theorem.

2. An Equilibrium Existence Theorem

In this section we give a proof of the existence of general equilibrium using the separating hyperplane theorem, under the added assumption that the economy's excess demand satisfies the weak axiom. This result is of course interesting in its own right, but we will also have occasion to use it in Section 3. In that section, we consider the problem of a consumer with not necessarily transitive preferences, choosing a preferred bundle from his budget set. It turns out that that problem could be thought of as the dual of the equilibrium existence problem considered here.

A standard approach to the equilibrium existence problem involves the construction of a correspondence, $Z : P \to R^l$, where P is a convex and pointed cone in R^l.[4] In this case, the economy has l goods, P is the set of price vectors, and Z is the excess demand (see Debreu (1982)). Typically Z will have a number of properties:

Property 1. Z satisfies Walras' Law, i.e., $p \cdot Z(p) = 0$ for all p in P.

Property 2. Z is a compact and convex valued, upper hemi-continuous correspondence.[5]

If one is investigating an exchange economy, then $P = R^l_{++}$, and Z will typically have two other properties:

Property 3. Z is bounded below.

Property 4. Z satisfies the following boundary condition: if p_n in R^l_{++} tends to $\bar{p}$ on the boundary of R^l_{++}, $\bar{p} \neq 0$, then $|Z(p_n)|$ tends to infinity.[6]

It is well known that if excess demand has Properties 1 to 4, then an equilibrium exists, i.e., there will be a price p^* in R^l_{++} such that $0 \in Z(p^*)$ (see Debreu (1982)). This result is usually established with Kakutani's fixed point theorem. We show here that another, quite instructive method, is available if Z also satisfies the weak axiom. We now give three increasingly strong definitions of the weak axiom for correspondences.

DEFINITION 2.1: The correspondence $G : R^l_{++} \to R^l$ satisfies W1(k) if

(i) $x \cdot G(x) = k$ for all x in R^l_{++} and

(ii) whenever x and y are in R^l_{++} and there exists $\bar{g}$ in $G(y)$ such that $x \cdot \bar{g} \leq k$, then

$y \cdot G(x) \geq k$.

DEFINITION 2.2: The correspondence $G : R^l_{++} \to R^l$ satisfies W2(k) if

(i) $x \cdot G(x) = k$ for all x in R^l_{++} and

(ii) whenever x and y are in R^l_{++}, $x \neq y$, and there exists $\bar{g}$ in $G(y)$ such that $x \cdot \bar{g} \leq k$, then $y \cdot g_x > k$ for all g_x in $G(x)$, $g_x \neq \bar{g}$.

DEFINITION 2.3: The correspondence $G : R^l_{++} \to R^l$ satisfies W3(k) if

(i) $x \cdot G(x) = k$ for all x in R^l_{++}; and

(ii) whenever x and y are in R^l_{++}, $x \neq y$, and there exists $\bar{g}$ in $G(y)$ such that $x \cdot \bar{g} \leq k$, then $y \cdot G(x) > k$.

Part (ii) of the definition of W2 is the standard definition of the weak axiom, at least when applied to functions; Part (ii) of the definition of W1 is a slightly weaker verson, which has been referred to as the weak weak axiom by some authors (for example, Jerison (1999)). The difference between W3 and W2 is that W3 guarantees that if $x \neq y$, then $G(x)$ and $G(y)$ are disjoint. If the correspondence G is a demand correspondence then k is simply the income (held fixed), while x is the price. If G is an excess demand function, then x is the price, and the standard definition of the weak axiom is W2(0), where $x \cdot G(x) = 0$ is simply Walras' Law. We have included (as part (i)) the budget identity or Walras' Law in our definition of the weak axiom, for the sole purpose of descriptive economy. This paper will not, in any case, consider demand that does not satisfy the budget identity.

We will now set out to show that an excess demand correspondence satisfying Properties 1 to 4, and W1(0) will have an equilibrium price. We begin with a lemma which guarantees that a *finite* set of excess demand vectors must have a supporting price.

LEMMA 2.4: Suppose that the correspondence $Z : P \to R^l$ satisfies Properties 1 and W1(0).[7] Then for any finite set $S = \{z_1, z_2, ..., z_n\}$ where z_i is an element of $Z(p_i)$, there is x^*, in the convex hull of $\{p_1, p_2, ...p_n\}$ such that $x^* \cdot S \geq 0$.

Proof: We proof by induction on n. If $n = 1$, choose $x^* = p_1$. If $n = 2$, then either $p_2 \cdot z_1$ or $p_1 \cdot z_2$ is non-negative. If it is the latter, choose, $x^* = p_1$.

Assume now that the proposition is true for n and assume that it is not true for $n + 1$. Consider the following constrained maximization problem:

$\max x \cdot z_k$ *subject to* x *satisfying the conditions:*

(a) $x \cdot z_i \geq 0$ *for* i *in* $I_k = \{1, 2, ...k - 1, k + 1, ..., n + 1\}$ *and*

(b) x *is in* P^**, the convex hull of* $\{p_1, p_2, ...p_n\}$.

By varying k, we have $n + 1$ problems of this sort.

Consider the case when $k = n + 1$. By the induction hypothesis, there is certainly x such that $x \cdot z_i \geq 0$ for all i in I_{n+1}, since this set has only n elements. Furthermore, P^* is compact, so the problem has at least one solution, which we denote by $\bar{x}_{n+1}$. Since we are proving by contradiction, we assume that $\bar{x}_{n+1} \cdot z_{n+1} < 0$.

We will now show that $\bar{x}_{n+1} \cdot z_i = 0$ for all i in I_{n+1}. If not, the set $J = \{i : \bar{x}_n \cdot z_i = 0\} \cup \{n + 1\}$ has n elements or less, and so there is $\bar{y}$ with $\bar{y} \cdot z_i \geq 0$ for all i in J. Consider now the vector $\theta\bar{y} + (1 - \theta)\bar{x}_{n+1}$, which is in P^*, provided θ is in $[0, 1]$. Then

(i) $[\theta\bar{y} + (1 - \theta)\bar{x}_{n+1}] \cdot z_i \geq 0$, for i in $J \setminus \{n + 1\}$

(ii) $[\theta\bar{y} + (1 - \theta)\bar{x}_{n+1}] \cdot z_i > 0$, for $i \notin J$ provided θ is sufficiently small

(iii) $[\theta\bar{y} + (1 - \theta)\bar{x}_{n+1}] \cdot z_{n+1} \geq (1 - \theta)\bar{x}_{n+1} \cdot z_{n+1} > \bar{x}_{n+1} \cdot z_{n+1}$.

This means that $\bar{x}_{n+1}$ does not solve the constrained maximization problem.

So the solution to this problem, $\bar{x}_{n+1}$, must satisfy $\bar{x}_{n+1} \cdot z_i = 0$ for i in I_{n+1} and $\bar{x}_{n+1} \cdot z_{n+1} < 0$. We can apply the same argument to a solution of the other problems. In this way, we obtain $\bar{x}_k$, for $k = 1, 2, ..., n+1$ with

(i) $\bar{x}_k \cdot z_i = 0$ for i in I_k and

(ii)$\bar{x}_k \cdot z_k < 0$.

Define $\bar{x} = [\sum_{i=1}^{n+1} \bar{x}_i]/(n+1)$; $\bar{x}$ is certainly in the convex hull of $\{p_1, p_2, ..., p_{n+1}\}$. Furthermore, $\bar{x} \cdot z_i < 0$, for $i = 1, 2, ..., n+1$. By W1(0), $p_i \cdot Z(\bar{x}) > 0$ for all i. Since $\bar{x}$ is in the convex hull of the p_is, we have $\bar{x} \cdot Z(\bar{x}) > 0$, which contradicts Walras' Law. QED

With this lemma, we could show that the range of Z has a supporting price.

PROPOSITION 2.5: Suppose that the correspondence $Z : P \to R^l$ satisfies Property 1 and W1(0). Then there is p^* in the closure of P such that $(p^* - p) \cdot Z(p) \geq 0$ for all p in P.

Proof: See Appendix.

We point out that p^* in Proposition 2.5 is really very close to an equilibrium price; essentially, it satisfies criterion (B) discussed in the introduction. To see that, let p be a price in P, with $p^i = p^{*i}$ for all i, except $i = k$. Proposition 2.5 tells us that $(p^{*k} - p^k)Z^k(p) < 0$. In other words, if p^k is greater than p^{*k} there will be excess supply of k; if it is lower, there will be excess demand of k. Note that we arrived at the existence of p^* relying exclusively on the geometric properties of Z. Continuity is not used at all. It is only needed to arrive at an equilibrium in the conventional sense.

LEMMA 2.6: Suppose that the correspondence $Z : P \to R^l$ satisfies Property 2 and that there exists a price p^* in the interior of the cone P such that $(p^* - p) \cdot Z(p) \geq 0$ for all

p in P. Then $0 \in Z(p^*)$.

Proof: Suppose not; then 0 and $Z(p^*)$ are disjoint and convex sets, and so by the separating hyperplane theorem, there is $v \neq 0$ such that $v \cdot Z(p^*) < 0$. Note that the strict inequality is guaranteed by the compactness of $Z(p^*)$. Define $p = p^* - \lambda v$, for some positive number λ. Since p^* is in the interior of P, for λ sufficiently small p is also in P; furthermore, $(p^* - p) \cdot Z(p) = \lambda v \cdot Z(p)$ which is strictly negative provided λ is sufficiently small so that $v \cdot Z(p) < 0$. The last condition is possible since $Z(p)$ is compact and Z is upper hemi-continuous. QED

The next theorem gathers together our results so far to establish the existence of an equilibrium in the case when the excess demand correspondence satisfies properties typically attained in an exchange economy.

THEOREM 2.7: *Suppose that the correspondence $Z : R^l_{++} \to R^l$ satisfies Properties 1 to 4, and W1(0). Then there is a price $p^* \gg 0$ such that $0 \in Z(p^*)$.*

Proof: Proposition 2.5 guarantees that a supporting price p^* exists in R^l_+. If we can show that p^* cannot be on the boundary, then Lemma 2.6 guarantees that p^* is an equilibrium price.

Assume to the contrary that $J = \{i : p^{*i} = 0\}$ is non-empty. We define the price vector p_n by $p^i_n = 1/n$ if i is in J, and $p^i_n = p^{*i}$ if i is not in J. So the sequence p_n tends to p^* on the boundary. Choose a sequence z_n, where z_n is in $Z(p_n)$. Then

$$\begin{aligned} (p^* - p_n) \cdot z_n &= (p^* - p_n) \cdot z_n \\ &= \sum_{i=1}^{l} (p^{*i} - p^i_n) z^i_n \end{aligned}$$

$$= -\frac{1}{n}\left[\sum_{i \in J} z_n^i\right]$$

If this term is negative, we have a contradiction. Indeed it is, because $[\sum_{i\in J} z_n^i]$ is positive when n is sufficiently large. To see this, note that z_n^i is bounded below, so in order for Walras' Law to be satisfied, z_n^i cannot tend to infinity if i is not in J; but the boundary condition requires $|z_n|$ to tend to infinity, so z_n^i must tend to infinity for some i in J. QED

3. Weakly Regular Preferences

In this section and the next, we examine different classes of non-transitive preferences which generate demand functions that are well-behaved, in a sense we will make specific. This is by no means the first attempt at addressing this issue. Following from Sonnenschein (1971), the literature on this problem has in fact become quite substantial; Kim and Richter (1986) provides a partial survey, together with an attempt at relating the seemingly different conditions that have been developed by different authors. It is not the intention here to come up with a set of conditions on preferences that will outdo all others in generality, though, in fact, we do end up with conditions that are different from, and no stronger than, conditions found by other authors. The assumptions and methods developed here are rather motivated by other considerations.

Firstly, we are not merely interested in a mix of assumptions that are sufficient to guarantee demand existence; the mix of assumptions we find should ideally also encompass a class of preferences that is rich enough to rationalize any demand function which obeys the weak axiom.

The second of these considerations is methodological. Suppose we find a set of preferences that have a well-behaved supporting price correspondence; then the agent's demand could be found by searching for the bundle on the budget plane where the supporting price coincides with the price the agent faces. This approach to the problem seems particularly appealing to an intuition developed from the case of transitive preferences. Furthermore, understood in this way, the problem is simply the dual analogue of the equilibrium existence problem considered in Section 2. This means that it could be solved by the same methods. In particular, in contrast to virtually all of the existing literature, the problem could be solved without resort to fixed point methods or their equivalent.

We assume the agent's consumption space is R^l_{++}. To each element x in R^l_{++} is associated a set, $R(x)$, also in R^l_{++}. The correspondence R is called a *preference correspondence* (or simply a *preference*) if

(i) for any x and y in R^l_{++}, either x is in $R(y)$ or y is in $R(x)$, and

(ii) for all x in R^l_{++}, x is in $R(x)$.

We will refer to the $R(x)$ as *the preferred set of* x. Whenever convenient, we will write "$y \succeq x$" or "y is preferred to x" to mean that y is in $R(x)$. We write "y is strictly preferred to x" or "$y \succ x$", when we mean that y is in $R(x)$ but x is not in $R(y)$. We also write $R^0(x) = \{y \in R^l_{++} : y \succ x\}$.

Properties (i) and (ii) are familiar: (i) just says that the preference is complete, while (ii) says that it is reflexive. The third, commonly assumed, property is transitivity, but this we do not assume.

DEFINITION 3.1: The preference R is *weakly regular* if to each x is associated a non-empty set $P(x)$, with the following properties:

(i) $P(x) \gg 0$;

(ii) for all x in R^l_{++}, $x \cdot P(x) = 1$ and $y \cdot P(x) > 1$ when y is in $R(x)$, $y \neq x$;

(iii) the correpondence P is compact and convex valued, and upper hemi-continuous; and

(iv) it satisfies the following *boundary condition*: if x_n tends to x_0 on the boundary of R^l_{++}, $x_0 \neq 0$, then $|P(x_n)|$ tends to infinity.

The interpretation of $P(x)$ is straightforward. It is simply the set of normalized supporting prices at each commodity bundle, so we will call P the *supporting price correspondence.* In the next section, we will impose assumpions on the sets $R(x)$ which guarantee the existence of the correspondence P with the nice properties listed in Definition 3.1. For now, we assume that such a P exists, and explore its implications. *All the results in this section assume that R is weakly regular.*

LEMMA 3.2: Suppose that x is in R^l_{++} and $y > x$. Then y is in the interior of $R^0(x)$.

Proof: If we can show that there is a neighborhood of y such that for all z in $N(y)$, there is p_z in $P(z)$ satisfying $x \cdot p_z \leq 1$, then we know that x is not in $R(z)$, so z is in $R^0(x)$, as required.

If no such neighborhood exists, we can find a sequence z_n tending to y, with $x \cdot p_n > 1$, p_n being an element taken from $P(z_n)$. By the upper hemi-continuity of P, we assume without loss of generality, that p_n has a limit of $\bar{p}$ in $P(y)$. So $x \cdot \bar{p} \geq 1$; but this cannot be true since $y > x$, $\bar{p} \gg 0$, and $y \cdot \bar{p} = 1$ implies that $x \cdot \bar{p} < 1$. QED

LEMMA 3.3: The correspondence P satisfies W3(1).

Proof: If for some p_x in $P(x)$ we have $y \cdot p_x \leq 1$, then y is not in $R(x)$. This means that x is in $R(y)$, so by the definition of P, $x \cdot P(y) > 1$. QED

An agent facing the price-income situation (p, w) in $R^l_+ \times R_+$, has a budget set, $B(p, w)$, defined by $B(p, w) = \{x \in R^l_{++} : p \cdot x \leq w\}$. A bundle x' in $B(p, w)$ is said to satisfy the *budget identity* if $p \cdot x' = w$. The bundle $\bar{x}$ is the *demand* at (p, w), if $\bar{x}$ is in $B(p, w)$, and $\bar{x} \succ y$ for all y in $B(p, w)$. Note that our definition of demand *requires* it to be unique.

PROPOSITION 3.4: (i) Demand exists at (p, w) in $R^l_+ \times R_+$ if and only if $(p, w) \gg 0$. (ii) If $\bar{x}$ is the demand at $(\bar{p}, \bar{w}) \gg 0$, then $\bar{x}$ has the following properties: (a) $\bar{x}$ satisfies the budget identity, and (b) of all the commodity bundles satisfying the budget identity, $\bar{x}$ is the *only* bundle for which $(\bar{p} \cdot x)^{-1}\bar{p}$ is in $P(x)$.

Proof: If $w = 0$ and $p \neq 0$, then the budget set is empty, so demand clearly does not exist; if $p = 0$ as well, then the budget set is the whole space, in which case demand certainly does not exist, because, by Lemma 3.2, we know that the preference is monotone. Suppose that w is strictly positive. Since $B(p, w) = B(p/w, 1)$, demand exists at (p, w) if and only if it exists at $(p/w, 1)$; furthermore if demand exists, then demand at (p, w) must coincide with demand at $(p/w, 1)$. For these reasons, we may assume without loss of generality that $w = 1$.

Suppose that there is $\bar{p}$ with $\bar{p}^1 = 0$ (i.e., the price of good 1 is zero), and $\bar{x}$ is the demand at $(\bar{p}, 1)$. Then $x' = \bar{x} + (1, 0, ..., 0)$ is within the budget set $B(\bar{p}, 1)$, and by Lemma 3.2 x' is strictly preferred to $\bar{x}$, which means that $\bar{x}$ cannot be the demand. So we have shown that demand cannot exist if either income or the price of some good equals zero. We will now go on to show that demand exists at $(\bar{p}, 1)$ provided $\bar{p} \gg 0$.

Define $E : R^l_{++} \to R^l$ by $E(y) = P(y) - \bar{p}$, if y satisfies the budget identity, and extending it to all of R^l_{++} by defining $E(y) = E(y/[\bar{p} \cdot y])$. It is trivial to check that E

satisfies Properties 1 to 4, typically associated with an excess demand function, as defined in Section 2. We now show that E satisfies W3(0). Since E is homogeneous of degree zero, we need only consider x and y satisfying the budget identity. Assume $x \cdot e_y \leq 0$, where $e_y = p_y - \bar{p}$ is in $E(y)$. If follows that $x \cdot p_y \leq 1$, and so $y \cdot P(x) > 1$, since P satisfies W3(1) (by Lemma 3.3). Re-writing the last inequaliy, we have $y \cdot E(x) > 0$.

Applying Theorem 2.7 to the correspondence E, we find there is $\bar{x}$ satisfying the budget identity with $0 \in E(\bar{x})$, or, in other words, $\bar{p} \in P(x)$. This means that if x is in $R(\bar{x})$, $x \cdot \bar{p} > 1$ so x is not in $B(\bar{p}, 1)$. Therefore, if y is in $B(\bar{p}, 1)$, $\bar{x} \succ y$. We have thus shown that $\bar{x}$ is the demand at $(\bar{p}, 1)$. In fact, we have shown that any bundle satisfying the budget identity with $\bar{p}$ as one of its supporting prices must be a demand at $(\bar{p}, 1)$. Since demand is unique, there can be only one bundle with this property. QED

As a result of the previous proposition, we can construct the agent's demand as a function of price and income, provided both are strictly positive. The final result of this section identifies the important properties of this function.

THEOREM 3.5: *Suppose that R is a minimally regular preference. Then it generates a demand function $f : R^l_{++} \times R_{++} \to R^l_{++}$ with the following properties:*

(i) it is homogeneous of degree zero,

(ii) it satisfies the budget identity,

(iii) it is continuous,

(iv) $f(\cdot, w)$ satisfies W2(w), and

(v) it has the following boundary property: if p_n tends to p' on the boundary of R^l_{++}, $p' \neq 0$, $|f(p_n, w)|$ tends to infinity.

Proof: By Proposition 3.4 demand exists and satisfies the budget identity. That it is homogeneous of degree zero comes simply from the fact that the budget is unchanged when prices and income are changed by a common factor. The proofs of (iii) to (v) are in the Appendix.

The properties identified in this theorem are the standard ones for a demand function. It is useful to give it a name.

DEFINITION 3.7: Any function $f : R^l_{++} \times R_{++} \rightarrow R^l_{++}$ is *consistent* if it satisfies conditions (i) to (v) in Theorem 3.5.

4. Regular Preferences

In the previous section we identified a class of preferences with sufficiently strong properties to guarantee the existence of a continuous demand function. In particular, weakly regular preferences are assumed to have well-behaved supporting prices at each commodity bundle. We now identify, with a more conventional definition, a class of preferences which is contained within the class of weakly regular preferences.

DEFINITION 4.1: The preference R is *regular* if it has the following properties:

(i) $R(x)$ is closed in R^l for all x in R^l_{++};

(ii) $R(x)$ is convex for all x;

(iii) if $y > x$, then y is in the interior of $R^0(x)$.

(iv) if for some $p \gg 0$, there is $y \neq x$ in $R(x)$ with $y \cdot p = 1$, then there is z in $R^0(x)$, with $z \cdot p < 1$.

It is standard in classical demand theory to assume that preferred sets are closed in the commodity space. It is also fairly standard to assume - as in (i) above - that it is closed in R^l (see, for example, Mas-Colell (1985), Definition 2.3.16). This has the effect of guaranteeing that all goods are demanded in strictly positive quantities. Properties (ii) and (iii) are standard and need little comment. Property (iv) serves two purposes. Firstly, it guarantees that the the preferred set of x, $R(x)$, does not have a locally flat boundary at x. This will be precisely what it says if we merely required z (in the definition) to be in $R(x)$. By requiring it to be in $R^0(x)$, we also guarantee that while $R^0(x)$ is contained in $R(x)$, there is a sense in which it takes up as much space as $R(x)$. In particular, we will show that $R^0(x)$ does not admit any supporting price at x that is not also a supporting price of $R(x)$.

It is perhaps also worth mentioning what we have *not* assumed. While we do assume that the preferred sets are closed, we have not assumed that the preference is continuous. Nor have we assumed that the preference is strongly convex in the following sense: if y and z are in $R(x)$, then $\alpha y + (1-\alpha)z$ is in $R^0(x)$, provided α is in $(0,1)$. Neither of these assumptions is crucial to the existence of a well-behaved supporting price correspondence.

We will now set out to demonstrate that if R is regular, each commodity bundle will have a non-empty set of supporting prices. It is useful to give two slightly different definitions of the set of prices supporting $R(x)$ at x. We define

$$\begin{aligned} P(x) &= \{p \in R^l : p \cdot x = 1 \text{ and for all } y \in R(x), p \cdot y > 1\} \text{ and} \\ \tilde{P}(x) &= \{p \in R^l : p \cdot x = 1 \text{ and for all } y \in R(x), p \cdot y \geq 1\}. \end{aligned}$$

In an analogous way, we define two different notions of the set of prices supporting $R^0(x)$ at x. We denote these by $P^0(x)$ and $\tilde{P}^0(x)$. The properties of these sets are explored in the next few lemmata.

LEMMA 4.2: If $P(x)$ [respectively $P^0(x)$] is non-empty and compact, then $P(x) = \tilde{P}(x)$ [respectively $P^0(x) = \tilde{P}^0(x)$].

Proof: We will show that $\tilde{P}^0(x) = P^0(x)$. The other case has exactly the same proof. Clearly, $P^0(x) \subseteq \tilde{P}^0(x)$. Choose $\bar{p}$ in $\tilde{P}^0(x)$ and pick p in $P^0(x)$ ($P^0(x)$ is non-empty by assumption). Then $[\alpha\bar{p} + (1-\alpha)p] \cdot x = 1$. Furthermore, if y is in R^0, $\bar{p} \cdot y \geq 1$ and $p \cdot y > 1$, so $[\alpha\bar{p} + (1-\alpha)p] \cdot y > 1$ provided $\alpha \in [0,1)$. So $\alpha\bar{p} + (1-\alpha)p$ is in $P^0(x)$ if $\alpha \in [0,1)$. Letting α tend to 1, $\bar{p}$ is in $P^0(x)$ since $P^0(x)$ is closed. QED

LEMMA 4.3: If R satisfies condition (iv) in Definition 4.1, then $\tilde{P}(x) = P(x)$.

Proof: Clearly, $P(x) \subseteq \tilde{P}(x)$. Suppose that p is in $\tilde{P}(x)$, and let y be in $R(x)$. If $p \cdot y = 1$, by assumption (iii) in our definition of regular preferences, there is z in $R^0(x)$, and hence $R(x)$, with $p \cdot z < 1$. But this will mean that p is not in $\tilde{P}(x)$, and so $p \cdot y > 1$. In other words, p is in $P(x)$. QED

LEMMA 4.4: Provided R satisfies conditions (ii) and (iii) in Definition 4.1, $\tilde{P}(x)$ is non-empty, compact and convex, with $\tilde{P}(x) \gg 0$.

Proof: By the convexity of R, we know that $\tilde{P}(x)$ is non-empty. It is clear that it must be compact and convex. Since any y, with $y > x$ is also contained in $R(x)$, we have $\tilde{P}(x) > 0$. In fact, $\tilde{P}(x) \gg 0$. Suppose otherwise; without loss of generality, assume that $p^1 = 0$ and $p^2 > 0$. By property (iii) in our definition of a regular preference, $\bar{x} = x + (1, -\epsilon, 0, ..., 0)$ is in $R(x)$ for a sufficiently small, positive ϵ. Then $p \cdot \bar{x} = 1 - \epsilon p^2 < 1$, which is a contradiction. QED

LEMMA 4.5: Suppose that R satisfies conditions (ii), (iii) and (iv) in Definition 4.1.

Then $P^0(x) = P(x)$.

Proof: Clearly, $P(x) \subseteq P^0(x)$. If p is in $P^0(x)$, $p > 0$, since any $y > x$ is also in $R^0(x)$. Suppose that p is not in $P(x)$. Then there exists $y' \in R(x)$ such that $p \cdot y' \leq 1$. Choose u with $u \gg x$ and $u \gg y'$. By the convexity of $R(x)$ and its monotone property, $x(\alpha) = \alpha y' + (1 - \alpha)u$ is in $R(x)$ for all α in $[0, 1]$. Since $p \cdot u > 1$, there must an $\bar{\alpha}$ for which $p \cdot x(\bar{\alpha}) = 1$. By property (iii) of regular preferences, there exists z in $R^0(x)$ such that $p \cdot z < 1$, so p cannot be in $P^0(x)$. QED

Lemmas 4.3 and 4.4 together imply that $P(x)$ is a non-empty, compact and convex set in R^l_{++}. Lemma 4.2 to 4.5 together show that, provided R satisfies conditons (ii), (iii), and (iv) in Definition 4.1, $P^0(x) = \tilde{P}^0(x) = P(x) = \tilde{P}(x)$ is a non-empty, compact, and convex set contained in R^l_{++}. We observe that condition (iv) is, in a sense, *just* the right one to use.

LEMMA 4.6: Suppose that for the preference R, $P(x) = \tilde{P}^0(x)$ for some x in R^l_{++}. Then condition (iv) in Definition 4.1 is satisfied at x.

Proof: If for some p there exists y in $R(x)$ with $y \cdot p = 1$, then p is not in $P(x) = \tilde{P}^0(x)$. This implies there is z in $R^0(x)$ such that $z \cdot p < 1$. QED

Notice that we have not yet used the fact that $R(x)$ is closed for all x. This condition is needed to establish the upper hemi-continuity of P.

LEMMA 4.7: Suppose that R satisfies condition (i) in Definition 4.1. Then R^0 is a lower hemi-continuous correspondence.[8]

Proof: Let x_n be a sequence tending to x, and let y be an element in $R^0(x)$. Assume, to the contrary, that no sequence in $R^0(x_n)$ tends to y. In particular, $R^0(x_{n_k}) \cap \{y\} = \phi$

for some subsequence x_{n_k}; since the preference is complete, $x_{n_k} \succeq y$, and by the closedness of $R(y)$, $x \succeq y$, which implies that y cannot be in $R^0(x)$. QED

PROPOSITION 4.8: Suppose that R is a regular preference. Then P is an upper hemi-continuous correspondence.

Proof: By Lemma 4.5, we may simply show that P^0 is an upper hemi-continuous correspondence. Let x_n tend to x, and let p_n be an element in $P^0(x_n)$. Without loss of generality, assume that p_n tends to $\bar{p}$. We want to show that $\bar{p}$ is in $P^0(x)$. By Lemma 4.7, for any y in $R^0(x)$, there is y_n tending to y, where y_n is in $R^0(x_n)$. By the definition of P^0, $y_n \cdot p_n > 1$; taking limits, we have $y \cdot \bar{p} \geq 1$. This means that $\bar{p}$ is in $\tilde{P}^0(x)$, which is also equal to $P^0(x)$ by our argument following Lemma 4.5. QED

So far we have shown that regular preferences must satisfy properties (i), (ii) and (iii) of weakly regular preferences (see Definition 3.1). Only property (iv) - the boundary condition - is left. We show this in the next proposition.

PROPOSITION 4.9: Suppose that R satisfies condition (i) in Definition 4.1 Then P has the following boundary property: if x_n tends to x_0 on the boundary of R^l_{++}, $x_0 \neq 0$; then $|P(x_n)|$ tends to infinity.

Proof: Suppose, to the contrary, that there is p_n in $P(x_n)$ tending to $\bar{p}$. Then there is a number M such that $p^i_n < M$ for all i and n. Defining $\tilde{x}^i = 1/2lM$, we see that $p_n \cdot \tilde{x} < 1$. This means that $\tilde{x}$ is not in $R(x_n)$, so x_n must be in $R(\tilde{x})$ for all n; but $R(\tilde{x})$ is closed in R^l and contained in R^l_{++}, so x_n cannot possibly have a limit on the boundary of R^l_{++}. QED

Assembling the results of this section and Theorem 3.5, we reach the following conclusion.

THEOREM 4.10: *Every regular preference is weakly regular and generates a demand function that is consistent.*

5. Rationalizing Demand

The section is devoted to proving the converse of Theorem 4.10: for any consistent function f we can find a regular preference which generates it. The proof proceeds by first finding a weakly regular preference that generates f, and then convexifying the preferred sets of the weakly regular preference to form a regular preference.

We begin by investigating the *equilibrating price correspondence* induced by f. This correspondence is defined in the following manner. For any commodity bundle x in R^l_{++}, we associate the *equilibrating price set of* x, defined as $Q(x) = \{p \in R^l_{++} : f(p,1) = x\}$. The next proposition identifies some important properties of the correspondence Q.

PROPOSITION 5.1: Suppose that f is a consistent function. Then the equilibrating price correspondence $Q : R^l_{++} \to R^l_{++}$ has the following properties:

(i) for all x, $Q(x)$ is non-empty, compact and convex;

(ii) it is upper hemi-continuous;

(iii) it satisfies W3(1);

(iv) it satisfies the boundary condition: if x_n tends to x_0 on the boundary of R^l_{++}, $x_0 \neq 0$, then $|Q(x_n)| \to \infty$.

Proof: (i) Define $Z_x(p) = f(p, p \cdot x) - x$. Z_x satisfes all the standard properties of an excess demand, Properties 1 to 4, as listed in Section 2. It also satisfes W2(0). By Theorem 2.7, there is p^* such that $Z_x(p^*) = 0$. Since Z_x is also homogeneous of degree zero, we may choose p^* to satisfy $p^* \cdot x = 1$. So p^* is in $Q(x)$, which must be non-empty.

To show convexity, let us assume that p^* and p^{**} are both in $Q(x)$. Since Z_x satisfies $W2(0)$, we have $p^* Z_x(p) \geq 0$ and $p^{**} Z_x(p) \geq 0$ for all $p \gg 0$. Therefore for any α in $[0,1]$, $(\alpha p^* + (1-\alpha)p^{**}) \cdot Z_x(p) \geq 0$. By Lemma 2.6, $Z_x(\alpha p^* + (1-\alpha)p^{**}) = 0$, so $\alpha p^* + (1-\alpha)p^{**}$ is in $Q(x)$.

To show closedness, let p_n^* be a sequence in $Q(x)$ converging to p^*. The boundary condition (see condition (v) in Theorem 3.5) guarantees that p^* cannot be on the boundary of R^l_{++}, so $p^* \gg 0$. Since $p_n^* \cdot Z_x(p) \geq 0$ for all p, taking limits, we have $p^* Z_x(p) \geq 0$ for all p, so by Lemma 2.6, p^* is in $Q(x)$.

(ii) Let x_n tend to x in R^l_{++}, and let p_n be a sequence extracted from $Q(x_n)$. Since the sequence is bounded, without loss of generality, let us assume that it converges to $\bar{p}$. Once again, the boundary condition guarantees that $\bar{p}$ is not on the boundary of R^l_+. Since Z_{x_n} satisfies W2(0) and $p \cdot Z_{x_n}(p_n) = 0$, we have

$$p_n \cdot Z_{x_n}(p) = p_n \cdot (f(p, p \cdot x_n) - x_n) \geq 0$$

for all $p \gg 0$. Fixing p and taking limits, we obtain

$$\bar{p} \cdot (f(p, p \cdot x) - x) = \bar{p} \cdot Z_x(p) \geq 0$$

for all $p \gg 0$. By Lemma 2.6, $\bar{p}$ is in $Q(x)$.

(iii) Suppose that $x \neq y$, and $y \cdot q_x \leq 1$ for some q_x in $Q(x)$. For any p in $Q(x)$, $y = f(p, 1)$, and since f satisfies W2(1), we have $f(q_x, 1) \cdot p = x \cdot p > 1$. Therefore, we have $x \cdot Q(y) > 1$.

(iv) Assume to the contrary that there is a sequence q_{x_n} extracted from $Q(x_n)$ with a finite limit of q. Since $x_n \cdot q_{x_n} = 1$ for all n, q cannot be zero. If $q \gg 0$, then $f(q_{x_n}, 1) = x_n$ has a limit of $f(q, 1) = x_0$, which is nonsense since x is on the boundary but the range of f is only R^l_{++}. If on the other hand, q is on the boundary, then $f(q_{x_n}, 1) = x_n$ cannot have a limit, by the boundary condition on f. QED

The next result is straightforward consequence of the previous proposition. It says that weakly regular preferences are the *largest* class of preferences that guarantee consistent demand functions. So our focus on this particular class of preferences turns out to be justified.[9]

THEOREM 5.2 : *Every preference that generates a consistent demand function must be weakly regular.*

Proof: Let f be a consistent demand function generated by R. We need only show that R has a well-defined supporting price correspondence satisfying the properties of weak regularity. In fact, we need only show that the supporting correspondence, denoted by $P(x)$ is none other than $Q(x)$, the equilibrating price correspondence induced by f, since by Proposition 5.1, $Q(x)$ satisfies all the properties of a supporting price correspondence.

If p is in $Q(x)$, we know that $f(p, 1) = x$, so by definition of demand, $x \succ y$ for all $y \cdot p \leq 1$. In other words, $z \succeq x$ must imply that $z \cdot p > 1$, so p is in $P(x)$. Therefore, $P(x)$ is non-empty since $Q(x)$ is non-empty.

If p is in $P(x)$, then for all y such that $y \cdot p \leq 1$, y is not in $R(x)$, so $x \succ y$. In other words, x must be the demand at $(p, 1)$, i.e., p is in $Q(x)$. QED

Theorem 5.2 falls short of what we want in this section, because it *assumes* rather than proves the existence of a rationalizing preference. We will now construct two preferences from a given consistent function f, both of which will rationalize f.

The Preference R_S

Define $S(x,y) = \min[y \cdot Q(x) - 1] + \max[1 - x \cdot Q(y)]$, where Q is the equilibrating price correspondence induced by f and, by definition, $\min[y \cdot Q(x) - 1] = \min\{y \cdot q_x - 1 : q_x \in Q(x)\}$ (with a similar definition for $\max[1 - x \cdot Q(y)]$). The function S satisfies $S(x,x) = 0$ and $S(x,y) = -S(y,x)$. It is clear this implies that the correspondence $R_S : R^l_{++} \to R^l_{++}$ defined by $R_S(x) = \{y \in R^l_{++} : S(x,y) \geq 0\}$ is a preference correspondence.

The Preference R

We define another preference correspondence, R, by the following rule: consider y to be in $R(x)$ if y is in the convex hull of $M_s(x) = \{y' \in R^l_{++} : y' > y \text{ for some } y \in R_S(x)\}$. Essentially $R(x)$ is simply $R_S(x)$ after it has been monotonized and convexified. That R is indeed a preference correspondence is also fairly obvious. Clearly, $R(x)$ is in R^l_{++} for all x in R^l_{++}. If y is not in $R(x)$, it is not in $R_S(x)$, and so $S(x,y) < 0$; this means that $S(y,x) = -S(x,y)$ is positive, so x is in $R(y)$.

We denote by P_S the supporting price correspondence of the preference R_S and denote by P the supporting price correspondence of the preference R. The corresponding supporting prices of the strictly preferred sets are denoted by P^0_S and P^0, and as in Section 4, we could define $\tilde{P}$, $\tilde{P}_S$, etc. The next theorem shows that all these notionally different sets coincide when f is consistent.

PROPOSITION 5.3: Suppose that f is a consistent function. Then

$P_S(x) = P^0_S(x) = \tilde{P}_S(x) = \tilde{P}^0_S(x) = Q(x)$ and $P(x) = P^0(x) = \tilde{P}(x) = \tilde{P}^0(x) = Q(x)$.

Proof: We will show the following:

(i) $Q(x) \subseteq P_S(x) \subseteq P_S^0(x)$,

(ii) $Q(x) \subseteq P(x) \subseteq P^0(x)$, and

(iii) $P_S^0(x) \subseteq P^0(x) \subseteq Q(x)$.

It follows from (i) and (iii) that $P_S(x) = P_S^0(x) = Q(x)$. That they are in turn equal to $\tilde{P}_S^0(x)$ and $\tilde{P}_S(x)$ follows from the compactness of $Q(x)$ (see Lemma 4.2). Similarly, (ii) and (iii) imply that $P(x) = P^0(x) = Q(x)$. Once again the compactness of $Q(x)$, guarantees that they are in turn equal to $\tilde{P}(x)$ and $\tilde{P}^0(x)$.

Since $R_S^0(x) \subseteq R_S(x)$, we must have $P_S(x) \subseteq P_S^0(x)$. So (i) is true if we show that $Q(x) \subseteq P_S(x)$. Let q_x be in $Q(x)$. Suppose $y \cdot q_x \leq 1$, $y \neq x$; then since Q satisfies W3(1) (by Proposition 5.1), $x \cdot Q(y) > 1$. The fact that $y \cdot q_x \leq 1$ implies that $\min[y \cdot Q(x) - 1] \leq 0$. Since Q is compact valued, the fact that $x \cdot Q(y) > 1$ implies that $\max[1 - x \cdot Q(y)] < 0$. So we have $S(x, y) < 0$ if $y \cdot q_x \leq 1$. In other words, y is in $R_S(x)$ only if $y \cdot q_x > 1$, or q_x is in $P_S(x)$.

We now show (ii). Since $R_S(x)$ is a subset of $R(x)$, we know that $P(x)$ must be a subset of $P_S(x) = Q(x)$. So we will be done if we can show that an element in $Q(x)$ must also be in $P(x)$. Let q_x be in $Q(x)$. If y is in $R(x)$, $y \neq x$, we write $y = \sum_{i=1}^{K} \beta^i y^i$, where the β^is are positive and add up to one, and $y^i \geq \bar{y}^i$, $\bar{y}^i$ being in $R_S(x)$. Since $Q(x) = P_S(x)$, $\bar{y}^i \cdot q_x \geq 1$, with a strict inequality if $\bar{y}_i \neq x$. This means that $y^i \cdot q_x \geq 1$ with a strict inequality for at least one y^i, since they cannot all be equal x. Therefore $y \cdot q_x > 1$. So q_x is in $P(x)$.

For (iii), first note that $R^0(x) \subseteq R_S^0(x)$ because $S(x, y) > 0$. (Otherwise, $S(y, x) \geq 0$, so x is in $R_S(y) \subseteq R(y)$, hence y is not in $R^0(x)$.) So $P_S^0(x) \subseteq P^0(x)$. So we need only show now that $P^0(x) \subseteq Q(x)$. If p is not in $Q(x)$, there is $z \neq x$ such that $f(p, 1) = z$, with $z \cdot p = x \cdot p = 1$. This means that p is in $Q(z)$, which by (ii), is a subset of $P(z)$. So x is not in $R(z)$, and therefore z is in $R^0(x)$. To recap, assuming that p is not in $Q(x)$, we have found z, with $z \cdot p = 1$ and z in $R^0(x)$: this means that p is not in $P_S^0(x)$. Therefore, $P^0(x) \subseteq Q(x)$. QED

The last proposition shows that R and R_S both generate the same supporting price correspondence, Q. Not surprisingly, both will generate the same demand function, which turns out to be f.

PROPOSITION 5.4: Suppose that f is a consistent function. Then the preferences R and R_S constructed from f are weakly regular, and will both generate f as their demand function.

Proof: Both R and R_S generate the supporting price correspondence Q. Comparing the properties of Q in Proposition 5.1 and Definition 3.1, we see that both R and R_S are weakly regular. By Proposition 3.4 (ii), the demand induced by a particular preference is completely determined by its supporting price correspondence, so R and R_S must have the same demand function. By Proposition 3.4 (ii), the demand at $(p, 1)$ is the commodity bundle x with p in $P_S(x)$. Since $P_S(x) = Q(x)$ by Proposition 5.3, p is in $Q(x)$; in other words, $x = f(p, 1)$. QED

So we have found, not one, but two, preferences which will ratonalize the function f. We will now go on to show that R is regular; the preference R_S generally is not, essentially because the preferred sets are not convex. The next result shows, however, that R_S does have closed preferred sets.

PROPOSITION 5.5: Suppose that f is a consistent function. Then for all x in R^l_{++}, the sets $R_S(x)$ are closed in R^l.

Proof: Let y_n be a sequence in $R_S(x)$ tending to y. If y is in R^l_{++}, $Q(y)$ exists and so $S(x,y)$ is defined. Since $Q(x)$ is compact, we may assume that $\min[y \cdot Q(x) - 1]$ is achieved at some q_x in $Q(x)$. Clearly, $\min[y_n \cdot Q(x) - 1] \leq y_n \cdot q_x - 1$. Assume that $\max[1 - x \cdot Q(y_n)]$ is achieved at some q_{y_n} in $Q(y_n)$, and assume that q_{y_n} tends to q. By the upper hemi-continuity of Q, q is in $Q(y)$. So,

$$
\begin{aligned}
0 \leq S(x, y_n) &= \min[y_n \cdot Q(x) - 1] + \max[1 - x \cdot Q(y_n)] \\
&\leq [y_n \cdot q_x - 1] + [1 - x \cdot q_{y_n}]
\end{aligned}
$$

Taking limits, we have

$$
\begin{aligned}
0 &\leq [y \cdot q_x - 1] + [1 - x \cdot q] \\
&\leq \min[y \cdot Q(x) - 1] + \max[1 - x \cdot Q(y)] = S(x, y)
\end{aligned}
$$

This means that y is indeed in $R_S(x)$.

Can y be on the boundary of R^l_{++}? We know that $y \neq 0$, since choosing any q_x in $Q(x) = P_S(x)$, we have $y_n \cdot q_x > 1$, so $y \cdot q \geq 1$. But then, by the boundary condition on Q (see Propostion 5.1 (iv)) we have $|Q(y_n)|$ tending to infinity, so $\max[1 - x \cdot Q(y)]$ must also tend to negative infinity, while $\min[y \cdot Q(x) - 1]$ remains bounded. Consequently, $S(x, y_n)$ cannot be non-negative for all n. So y is not on the boundary. QED

Finally we state the main result of this section.

THEOREM 5.6: *Every consistent function may be rationalized by a regular preference, which also has the following property: if y is in $R(x)$, then for any $y' \geq y$, y' is in $R(x)$.*

Proof: In Proposition 5.4, we have already shown that f is generated by R. So we need only show that R is indeed regular. We will relegate to the Appendix the straightforward but tedious proof that $R(x)$ is closed in R^l for all x. That $R(x)$ is convex is true by construction. This leaves us here, with just conditions (iii) and (iv) in Definition 4.1. By Proposition 5.4, R is weakly regular, so (iii) follows from Lemma 3.2. Condition (iv) is true by Proposition 5.3 and Lemma 4.5. QED

It is a fairly common practice to assume that preferences are continuous. In our setting this means that the set $\{(x, y) \in R^l_{++} \times R^l_{++} : y \in R(x)\}$ is closed in $R^l_{++} \times R^l_{++}$. We have not included this in our definition of regular preferences; there is no compelling need to do so, since, as we have shown in Section 4, it is not needed to guarantee the generation of consistent demand functions. Nevertheless, it seems a nice property to have, not least because of its intuitive consequences: it means, for example, that strictly preferred sets are open in R^l.

In Kim and Richter (1986), the rationalizing preference is continuous, but the preference is not regular, and in particular, not convex, in our sense. We do not know if every consistent demand function can be rationalized by a preference that is both regular and continuous, but we do know that the continuity of the preference is guaranteed if we impose an additional (and mild) condition on the consistent function f. In particular, we require the equilibrating price at each commodity bundle to be *unique*; in other words, the equilibrating price correspondence should be a *function*. Put still another way, it means that for a rationalizing preference, the preferred set at some bundle x cannot have a kink at x. We state this result formally.

THEOREM 5.7: *Suppose that f is a consistent function, with a unique equilibrating price at each commodity bundle. Then f can be rationalized by a preference that is continuous, regular, and satisfies the property: if y is in $R(x)$, then for any $y' \geq y$, y' is in $R(x)$.*

Proof: The proof is in the Appendix. It proceeds by showing that the preference R constructed for Theorem 5.6 is also continuous. This in turn rests on the fact that R_S is continuous, which follows from the continuity of the function $S(x, y) = y \cdot Q(x) - x \cdot Q(y)$. Note that since Q is now a function, we could dispense with the minimizing/maximizing operations in the more general definition of S. The continuity of S follows simply from the continuity of Q.

6. Conclusion

We showed in the last section that any consistent function could be rationalized by a regular preference. It is well-known that if the consistent function satisfied the *strong*, rather than just the weak axiom, it could be rationalized by a transitive preference; however, the procedure described in the last section will not, even in this case, yield a transitive preference.

The rationalizing preference we constructed has preferred sets that are closed in R^l. This may not initially come across as surprising, given that the demand for every good is assumed to be strictly positive at all prices, at least not until we recall the situation depicted in Figure 2. In this case, the agent has a commodity space of R^2_{++} and a transitive preference, with indifference curves that are smooth and eventually flat along the axes. Note, however, that the preferred sets are not closed in R^l. All goods are demanded in positive quantities at all strictly positive prices, essentially because the indifference curves become flat as they approach the axes. Subject to certain mild assumptions, the preference captured by Figure 2 is also the only *transitive* preference generating its demand. Yet the procedure developed in the last section guarantees that this demand could be generated by another preference, with preferred sets that are closed in R^l, so this preference *must* be non-transitive.

Kihlstrom, Mas-Colell, Sonnenschein and Shafer (1976), conjectured that any demand satisfying the weak axiom could rationalized by a preference R satisfying *strong convexity* in the following sense: if x and y are in $R(z)$, then $\alpha x + (1-\alpha)y$ is in $R^0(z)$, for α in $(0,1)$. No one has succeeded in finding a rationalizing preference with convexity of this sort. In this paper, we could guarantee that $\alpha x + (1-\alpha)y$ is in $R(z)$, but not in $R^0(z)$, while Kim and Richter (1986) obtained a rationalizing preference that satisfies convexity in the following two senses:

(i) if y is in $R(x)$, then $\alpha x + (1-\alpha)y$ is in $R^0(x)$, provided α is in $(0,1)$ and

(ii) if $x = \sum_{i=1}^{K} \alpha_i y_i$, for positive α_is that add up to 1, then x is in $R(y_i)$ for some i .

This raises the question of whether it is in fact possible to obtain convexity of the type proposed by Kihlstrom et al. We give an example that *suggests* it is not.

Consider the situation depicted in Figure 3. The commodity bundles a and a' have supporting prices $P(a)$ and $P(a')$ (put another way, a is the demand when price is $P(a)$ and income is 1, etc). Both budget lines pass through the point b'', so a and a' are both revealed preferred to b''. This means, if preferred sets are convex, that a'' is preferred to b''. If there is strong convexity in the sense of Kihlstrom et al, then a'' is *strictly* preferred to b'', but this cannot be the case. From the figure again, we see that b and b' are both revealed preferred to a'', and so b'' must be preferred to a''. In short the agent must be indifferent between a'' and b''.

As is obvious from the figure, the situation we described cannot happen in two dimensions, because the weak axiom is violated. On the other hand, with three commodities, an example can be constructed. Let

$$\begin{aligned}
a &= \left(\frac{1}{12}, \frac{1}{9}, \frac{3}{2}\right) \text{ and } P(a) = \left(\frac{2}{3}, 4, \frac{1}{3}\right), \\
a' &= \left(\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right) \text{ and } P(a') = (1, 1, 1), \\
b &= \left(\frac{1}{4}, \frac{13}{60}, \frac{1}{5}\right) \text{ and } P(b) = \left(2, 2, \frac{1}{3}\right), \\
b' &= \left(\frac{31}{40}, \frac{1}{40}, \frac{28}{40}\right) \text{ and } P(b') = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right).
\end{aligned}$$

It is easy to check that this finite set of data obeys the weak axiom: specifically, $b' \cdot P(b)$, $a \cdot P(b')$, $b' \cdot P(a')$, $b \cdot P(a)$, $a' \cdot P(b)$, and $a \cdot P(a')$ are all greater than 1. Defining

$$b'' = \frac{3}{5}b + \frac{2}{5}b' \text{ and } a'' = \frac{18}{25}a + \frac{7}{25}a',$$

we can check that $b'' \cdot P(a) = b'' \cdot P(a') = a'' \cdot P(b) = a'' \cdot P(b') = 1$. So the situation is exactly as we have described it above.

This example does not offer conclusive proof that there are consistent functions which cannot be rationalized by strongly convex preferences. This is because we have not defined a consistent function for all price-income situations; we have merely constructed a finite set of data on which the weak axiom is not violated. It is well known that if a finite data set obeys the strong axiom, then it could be rationalized by a transitive preference (this is Afriat's theorem; a proof, together with further results could be found in Varian (1982)). To our knowledge, no one has determined whether every finite data set that obeys the weak axiom could be generated by a possibly non-transitive preference. Suppose that this can be done; in fact, suppose that every finite data set obeying the weak axiom, like the one above, can be extended to a consistent function defined for all (p, w) in $R^l_{++} \times R_{++}$. If this is true (and we are inclined to think so), then rationalization with a strongly convex preference is not generally possible. Otherwise, we would have just provided an example of a finite data set which cannot be so extended - which is also not without interest.

St Hugh's College, Oxford, OX2 6LE, UNITED KINGDOM; john.quah@economics.ox.ac.uk

APPENDIX

Before we prove Proposition 2.5 it is useful to review some basic results on cones. Suppose that A is a convex and pointed cone in R^l. Define the set A^* by

$$A^* = \{v \in R^l : v \cdot a < 0 \text{ for all } a \in A, a \neq 0\},$$

and the set A^0 by

$$A^0 = \{v \in R^l : v \cdot a \leq 0 \text{ for all } a \in A, \}.$$

The set A^0 is usually referred to as the polar cone or negative polar cone of A. A^* is defined similarly, except that the inequality is strict rather than weak. We denote the closure of any set S by $\text{cl}S$.

LEMMA A1: (i) $\text{cl}A^* = (\text{cl}A)^0$ and (ii) If A is closed, $(A^0)^0 = A$.

Proof: (i) If x is in $\text{cl}(A^*)$, there is x_n such that $x_n \cdot a < 0$ for all a in $A \setminus \{0\}$. Taking limits, we have $x \cdot \bar{a} \leq 0$ for all $\bar{a}$ in $\text{cl}A$. So x is in $(\text{cl}A)^0$.

We will now show that $(\text{cl}A)^0 \subseteq \text{cl}(A^*)$. In fact, we will show something a little stronger, that $A^0 \subseteq \text{cl}(A^*)$. If x is in A^0, by definition, $x \cdot a \leq 0$ for all a in A. Since A is convex and pointed, by the separating hyperplane theorem, there is $w \neq 0$ such that $w \cdot A > 0$, for all a in $A \setminus \{0\}$. Since $[x - (w/n)] \cdot a < 0$ for all a, $x - (w/n)$ is in A^*. Letting n go to infinity, we see that x is in $\text{cl}(A^*)$.

(ii) If a is in A, for all v in A^0, $v \cdot a \leq 0$, so a is certainly in $(A^0)^0$. On the other hand, if a is not in A, then by the separating hyperplane theorem, there is w such that $w \cdot a > w \cdot A$. (Note that the inequality is strict because A is closed and pointed.) This means that $w \cdot A \leq 0$; otherwise the right hand side of the inequality is unbounded above. So w is A^0. We also have $w \cdot a > 0$, so this means that a is not in $(A^0)^0$. QED

Proof of Proposition 2.5: We claim that $\text{co}Z \cap P^* = \phi$, where $\text{co}Z$ is the convex hull of the set $\{Z(p) \in R^l : p \in P\}$ and $P^* = \{v \in R^l : v \cdot p < 0 \text{ for all } p \in P, p \neq 0\}$. If not, we can find $\sum_{i=1}^{K} \beta_i z_i$ in P^*, where z_i is in $Z(p_i)$ for some p_i, and the β_is are non-negative numbers that up to 1. By Lemma 2.4, there is x in P, with $x \cdot z_i \geq 0$ for all i, and consequently, $x \cdot [\sum_{i=1}^{K} \beta_i z_i] \geq 0$, contradicting the definition of P^*. So our claim is true. The separating hyperplane theorem guarantees that there is $p^* \neq 0$ such that $p^* \cdot Z(p) \geq p^* \cdot P^*$. Since P^* is a cone, the right hand side of this inequality could be bounded above only if is non-positive, so we have $p^* \cdot Z(p) \geq 0$ for all p in P.

We also claim that p^* is in cl$P$. Since $p^* \cdot P^* \leq 0$, we also have $p^* \cdot (\mathrm{cl}P^*) \leq 0$. By part (i) of the lemma, $p^* \cdot (\mathrm{cl}P)^0 \leq 0$, so p^* is in $(P^0)^0$, which is equal to cl$P$ by part (ii) of the Lemma. QED

Proof of Thoerem 3.5: (iii). Assume that p_n tends to $p \gg 0$, and write $x_n = f(p_n, 1)$. Since $x_n \cdot p_n = 1$, x_n is bounded, and we can assume without loss of generality that x_n tends to x. Clearly, $x \cdot p = 1$. Since P is upper hemi-continuous, and p_n is in $P(x_n)$, we have p in $P(x)$. So $x = f(p, 1)$.

(iv) Suppose that $p \neq q$ and $q \cdot f(p,1) \leq 1$. If $f(p,1) \neq f(q,1)$, then since P satisfies W3(1) (by Lemma 3.3), we have $f(q,1) \cdot P(f(p,1)) > 1$. In particular, p is in $P(f(p,1))$, so $p \cdot f(q,1) > 1$.

(v)Let p_n tend to $p' \neq 0$, on the boundary of R^l_{++}. Write $x_n = f(p_n, 1)$ and suppose, on the contrary, that $|x_n|$ does not tend to infinity. In that case, we may assume that it has a limit of x'. We know that $x' \neq 0$, since $x' \cdot p' = 1$. If x' is on the boundary of R^l_+, by the boundary condition on P (see Definition 3.1 (iv)), $|P(x_n)|$ tends to infinity, so $|p_n|$ tends to infinity and cannot have a limit of p'. Therefore x' must be in R^l_{++}; in which case, by the upper hemi-continuity of P, p' is in $P(x')$. This is impossible since $P(x')$ is in R^l_{++}. QED

We have proven all the claims in Theorem 5.6 except the closedness of $R(x)$. This is best understood by first proving the next two lemmas. For any set A, we write $A' = \{a' \in R^l : a' \geq a \text{ for some } a \in A\}$.

LEMMA A2: If A is closed and bounded below, A' is also closed and bounded below.

Proof: That A' is bounded below is obvious, so we need only show that it is closed.

Consider a sequence a'_n in A' converging to a'. Then there exists a_n in A, with $a_n \leq a'_n$. The sequence a'_n has a limit, since it is bounded both above and below. Assuming that its limit is $\tilde{a}$ in A, we have $\tilde{a} \leq a'$. QED

LEMMA A3: (i) If A is closed and contained in R^l_+, then any a in cl(co$A$) can be written as $a = \bar{a} + \bar{b}$, where $\bar{a}$ is in co$A$ and $\bar{b} \geq 0$. (ii) Suppose that A is closed and contained in R^l_+ , and satisfies the property $\mathcal{M}$: if a is in A then any $a' \geq a$ is also in A. Then co$A$ is closed.

(For any set S, co$S$ denotes its convex hull.)

Proof: (i) Suppose that a_n in co$A$ has a limit of a. By Caratheodory's Theorem, a_n can all be written $a_n = \sum_{i=1}^{l+1} \beta^i_n a^i_n$, where all the β^i_ns are non-negative and $\sum_{i=1}^{l+1} \beta^i_n = 1$. Without loss of generality assume that β^i_n has a limit of β^i. Since $\sum_i^{l+1} \beta^i = 1$, the set $I_1 = \{i : \beta_i > 0\}$ is non-empty. Call the complement of this set I_2. All the a^i_ns are bounded below since they are in R^l_+; furthermore, for i in I_1, a^i_n must also be bounded above, otherwise, a_n becomes an undounded sequence. So let us assume that a^i_n converges to a^i for i in I_1. By the closure of A, a^i is in A. Without loss of generality, assume that $I_1 = \{1, ..., k\}$. The sequence $\bar{a}_n = \sum_{i=1}^{k} \beta^i_n a^i_n$ has a limit of $\bar{a} = \sum_{i=1}^{k} \beta^i a^i$, which is in co$A$. The sequence $\bar{b}_n = \sum_{i=k+1}^{l+1} \beta^i_n a^i_n = a_n - \bar{a}_n$ must also have a limit since both a_n and $\bar{a}_n$ have limits. We denote this limit by $\bar{b}$. Clearly, $\bar{b} \geq 0$. So the sequence $a_n = \bar{a}_n + \bar{b}_n$ has a limit of $a = \bar{a} + \bar{b}$. QED

(ii) By part (i) of this lemma, any element a in cl(co$A$) may be written as $a = \sum_{i=1}^{l+1} \gamma^i a^i + \bar{b}$, where the a^is are in A and $b \geq 0$. Re-writing this, we have $a = \sum_{i=1}^{l+1} \gamma^i [a^i + \bar{b}]$. Since A satisfies property $\mathcal{M}$, $a^i + \bar{b}$ is also in A, and we find that a is in co$A$. QED

Proof of Theorem 5.6 (the closedness of $R(x)$): Define

$$A = \{a \in R^l_{++} : a \geq y, \text{ where } S(x,y) \geq 0\}.$$

By Proposition 5.5 and Lemma A2, A is a closed set. By Lemma A3(ii), $R(x) = \text{co}A$ is also closed, since A clearly satisfies property $\mathcal{M}$. QED

Proof of Theorem 5.7: Suppose that y_n is in $R(x_n)$, and (x_n, y_n) has a limit of (x, y) in $R^l_{++} \times R^l_{++}$. We wish to show that y is in $R(x)$.

By the definition of $R(x_n)$ and Caratheodory's Theorem, $y_n = \sum_{i=1}^{l+1} \beta^i_n y^i_n$, where $y^i_n \geq z^i_n$, with $S(x_n, z^i_n) \geq 0$. Define $z_n = \sum_{i=1}^{l+1} \beta^i_n z^i_n$. Then we may write $y_n = z_n + c_n$, where $z_n, c_n \geq 0$. Without loss of generality, assume that β^i_n, z_n and c_n have limits β^i, z, and c respectively. Since $\sum_{i=1}^{l+1} \beta^i = 1$, the set $I_1 = \{i : \beta^i > 0\}$ is non-empty. Assume that it is $\{1, 2, ..., k\}$. For i in I_1, the sequences z^i_n are bounded, so we assume that z^i_n has a limit of z^i. Clearly, $z^i \neq 0$. If it were, $x_n \gg z^i_n$ for n sufficiently large, and $S(x_n, z^i_n) < 0$. Nor could z^i be anywhere else on the boundary of R^l_{++}; if z^i_n tended to the boundary, $|Q(z^i_n)|$ will tend to infinity (Proposition 5.1 (iv)), so that $S(x, z^i_n) = z^i_n \cdot Q(x) - x \cdot Q(z^i_n)$ will eventually be negative, contradicting the fact that it is always non-negative. So $z^i \gg 0$. Since S is continuous in $R^l_{++} \times R^l_{++}$, we have $S(x, z^i) \geq 0$.

So $z_n = \sum_{i=1}^{k} \beta^i_n z^i_n + \sum_{i=k+1}^{l+1} \beta^i_n z^i_n$. The first term on the right hand side has a limit of $\sum i = 1^k \beta^i z^i$, with $\sum i = 1^k \beta^i = 1$, and $R(x, z^i) \geq 0$; the second term must also have a limit, denoted by b, which must satisfy $b \geq 0$. So we have $y = \sum_{i=1}^{k} \beta^i (z^i + b + c)$, which means that y is in $R(x)$. QED

Footnotes:

1. I would like to thank Robert Anderson for helpful discussions.

2. Kihlstrom, Mas-Colell and Sonnenschein (1976) showed that the negative semi-definiteness of the Slutsky matrix is essentially equivalent to the weak axiom.

3. An exception is Moldau (1996), who gives an interesting inductive proof in the case when the preference is both strongly convex and continuous.

4. A set P is a *cone* if whenever p is in P, λp is in P, for $\lambda > 0$. It is *pointed* if, whenever $p \neq 0$ is in P, $-p$ is not in P.

5. A correspondence $F : X \to Y$ is *upper hemi-continuous* if, for every sequence x_n tending to x in X, and every sequence y_n in $F(x_n)$, there is a subsequence of y_n with a limit in $F(x)$.

6. For any compact set S in R^l, by $|S|$, we mean $\min\{|s| : s \in S\}$.

7. In fact, Property 1 is part of the definition of W1(0). We write it this way for emphasis.

8. The correspondence $F : X \to Y$ is *lower hemi-continuous* if, for every sequence x_n converging to x in X, and every y in $F(x)$, there is y_n in $F(x_n)$ converging to y.

9. Note that the proof uses in an essential way our more stringent definition of demand, in which the demanded bundle is strictly (and not just weakly) preferred to other bundles in the budget set.

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