Comparative Statics and Welfare Theorems When Goods are Normal

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Abstract: We examine the impact of the normality assumption, together with the weak axiom, in three related areas of general equilibrium theory. Most obviously, these properties have important implications for equilibrium comparative statics, in the context of exchange, production or (incomplete) financial economies. They also shed light on the relationship between comparative statics and the structure of the excess demand function, which could be thought of as an aspect of the correspondence principle (Samuelson (1947)). Lastly, these properties permit the construction of welfare-like theorems which do not rely on the classical assumptions of individual rationality.

Keywords: normal goods, weak axiom, comparative statics, correspondence principle, welfare theorem

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1. Introduction

While there is a rich theory of consumer demand under the utility maximization hypothesis, the structural properties of demand derived from this hypothesis - like the weak and strong axioms - are not generally preserved with aggregation. For the theory of aggregate or market demand to go beyond essentially negative disaggregation results (see Shafer and Sonnenschein (1982) and Chiappori and Ekeland (1999)), it is necessary to impose some additional structure on demand beyond those which are implied by utility maximization. The objective of this paper is to give a thorough study of the consequences flowing from one of these additional properties.

We say that a consumer's demand function is *normal* if, with prices held fixed, demand for all goods increase as income increases. This property is not implied by utility maximization, but it is clearly a very mild assumption to impose, provided goods are thought of in sufficiently broad categories. We show that this property and its variants could be employed to address a number of interesting issues in demand and general equilibrium theory. They fall under three broad headings.

a. Comparative Statics

A basic problem in comparative statics is the following. Imagine that an exchange economy with l goods has a mean endowment of ω and an equilibrium price of p, with both ω and p in R^l_{++}. While the preferences of all agents remain unchanged, an endowment perturbation causes mean endowment to change to ω' and the equilibrium price to p'. When will it be true that the price of a good falls as its endowment increases? Since a scalar multiple of an equilibrium price is also an equilibrium price, the question implicitly assumes some suitable choice of scales. One way of formulating this is to require the two prices p and p' to preserve the value of some good or bundle of goods.

Formally, we are requiring that there exists a bundle of goods b (understood as an element in R^l_+), so that with prices normalized (by scalar multiplication) to satisfy (a) $p \cdot b = p' \cdot b$, we have (b) $(p - p') \cdot (\omega - \omega') < 0$. Condition (a) equalizes the value of the bundle b at the two prices so it effectively designates b as the *numeraire*. Condition (b) says that prices and endowments move in oppositive direction. In particular, if the price change has resulted from an increase in the endowment of good 1 with the endowment of other goods remaining the same, the price of good 1 before the perturbation, p^1, will be greater than its value after the perturbation, $(p')^1$. Clearly, we also have $p^1/p \cdot b < (p')^1/p' \cdot b$, i.e., the price of good 1, relative to the price of the bundle b, has fallen. We say that the pair (p, ω) and (p', ω') satisfies *N-monotonicity* if conditions 1 and 2 are satisfied.

The slightly odd name we have given this property is partly explained by the first major result in this paper: if the economy is a single agent economy, then the N-monotonic property is equivalent to the agent having a demand function which obeys the weak axiom and is normal. The letter 'N' in N-monotonicity could stand for the numeraire bundle, whose existence is guaranteed by the property; it could stand for normality, since this condition is crucial to the property; and it could stand for Nachbar, whose 1999 paper is the first to study this property, though he did not call it as such. (A detailed discussion of the relationship between this paper and Nachbar's is found in Section 4 of this paper.)

The paper then extends this result to a multi-agent setting. The main complication in this case arises from the fact that the income distribution in the economy before the endowment change will typically be different from the income distribution after the change in endowments and at the new equilibrium price. When comparing two situations like this, the crucial condition guaranteeing N-monotonicity is *aggregate normality*: two income distributions satisfy this condition at some fixed price vector if the aggregate demand generated at the income distribution with the higher mean income has greater demand for *all* goods. (The term 'aggregate normality' was first used by Nachbar (1999) and is obviously appropriate.) Since we show that aggregate normality is also in some sense necessary for N-monotonicity, the condition effectively identifies those equilibrium comparisons where N-monotoncity holds and where it fails. Unless all the agents in the economy have linear and parallel income expansion paths, it is quite clear that distributional changes of income which violate aggregate normality *must* exist, and this in turn will lead to violations of N-monotonicity for some pair of equilibrium outcomes. Nevertheless, aggregate normality is a sufficiently mild condition that the equilibrium comparisions that one is likely to encounter empirically may very well satisfy this condition and therefore, N-monotonicity.

Analogous conditions for N-monotonic equilibrium comparisons can also be formulated for financial economies (with a possibly incomplete asset structure) and production economies. In the former, N-monotonicity guarantees that the prices and endowment of securities move in opposite directions, when prices are compared against a numeraire securities portfolio. In production economies, we show that if aggregate demand in the household sector satisfies N-monotonicity for a pair of equilibrium outcomes, then the property must also hold when comparing both economies in its entirety. This result follows from the profit maximization hypothesis. In particular, the result guarantees that the price of a factor, i.e., an endowed good which do not give utility but which constribute to production, will also move in a direction opposite to that of its endowment. This price is measured relative to some bundle of utility giving goods, in other words, we are referring to movements in the 'real' factor price. This result also has implications for factor augmenting technology changes. For example, if a techonology change causes each unit of a particular factor to behave like it was two, then the real factor price will not increase by more than two.

b. The Correspondence Principle

The correspondence principle is one of the major themes in Samuelson's *Foundations* (1947). Very loosely speaking, it says that there is a formal connection between the conditions needed for the stability of equilibrium and the conditions needed for 'nice' comparative statics. In the context of an exchange eonomy, the stability of the equilibrium price with respect to to Walras' tatonnement hinges on the structure of the excess demand function; in particular, it must satisfy what we call the *equilibrium weak axiom*, i.e., if Z is the excess demand function and p the equilibrium price, then $(p - p') \cdot Z(p') > 0$ for any price p' non-collinear with p. A natural question inspired by the correspondence principle is the following: given that the structure of the excess demand function is not directly observable, what information about its structure can one draw from observations of equilibrium price changes arising from endowment perturbations?

More narrowly, we ask whether equilibrium comparisons that obey N-monotonicity offer support for the equilibrium weak axiom, and whether violations of N-monotonicty negate that property. It turns out that even when an excess demand function satisfies the equilibrium weak axiom, there will still exist equilibrium comparisons which violate

N-monotonicity, and when an excess demand function violates the weak axiom in the direction of the price change $p - p'$, i.e., $(p - p') \cdot Z(p') \leq 0$, one could still find an equilibrium comparison with the same price change which obeys N-monotonicity. So the connection between comparative statics and the properties of the excess demand function is not completely straightforward. However, when a suitable normality condition is imposed on the economy's demand behavior, it is possible to classify comparative statics observations into those which convey information about the excess demand function and those which do not. Furthermore, for any price p', there is an equilibrium comparison which will reveal the sign of $(p - p') \cdot Z(p')$.

c. Welfare Theorems

An allocation of consumption across households in a production economy is called *Walrasian* if it can be achieved via a Walrasian/competitive equilibrium, after suitable lump sum transfers of income. Let A and B be two such Walrasian allocations, and let p be the equilibrium price supporting the allocation A. Then one could easily show that because of the profit-maximization hypothesis, if both allocations are valued using the price p, the mean value of allocation A must be (weakly) greater than that of B. It follows that, with such a valuation, the *average* income of agents in A must be higher than in B. The question we ask is whether it is possible for *every household* to have a higher income in A compared to B. We show that under some some fairly mild assumptions on aggregate household demand, including a normality assumption, this is not possible. In fact in these situations there are at least two sensible ways of computing a household's income, and in both cases, the income distributions at A and B cannot be unambiguously ranked.

These results should be understood as versions of the first welfare theorem. Just as the first welfare theorem says that the allocations A and B cannot be unambiguously ranked by the utilities they achieve, so our results say that they cannot be ranked by their incomes either. A novel feature of this approach is that it makes absolutely no reference to preferences. The conclusions are about the income distributions, while conditions are imposed on the aggregate structure of demand. This structure could be motivated in whole or in part by the utility maximization hypothesis, or they may not be. It is possible to achieve aggregate structure in demand through distributional assumptions on behavior or income and without assuming utility maximization at all (see, e.g., Becker (1962) or Grandmont (1992)), and indeed there is a school of thought which argues that the traditional view of aggregate behavior as the sum of atomistic individuals maximizing their individual utilities is deeply inadequate (see Kirman (1992)). What we have shown is that, even if this is true of aggregate household demand, suitably modified versions of the first welfare theorem could still apply.

The paper is organized as follows. Sections 2 and 3 deal with individual and market demand respectively, and formulate conditions under which N-monotonicity is satisfied. These results are then applied in Section 4 to comparative statics in exchange economies, which leads naturally to a discussion of the correspondence principle in Section 5. Sections 6 and 7 are on production economies, with the first on comparative statics and the second on welfare theorems. Comparative statics in financial economies are discussed in Section 8.

2. N-Monotonic Individual Demand

We begin with a study of the demand of an individual consumer. We assume that the commodity space has l goods and the agent's demand is a function of price and income. Formally, a function $f : R^l_{++} \times R_+ \to R^l_{++}$ is a *demand function* if it is homogeneous of degree zero, ie., $f(p, y) = f(kp, ky)$ for all $k > 0$, and satisfies $p \cdot f(p, y) = y$. These properties are standard: interpreting $f(p, y)$ as the agent's choice from the budget set $\{x \in R^l_{++} : p \cdot x \leq y\}$, the first condition says that this choice is unchanged when the budget set is unchanged, while the second says that the budget constraint is binding. There are other properties that demand functions could have, depending on the rationality requirements imposed on the consumer. We list three.

A pair (p, x) and (p', x') in $R^l_{++} \times R^l$ satisfies

i. *the weak axiom* if either $p \cdot x' > p \cdot x$ or $p' \cdot x > p' \cdot x'$;

ii. *the limited weak axiom* if any of these three cases is true: $p \cdot x' > p \cdot x$, $p' \cdot x > p' \cdot x'$, or $p \cdot x' < p \cdot x$ and $p' \cdot x > p' \cdot x'$; and

iii. *monotonicity* if $(p - \lambda p') \cdot (x - x') < 0$ for λ chosen to satisfy $p \cdot x = \lambda p' \cdot x'$.

A function $g : R^l_{++} \to R^l$ is said to satisfy the weak axiom (limited weak axiom, monotonicity) if the pair $(p, g(p))$ and $(p', g(p'))$ satisfies the weak axiom (respectively limited weak axiom, monotonicity) whenever $p \neq p'$. A demand function f is said to satisfy one of these properties if $f(\cdot, y)$ satisfies the property for some positive income y. Note that because f is homogeneous of degree zero, if $f(\cdot, y)$ satisfies one of these properties for some y, then it must satisfy the same property for every positive y.

The weak axiom is a standard property for a demand function to have; it is implied by, and is weaker than, utility maximization. Monotonicity can be more familiarly written as the condition requiring that $(p-p')\cdot(f(p,y)-f(p',y))<0$ whenever $p\neq p'$. It is just the multi-market version of the one good law of demand: when the price of a good goes up and income is held fixed, demand falls. It is known that Giffen goods are compatible with utility maximization, so clearly monotonicity is not implied by utility maximization, nor indeed does monotonicity imply utility maximization. But the property is a natural one to employ in many situations and fairly weak utility conditions implying monotonicity (due to Milleron, Mitjuschin and Polterovich) are known (see Mas-Colell (1991) or Hildenbrand (1994) for a proof of this result; for analogous conditions on the indirect utility function, see Quah (1997)). It is not hard to check that if f is monotonic then it also satisfies the weak axiom. Finally, the limited weak axiom is just a weaker version of the weak axiom, and we introduce it here because it is precisely what we need. Note that the limited weak axiom can be equivalently defined in the following way: a demand function f satisfies the limited weak axiom if for any $p\neq p'$, whenever $p\cdot f(p',y')=y$, we have $p'\cdot f(p,y)>y'$.

The focus of this paper is on a different property. The pair (p,x) and (p',x') in $R^l_{++}\times R^l$ satisfies *N-monotonicity* if there exists a positive number λ such that $R^l_+\setminus\{0\}$ such that $(p'-\lambda p)\cdot(x'-x)<0$, and when p and p' are not collinear, we may choose λ to satisfy the additional condition $p'\cdot b=\lambda p\cdot b$ where b is in R^l_+. We refer to b as *the normalizing vector*; as we had pointed out in the introduction, b can be interpreted as a bundle of goods serving as a numeraire. A function $g: R^l_{++}\to R^l$ is N-monotonic if the pair $(p,g(p))$ and $(p',g(p'))$ is N-monotonic for all $p\neq p'$. (Note that the normalizing vector need not be common across different pairs.) A demand function f is N-monotonic if $f(\cdot,y)$ is N-monotonic for some (and therefore by the zero-homogeneity of f, for any) positive income level y.

It is not hard to check that if the pair (p,x) and (p',x') satisfies N-monotonicity, then it also satisfies the weak axiom. If the pair satisfies the weak axiom, then without loss of generality, we can assume that $p \cdot (x'-x) > 0$, in which case, for some positive number λ sufficiently large, we must have $(p' - \lambda p) \cdot (x' - x) < 0$. Given any p' and λp, we can always find a vector $b \neq 0$ such that $p' \cdot b = \lambda p \cdot b$, so what distinguishes N-monotonicity and makes it stronger than the weak axiom is the requirement that $b > 0$.

Our first result gives the precise conditions under which a pair (p,x) and (p',x') satisfies N-monotonicity. It is fundamental to virtually everything else in this paper.

LEMMA 2.1: *Consider a pair (p,x) and (p',x') in $R^l_{++} \times R^l$, where p and p' are not collinear and $p' \cdot x \neq p' \cdot x'$. Then the pair satisfies N-monotonicity with the normalizing vector $b > 0$ if and only if there exists c in R^l such that (i) b is collinear with $x' - c$, (ii) $p' \cdot c = p' \cdot x$, (iii) (p',c) and (p,x) satisfies the limited weak axiom.*

Proof: We first proof the sufficiency of the conditions. Assume that $p' \cdot x < p' \cdot x'$ (the proof for the other case is the same). Choose c such that it satisfies (i) to (iii). Choose λ such that $p' \cdot b = \lambda p \cdot b$. Then

$$\begin{aligned}
(p' - \lambda p) \cdot (x' - x) &= (p' - \lambda p) \cdot (x' - c + c - x) \\
&= (p' - \lambda p) \cdot (c - x) \\
&= -\lambda p \cdot (c - x) < 0.
\end{aligned}$$

Note that the second equality follows from our choice of b and λ, the final equality follows from (ii), and the final inequality from (iii).

To proof necessity: suppose that $(p'-\lambda p)\cdot(x'-x) < 0$ for b satisfying $p'\cdot b = \lambda p\cdot b$. Choose c such that $p'\cdot c = p'\cdot x$ and $x'-c$ is collinear with b. Then clearly c satisfies (i) and (ii). Furthermore, since the pair is N-monotonic, the expression $(p'-\lambda p)\cdot(x'-x) = -\lambda p\cdot(c-x)$ must be negative. This implies that $p\cdot c > p\cdot x$, which is condition (iii). QED

Perhaps the best way of understanding conditions (i) to (iii) is to look at Figure 1. Condition (i) says that c should be on the budget line α, i.e., with price p' and bundle x just affordable. Condition (ii) says that c rests on the line α and to the left of the bundle x. Condition (iii) says that c must lie between the points m and n. In short, (p, x) and (p', x') is N-monotonic if and only if a point c like the one in Figure 1 can be found - on the line α, between m and x. The normalizing vector will then be $x' - c$.

Given a demand function f, the lemma says that whether or not the pair $(p, f(p, y))$ and $(p', f(p', y'))$ is N-monotonic depends on the existence of a vector c satisfying conditions (i) to (iii). The existence of c is guaranteed by a familiar condition on demand.

We say that the demand function f is *normal at* (p', y') if $f(p', y) > f(p', y')$ if $y > y'$ and $f(p', y) < f(p', y')$ if $y < y'$. A demand function is normal if it is normal at all (p', y') in $R^l_{++}\times R_+$. The next proposition says that the demand function f is N-monotonic if it is normal and satisfies the weak axiom.

PROPOSITION 2.2: *Suppose that the demand function f satisfies the limited weak axiom, and is normal at (p', y'). Then the pair $(p', f(p', y'))$ and $(p, f(p, y))$ is N-monotonic for any (p, y) in $R^l_{++}\times R_+$ which is not collinear with (p', y').*

Proof: If p and p' are collinear but (p, y) and (p', y') are not collinear, then $p\cdot(f(p, y) - f(p', y'))$ and $p'\cdot(f(p, y) - f(p', y'))$ must be non-zero and of the same sign. In this case clearly, there exists λ such that $(p' - \lambda p) \cdot (f(p,y) - f(p',y')) < 0$.

So we consider the case where p and p' are not collinear. If $p' \cdot f(p,y) = y'$ then, by the limited weak axiom, $p \cdot f(p',y') > y$. In this case, for any $\lambda > 0$,

$$\begin{aligned}(p' - \lambda p) \cdot (f(p',y') - f(p,y)) &= -\lambda p \cdot (f(p',y') - f(p,y)) \\ &= \lambda(y - p \cdot f(p',y')) < 0,\end{aligned}$$

so any vector $b > 0$ can serve as the normalizing vector. In the case where $p' \cdot f(p,y) \neq y'$ we apply Lemma 2.1, with $c = f(p', p' \cdot f(p,y))$. All three conditions imposed on c in that lemma are satisfied: (i) is true since f is normal at (p',y'), (ii) follows from the niceness of f, and (iii) follows from the limited weak axiom property on f. QED

We now show that, under some completely standard assumptions, normality is also a necessary condition for N-monotonicity. A function $g : R^l_{++} \to R^l_{++}$ is *nice* if it is continuous and satisfies the boundary condition: if the sequence $p_n \gg 0$ tends to $p \neq 0$ and on the boundary of R^l_+, then $|g(p_n)|$ tends to infinity. A demand function f is nice if $f(\cdot, y)$ is nice for all positive y. Continuity of f is guaranteed if f is generated by a continuous preference or utility function, while the boundary condition holds if the agent's preference is strongly monotone (see Mas-Colell et al (1995)).

PROPOSITION 2.3: *Suppose that the demand function f is nice and satisfies the limited weak axiom. If f violates normality at (p',y') then there is (p,y) such that $(p, f(p,y))$ and $(p', f(p',y'))$ violates N-monotonicity.*

The proof of Proposition 2.3 is rather long and is presented in the Appendix. In summary, this section has established the following result.

THEOREM 2.4: *A nice demand function f is N-monotonic if and only if it is normal and satisfies the weak axiom.*

3. N-Monotonic Market Demand

The ideas of the previous section could be extended to a multi-agent setting. We consider a market in which the agents form a finite set A. This market has l commodities; each agent a has an income y_a and a demand function f_a. We refer to the vector $y = (y_a)_{a \in A} \gg 0$ as the market's *income distribution* and denote the mean income by $\bar{y}$. We say that the income distrbution y *dominates* the income distribution y' if $y > y'$. An ordered pair (p, y) will be called a *market situation.* The (mean) market demand at the market situation (p, y) will be denoted by $F(p, y)$, so $F(p, y) = \sum_{a \in A} f_a(p, y_a)/|A|$. If we assume that f_a is nice, then F is also homogeneous of degree zero, and will satisfy $p \cdot F(p, y) = \bar{y}$. *For the rest of this paper, we shall assume that F satisfies these properties.*

The issue we wish to address is the structural property of F. In particular, we ask when will a pair $(p, F(p, y))$ and $(p', F(p', y'))$ satisfy N-monotonicity or the weak axiom. An important and simpler version of this problem considers the case where $y = y'$, i.e., the income distribution is fixed. It turns out that even in this case, the problem is not straightforward. It is well-known that the weak axiom is not preserved with aggregation even if all agents in the market have normal demand functions. This means in particular, that while N-monotonicity may hold at an individual level, market demand can still violate the weak axiom and therefore N-monotonicity. However, there is also a very large body of results, partially surveyed in Mas-Colell et al (1995), which gives conditions under which the fixed-income (distribution) version of the aggregate weak axiom is satisfied.

Conditions guaranteeing this property fall into a two broad categories. Most obviously, one could impose conditions on individual preferences or demand which guarantee nice aggregate properties. In particular, if agents each have a demand function which is monotonic, then one could show easily that the aggregate demand is also monotonic, and hence will satisfy the weak axiom. At the other extreme are conditions which minimize restrictions on individual demand and emphasize the distributional characteristics of the market. Perhaps the most famous examples of this approach are Hildenbrand (1983) and Grandmont (1992). In the first case, restrictions are imposed on the income distribution; in the second case, conditions are imposed on the distribution of preferences. The individualistic and distributional approaches to this problem are not mutually exclusive, and indeed one could construct models where a mix of assumptions of both types could be combined to guarantee the weak axiom for market demand (see Quah (2000)). The extensive empirical work reported in Hildenbrand (1994), also lends support to the view that the weak axiom is true in the case of a fixed income distribution.

So we can safely conclude that there is strong theoretical and empirical support for the view that $F(\cdot, y)$ satisfies the weak axiom. Happily, this is just a small step away from saying that $F(\cdot, y)$ obeys N-monotonicity.

Corollary 3.1: *Suppose that for some distribution y, $F(\cdot, y)$ satisfies the limited weak axiom, and that all agents in the market have normal demand functions, i.e., f_a is normal for all a in A. Then $F(\cdot, y)$ is N-monotonic.*

Proof: Without loss of generality assume that $\bar{y} = 1$. The function $h : R^l_{++} \times R_+ \to R^l_{++}$

defined by $h(p, \phi) = F(p, \phi y)$ is nice, normal, and satisfies the limited weak axiom. The result follows from Proposition 2.2. QED

So we see that for fixed income situations, the weak axiom and N-monotonicity are both fairly general properties of market demand, with the latter requiring the additional, and mild, assumption that all agents have normal demand functions. When the income distribution is not fixed - so we are considering the pair $(p, F(p, y))$ and $(p', F(p', y'))$ where y need not be equal to y' - it turns out that N-monotonicity will follow from some version of the weak axiom together with a generalized assumption of normality.

The function F satisfies *aggregate normality between* (p, y) *and* (p, y') if the following holds: when $\bar{y} > \bar{y}'$, $F(p, y) > F(p, \bar{y}')$; when $\bar{y} < \bar{y}'$, $F(p, y) < F(p, \bar{y}')$. It is easy to imagine situations where an income change will violate this condition. An increase in average income does not preclude a fall in income for some agents; so long as agents have non-parallel income expansion paths, an increase in average income can lead to a fall in demand for some good if those agents whose incomes have fallen decrease their consumption of the good by more than the increase in consumption of that good among those agents who have enjoyed an increase in income. At any given price, aggregate normality will be more likely for large changes in average income than small ones. For example, if there is a large rise in average income levels, then the fall in demand among those whose incomes have fallen (and it cannot fall below zero) will be more than compensated by the increase in demand among those agents whose incomes have gone up. In general, information on income expansion paths and the income distributions will allow us to say whether aggregate normality will hold for any particular pair of market situations.

Theorem 3.2: *Let (p, y) and (p', y') be two market situations and by multiplying (p', y') with a scalar if necessary, assume that $p' \cdot F(p, y) = \bar{y}$. Then the pair $(p, F(p, y))$ and $(p', F(p', y'))$ is N-monotonic if there exists some distribution y'' such that*

(1) $\bar{y}' \neq \bar{y} = \bar{y}''$ and F satisfies aggregate normality between (p', y'') and (p', y'); and

(2) F satisfies the limited weak axiom between $(p', F(p', y''))$ and $(p, F(p, y))$.

Proof: If p and p' are collinear N-monotonicity is trivially true. So we assume that they are not collinear. In that case, the problem is a straightforward application of Lemma 2.1, with $c = F(p', y'')$. The fact that $\bar{y} \neq \bar{y}'$ means that $p' \cdot F(p, y) \neq p' \cdot F(p', y')$ as Lemma 2.1 requires. Condition (ii) of Lemma 2.1 is satisfied since

$$p' \cdot c = p' \cdot F(p', y'') = \bar{y}'' = \bar{y} = p' \cdot F(p, y).$$

Condition (iii) of Lemma 2.1 is satisfied because of condition (2) in this theorem. Finally, condition (i) of Lemma 2.1 is satisfied because F satisfies aggregate normality between (p', y'') and (p', y'). QED

Clearly, Theorem 3.2 can permit the case where $\bar{y}' = \bar{y}$ if the definition of aggregate normality is strengthened to include the following case: for any two income distributions y and y' with $\bar{y} = \bar{y}'$, $F(p, y) = F(p, y')$. Unless $y = y'$, in which case it is trivially true, the condition is very strong, so we have decided not to include it.

Theorem 3.2 allows for different y'' to be chosen. There are at least three natural choices for y'' depending on the context and issues we wish to address. We point out two now, and a third in the next section. The conditions (A1) and (A2) below are more specific versions of (1) and (2) in Theorem 3.2.

(A1) $\bar{y}' \neq \bar{y}$ *and F satisfies aggregate normality between (p', y'') and (p', y') where* $y'' =$

$(p' \cdot f_a(p, y_a))_{a \in A}$*; and*

(A2) *F satisfies the limited weak axiom between $(p', F(p', y''))$ and $(p, F(p, y))$.*

In this case, we have given each agent a just enough income to purchase the bundle $f_a(p, y_a)$ at the price p'. (This is essentially the Slutsky decomposition performed on each agent.) Note that so long as agent a obeys the limited weak axiom, we have $p \cdot f(p', y_a'') > p \cdot f(p, y_a)$. If this is true for all agents, in the aggregate $p \cdot F(p', y'') > p \cdot F(p, y)$. In other words, condition (A2) must be satisfied. With this choice of y'', Theorem 3.2 tells us that in a market consisting of agents who individually obey the weak axiom, violations of the weak axiom must imply a violation of the aggregate normality condition (A1), i.e., $F(p', y') - F(p', y'')$ is not in R_+^l or R_-^l.

Assuming that agents individually have normal demand functions, an important special case where (A1) (and hence N-monotonicity) is guaranteed is when $y' > y''$ or $y'' > y'$. In particular, suppose that each agent's demand function is generated by a locally non-satiated preference, and that the market situation (p, y) Pareto dominates (p', y'). By this we mean that every agent a weakly prefers $f_a(p, y_a)$ to $f_a(p', y_a')$, with at least one agent having a strong preference. In that case, $p' \cdot f_a(p, y_a) = y_a'' \geq y_a'$, with a strict inequality for at least one agent so it follows that the pair $(p, F(p, y))$ and $(p', F(p', y'))$ is N-monotonic.

Another natural choice for y'' is simply to set it equal to y, in which case we obtain

(B1) *$\bar{y}' \neq \bar{y}$ and F satisfies aggregate normality between (p', y') and (p', y);*

(B2) *F satisfies the limited weak axiom between $(p', F(p', y))$ and $(p, F(p, y))$.*

In this case, condition (B2) is just the fixed income version of the weak axiom. Conditions (B1) and (B2) for N-monotonicity generalize those of Corollary 3.1; the fixed income version of the weak axiom (B2) is assumed in both cases, while (B1) generalizes the normality condition in that corollary to account for the difference in the income distribution. In particular, if all agents have normal demand functions, (B1) (and hence N-monotonicity) is satisfied if the income distribution y dominates y' or vice versa.

In the case of a single agent, we have shown that normality is a necessary condition for N-monotonicity. That result generalizes easily to a multi-agent context. In the next proposition, we show that assuming (B2), condition (B1) is a necessary condition for N-monotonicity. Its proof is in the Appendix.

Proposition 3.3: *Suppose that F violates aggregate normality between (p', y) and (p', y') and that $F(\cdot, y)$ is nice and satisfies the limited weak axiom. Then there is p such that $p' \cdot F(p, y) = \bar{y}$ and $(p, F(p, y))$ and $(p', F(p', y'))$ violates N-monotonicity.*

Two conclusions can be safely drawn from our discussion of N-monotonicity in the case where the income distribution is not fixed. First, we should not expect N-monotonicity to hold between *any* two market situations. This is because even when condition (B2) is satisfied, a violation of the aggregate normality condition (B1) is sufficient to guarantee a violation of N-monotonicity, and violations of (B1) *must* exist when agents have non-parallel income expansion paths. Second, even though aggregate normality cannot always hold, the condition is sufficiently mild that it may very well be valid for the market situations we are likely to encounter empirically.

The rest of this paper is devoted to applications of the basic results developed in this and the previous section.

4. Comparative Statics of Exchange Economies

The most obvious application of Theorem 3.1 is to the comparative statics of exchange economies. Suppose that the market is in fact an exchange economy and that the agent a has an endowment of ω_a in R^l_+. We write $\omega = (\omega_a)_{a\in A}$ and the mean endowment as $\bar{\omega}$. We assume that $\bar{\omega} \gg 0$. We denote this exchange economy by $\mathcal{E}(\omega)$ and the collection of such economies by $\mathcal{M}$, so $\mathcal{M} = \{\mathcal{E}(\omega) : \omega \in R^{l|A|}_+, \bar{\omega} \gg 0\}$. Two economies in $\mathcal{M}$ have the same agents with the same demand functions, but different endowments. The *equilibrium set*, denoted by E, is the set of ordered pairs (p,ω) where p is an equilibrium price of $\mathcal{E}(\omega)$ in $\mathcal{M}$. Abusing our terminology a little, we shall say that (p,ω) and (p',ω') in E satisfies N-monotonicity if $(p,\bar{\omega})$ and $(p',\bar{\omega}')$ satisfies N-monotonicity. The next result identifies conditions which guarantee that a pair in E satisfies N-monotonicity.

Corollary 4.1: *Let (p,ω) and (p',ω') be two elements in E and by multiplying p' with a scalar if necessary, assume that $p'\cdot\bar{\omega} = p\cdot\bar{\omega}$. The pair (p,ω) and (p',ω') is N-monotonic if*

(B1) $p'\cdot\bar{\omega}' \neq p'\cdot\bar{\omega}$ and F satisfies aggregate normality between (p',y) and (p',y') where $y = (p\cdot\omega_a)_{a\in A}$ and $y' = (p'\cdot\omega'_a)_{a\in A}$; and

(B2) F satisfies the limited weak axiom between $(p',F(p',y))$ and $(p,F(p,y))$.

Proof: This follows immediately from Theorem 3.2, specialized to the case of $y''=y$, i.e., condition (B1) and (B2) in Section 3 and using the fact that, at equilibrium, $F(p,y)=\bar{\omega}$ and $F(p',y')=\bar{\omega}'$. QED

A little explanation of what the two conditions say will be helpful. Imagine that the economy's endowment is perturbed from ω to ω', causing equilibrium price to change from p to p'. The income distribution at the original economy induced by the equilibrium price p is

y; that at the new economy is y'. The aggregate normality condition (B1) is a condition on $F(p', y)$ - the mean demand at the new equilibrium price p' if the income distribution is held at that of the original economy (which is y). Aggregate normality requires $\bar{\omega}' = F(p', y') > F(p', y)$ if the new economy is valued more than the original economy at the price p', i.e., $\bar{y}' > \bar{y}$, and $\bar{\omega} = F(p', y') < F(p', y)$ if the new economy is valued less. Condition (B2) simply says that average demand F must satisfy the limited weak axiom when the income distribution is held fixed at the distribution of the original economy, y.

Corollary 4.1 is clearly not the only way of constructing sufficient conditions for N-monotonicity between exchange economies; for example, the conditions (B1) and (B2) could be replaced with (A1) and (A2) (see Section 3), which involves a different specification of the distribution y'' in Theorem 3.2. There is a third choice for y'' that is natural in the context of an exchange economy. Suppose that, as in the Corollary 4.1, we normalize p' so that $p' \cdot \bar{\omega} = p \cdot \bar{\omega}$ and choose $y'' = (p' \cdot \omega_a)_{a \in A}$, i.e., the income distribution induced by the equilibrium price of $\mathcal{E}(\omega')$, i.e., p', at the endowment distribution of $\mathcal{E}(\omega)$. If we do this, applying Theorem 3.2 again, the conditions of Corollary 4.1 can be replaced by

(C1) $p' \cdot \bar{\omega}' \neq p' \cdot \bar{\omega}$ *and* F *satisfies aggregate normality between* (p', y'') *and* (p', y'),

(C2) F *satisfies the limited weak axiom between* $(p, F(p, y)) = (p, \bar{\omega})$ *and* $(p', F(p', y''))$.

Since $p' \cdot \bar{\omega} = \bar{y}'' = \bar{y}$, condition (C2) says that $p \cdot F(p', y'') > p \cdot \bar{\omega}$. Note that $F(p', y'') - \bar{\omega}$ is the excess demand at price p' of the economy $\mathcal{E}(\omega)$, so condition (C2) can be re-written as requiring $p \cdot Z_\omega(p') > 0$, where Z_ω is the excess demand function of $\mathcal{E}(\omega)$.

We say that the excess demand function function Z_ω satisfies *the equilibrium weak axiom* if $p \cdot Z_\omega(p') > 0$, where p is the equilibrium price and p' is any price not collinear with p.

With this definition, (C2) is implied by

(C2)* Z_ω *satisfies the equilibrium weak axiom.*

Conditions (C1) and (C2)* are the exact discrete analogs to the local and differentiable conditions given by Nachbar (1999) for N-monotonicity. This discrete version of Nachbar's result was independently obtained by Nachbar (2001) and Quah (2001).

In the infinitesimal version of this result (Nachbar (1999)), the analog to (C2)* is to assume that $v^T \partial_p Z_\omega(p) v < 0$ for all vectors v in R^l, non-zero and non-collinear with the locally unique equilibrium price p. It is easy to show that this implies that $p \cdot Z_\omega(p') > 0$ for all p' not collinear with p and in some open neighborhood of p. Quite clearly, this local condition, together with (C1) will give N-monotonicity between (p, ω) and (p', ω') when the endowment (and hence equilibrium price) change is sufficiently small.

The advantage of conditions (C1) and (C2)* is that (C1) (unlike conditions (A1) or (B1)) can be verified with information on the endowment perturbation and agents' Engel curves. Note that $y''_a = y'_a + p' \cdot (\omega_a - \omega'_a)$, so if p' and y' (which we could think of as the initial equilibrium price and income distribution) along with the endowment change $\omega_a - \omega'_a$ are known, y''_a can be calculated. Knowledge of the Engel curves at p' will then make it possible for one to determine the value of $F(p', y'')$ and therefore check if (C1) is satisfied. Put another way, so long as (C2) holds, it is possible to identify a set of endowment perturbations for which (C1) holds and N-monotonicity is guaranteed.

The disadvantage of conditions (C1) and (C2)* is that (C2)*, the equilibrium weak axiom, is a considerably stronger assumption compared to the fixed-income version of the weak axiom (assumption (B2)). There *are* theoretical work which studies the conditions under which the excess demand function of an exchange economy obeys the equilibrium weak axiom either locally or globally (see for example, Grandmont (1992), Quah (1997) and Jerison (1999)). However, essentially because the income distribution varies with price in an exchange economy, these conditions are considerably stronger than those guaranteeing the weak axiom for market demand when the income distribution is fixed. Furthermore, (C2)* is not generally necessary for N-monotonicity. It is easy to construct scenarioes where N-monotonicity holds, but where the excess demand function misbehaves.

PROPOSITION 4.2: *Suppose that in the economy $\mathcal{E}(\omega)$, all agents have monotonic and normal demand functions Let p be an equlibrium price of this economy. Then for every p' not collinear to p, there is an economy $\mathcal{E}(\omega')$ with the following characteristics:*

(i) it has an equilibrium price p' such that the pair (p, ω) and (p', ω') is N-monotonic,

(ii) the endowment change from ω to ω' is collinear, i.e., for all a, $\omega'_a = \omega_a + k_a\theta$, where $k_a \geq 0$, $\sum_{a\in A} k_a = 1$, and $\theta \neq 0$; and (iii) $y' = (p' \cdot \omega'_a)_{a\in A}$ is collinear with $y = (p \cdot \omega_a)_{a\in A}$. Furthermore, so long as all agents have continuous demand functions, if p' tends to p, then ω' can be chosen such that ω' tends to ω.

The proof of Proposition 4.2 is in the Appendix. The proposition considers an economy $\mathcal{E}(\omega)$ which is inhabited by agents with demand functions that are normal and monotonic. Notice that these behavioral assumptions are rather mild; at their very worst, they still embrace all homothetic preferences, which means, according to Mantel's indeterminacy theorem (1976), that the excess demand function, Z_ω, of this economy can seriously misbehave, and certainly need not obey the limited weak axiom. This means that there could exist p' such that $p \cdot Z_\omega(p') < 0$, i.e., condition (C2) is violated. In spite of this, the proposition states that for *any* price p' there is another economy $\mathcal{E}(\omega')$ with p' as its equilibrium such that the pair (p, ω) and (p', ω') is N-monotonic. Since the endowment change from ω to ω' is collinear, with $y''_a = y'_a - k_a p' \cdot \theta$, either $y'' \geq y$ or $y' > y''$, depending on the sign of $p' \cdot \theta$. Together with the assumption that all agents have normal demand functions, we know that the aggregate normality condition (C1) is satisfied.

Comparative statics results in general equilibrium are often formulated in terms of conditions on the excess demand function, of which (C2)* is an example (see also the results in Arrow and Hahn (1971)). Since these restrictions on the excess demand function are often considered to be strong, the same view has been extended (not always logically, but certainly impressionistically) to the comparative equilibrium restrictions that they imply. This view is not always warranted.

We have seen that a misbehaving excess demand function does not always preclude N-monotonic equilibrium comparisons. If we wish to have some intuition on the ubiquity of N-monotonicity as an observable equilibrium phenomenon, we could just as easily look at, say, conditions (B1) and (B2) in Corollary 4.1. If we do that, our conclusion will essentially be the same as the one we reached in the last section. Even when agents' preferences are such that the fixed income version of the weak axiom, i.e., condition (B2), is satisfied, we cannot expect N-monotonicity to hold for every pair in E. This is because so long as agents do not have parallel income expansion paths, a change in income distribution which violates (B1) always exists, and by Proposition 3.3, we can find a pair of market situations which violates N-monotonicity. This in turn could be re-formulated as a violation of N-monotonicity for a pair of economies in E. In spite of this, the aggregate normality condition (B1) is sufficiently mild that we could still expect N-monotonic equilibrium observations to be a common empirical phenomenon, or at least there is no reason to believe that it will be rare, whatever our views may be on the structure of the excess demand function.

In the next section, we will examine more closely the relationship between the excess demand function and N-monotonic comparative statics, not because we wish to use conditions on the former to motivate the latter, but for a quite different reason.

5. The Correspondence Principle

In his *Foundations* (1947), Samuelson argued that there is a close connection between the conditions needed to guarantee stability of equilibria and those guaranteeing comparative statics. He called this *the correspondence principle* and illustrated its validity in a variety of settings. A comprehensive discussion of what this principle could mean will take us too far afield, but the comparative statics we have been examining suggests one interpretation of this principle which is worthy of investigation.

For the economy $\mathcal{E}(\omega)$, the ability of Walras' tatonnement to converge to an equilibrium price p depends on the structure of the excess demand function Z_ω. A sufficient and essentially necessary condition for convergence is that Z_ω satisfies the equilibrium weak axiom - globally, if we wish to ensure global convergence; in some neighborhood of an equilibrium price p if we wish to ensure local convergence. Therefore, one is led to ask if properties of the excess demand function Z_ω, which we assume are not directly observable, could be inferred from equilibrium observations. In particular, could these observations negate or confirm the hypothesis that Z_ω satisfies the equilibrium weak axiom?

More specifically, using the notation of the previous section, the problem we consider here is the following:

Let (p, ω) and (p', ω') be two elements in E. Suppose an observer observes the pair $(p, \bar{\omega})$ and $(p', \bar{\omega}')$ and suppose that he knows that the demand function F satisfies aggregate normality between (p', y') and (p', y''). When can he be certain about the sign of $p \cdot Z_\omega(p')$?

Note that $y' = (p' \cdot \omega'_a)$ and $y'' = (p' \cdot \omega_a)$, and the aggregate normality condition is just the condition (C1) introduced in the last section. To assume that an observer 'knows' (C1) must imply that he knows (or is willing to assume) something about the endowment perturbation and about F, but clearly we are also assuming effectively that he does not know so much that he could actually work out $Z_\omega(p')$ and render the problem trivial. A simple situation where that is true is when the observer could observe $\delta_a = \omega_a - \omega'_a$ for all a, and finds that $p' \cdot \delta_a$ is (weakly) of the same sign for all a. This implies that either $y' \geq y''$ or $y'' > y'$. If the observer also assumes that all agents have normal demand functions, he will be able to assume that (C1) is satisfied, but will not have enough information to determine exactly the value of $Z_\omega(p')$. Nevertheless, will he be able to determine the sign of $p \cdot Z_\omega(p')$?

We already know part of the answer to the question we posed. We have shown that (p, ω) and (p', ω') will satisfy N-monotonicity if in addition to (C1), condition (C2) is satisfied, i.e., $p \cdot Z_\omega(p') > 0$; or to put it another way, if it is observed that the pair (p, ω) and (p', ω') violates N-monotonicity, then the observer may infer that $p \cdot Z_\omega(p') \leq 0$, i.e., Z_ω has violated the limited weak axiom. This is information, albeit of a negative sort, that has been gleaned from a pair of observations in E.

On the other hand, observing N-monotonicity does not say nearly as much. Note that condition (C1) is satisfied in Proposition 4.3 since the endowment change is collinear, but as we have shown in that case, the pair (p, ω) and (p', ω') can be N-monotonic even if $p \cdot Z_\omega(p') < 0$. To exclude this possibility, something stronger than N-monotonicity must be observed.

We say that the pair (p, x) and (p', x') in $R^l_{++} \times R^l_{++}$ is *strongly N-monotonic* if p and p' are not collinear and for all $b > 0$, $(p' - \lambda p) \cdot (x' - x) < 0$ where λ is chosen to satisfy $p' \cdot b = \lambda p \cdot b$. In other words, strong N-monotonicity requires that every vector in the positive orthant could serve as a normalizing vector. An even stronger requirement is *super N-monotonicity*, which holds when the pair (p, x) and (p', x') in $R^l_{++} \times R^l_{++}$, with p and p' non-collinear satisfies $(p' - \lambda p) \cdot (x' - x) < 0$ for all $\lambda > 0$. This definition is equivalent to saying that $p \cdot (x - x') \leq 0$ and $p' \cdot (x' - x) \leq 0$, with strict inequality in at least one case. In Figure 2, the pair (p', x') and (p_1, x_1) is N-monotonic but not strongly N-monotonic, since $x' - c$ cannot be a normalizing vector (see also Figure 1 and the discussion following Lemma 2.1). The pair (p', x') and (p_2, x_2) is strongly monotonic since it is clear that any vector $x' - c$ for c on the budget line between m and n can be a normalizing vector. The pair (p', x') and (p_3, x_3) is also strongly N-monotonic; in fact, it is super N-monotonic since $p' \cdot (x' - x_3) < 0$ and $p_3 \cdot (x_3 - x') < 0$.

PROPOSITION 5.1: *Let (p, ω) and (p', ω') be a pair in E and assume that (C1) is satisfied, i.e., F satisfies aggregate normality between (p', y') and (p', y''), where $y' = (p \cdot \omega_a)_{a \in A}$ and $y'' = (p' \cdot \omega_a)_{a \in A}$, with $\bar{y}'' \neq \bar{y}'$.*

(i) If (p, ω) and (p', ω') are not N-monotonic, then $p \cdot Z_\omega(p') \leq 0$.

(ii) if (p, ω) and (p', ω') are strongly N-monotonic then $p \cdot Z_\omega(p') > 0$.

Proof: We have already proved (i); we now show (ii). We consider the case where $\bar{y}'' < \bar{y}'$; the other case can be dealt with similarly. With this assumption, $\bar{\omega}' = F(p', y') > F(p', y'')$. Since (p, ω) and (p', ω') is a strongly N-monotonic pair, the vector $b = \bar{\omega}' - F(p', y'')$ can be used as a normalizing vector vector. Choose λ so that $\lambda p \cdot b = p' \cdot b$. Note that

$$\begin{aligned}(p' - \lambda p) \cdot (\bar{\omega}' - \bar{\omega}) &= (p' - \lambda p) \cdot (\bar{\omega}' - F(p', y'') + F(p', y'') - \bar{\omega}) \\ &= (p' - \lambda p) \cdot (F(p', y'') - \bar{\omega}) \\ &= (p' - \lambda p) \cdot Z_\omega(p').\end{aligned}$$

So, by strong N-monotonicity, $p \cdot Z_\omega(p') > 0$. QED

This proposition tell us when a pair of observations in E will convey information about excess demand. As an application, consider once again the scenario in Proposition 4.3 where all agents in the economy $\mathcal{E}(\omega)$ have normal demand functions and the economy is subjected to a collinear endowment change. In this case, condition (C1) must be satisfied. Suppose in addition that the endowment change preserves the value of the "original" economy $\mathcal{E}(\omega)$, i.e., $\omega'_a = \omega_a + k_a \theta$, where $k_a \geq 0$, $\sum_{a \in A} k_a = 1$, and θ satisfies $p \cdot \theta = 0$. We claim that endowment changes of this sort are *always* informative. Note that the economy $\mathcal{E}(\omega')$ has a mean endowment of $\bar{\omega} + \theta$. By Proposition 5.1(i), if the pair $(p, \bar{\omega})$ and $(p', \bar{\omega} + \theta)$ violates N-monotonicity, an observer may conclude that $p \cdot Z_\omega(p') \leq 0$. On the other hand, if it is N-monotonic then it must also be super N-monotonic, since $p \cdot (\bar{\omega} - (\bar{\omega} + \theta)) = 0$ and $p' \cdot (\bar{\omega} + \theta - \bar{\omega}) < 0$. By Proposition 5.1(ii), $p \cdot Z_\omega(p') > 0$.

If the equilibrium price p of $\mathcal{E}(\omega)$ is *regular* (see Mas-Colell et al (1995) for the definition), then standard arguments will show that for p' not collinear with p and in some open neighborhood of p, there is θ, with $p \cdot \theta = 0$ such that the economy $\mathcal{E}(\omega')$ with $\omega'_a = \omega_a + \theta$ for all a will have p' as an equilibrium price. In other words, for p' sufficiently close to p, there is an endowment change which is guaranteed to reveal the sign of $p \cdot Z_\omega(p')$. As the next proposition shows, not all endowment changes are similarly informative.

PROPOSITION 5.2: *Consider a pair (p, ω) and (p', ω') in $R^l_{++} \times (R^l_+ \setminus \{0\})^{|A|}$, with $p' \cdot \bar{\omega} \neq p' \cdot \bar{\omega}'$. Suppose there is a vector $b > 0$ such that $(p' - \lambda p) \cdot (\bar{\omega} - \bar{\omega}') < (>) 0$, when $p' \cdot b = \lambda p \cdot b$. Then*

(a) there exists x_a, x'_a, and x''_a in R^l such that $p \cdot x_a = p \cdot \omega_a$, $p' \cdot x'_a = p' \cdot \omega'_a$, $p' \cdot x''_a = p' \cdot \omega_a$, and $S_a = \{(p, x_a), (p', x'_a), (p', x''_a)\}$ satisfies the strong axiom of revealed preference for $1 \leq a \leq |A|$;

(b) $\sum_{a \in A} x_a = \bar{\omega}$ and $\sum_{a \in A} x'_a = \bar{\omega}'$;

(c) $\sum_{a \in A} x''_a < \bar{\omega}'$ if $p' \cdot \bar{\omega} < p' \cdot \bar{\omega}'$ and $\sum_{a \in A} x''_a > \bar{\omega}'$ if $p' \cdot \bar{\omega} > p' \cdot \bar{\omega}'$; and

(d) $p \cdot (\sum_{a \in A} x''_a - \bar{\omega}) > (<) 0$.

The proof of this proposition is in the Appendix and is straightforward once we apply a disaggregation result of Andreu (1982). The proposition says that the mean endowments at the two observations - $\bar{\omega}$ and $\bar{\omega}'$ could be disaggregated in such a way that it is compatible with individual rationality (condition (a)) and the aggregate normality condition (C1) (condition (c) in the proposition). Furthermore, $(p' - \lambda p) \cdot (\bar{\omega}' - \bar{\omega})$ and $p \cdot (\sum_{a \in A} x''_a - \bar{\omega})$ have opposite signs, where $\sum_{a \in A} x''_a - \bar{\omega}$ is just the excess demand at price p' when ω is the economy's endowment (in more familiar notation, it is $Z_\omega(p')$). When the pair is N-monotonic but not strongly N-monotonic, the sign of $(p' - \lambda p) \cdot (\bar{\omega}' - \bar{\omega})$ may be positive or negative depending on the choice of v (and hence λ), so the observations are compatible with both $p \cdot Z_\omega(p') > 0$ and $p \cdot Z_\omega(p') \leq 0$.

To sum up, we have shown that for any price p' which is sufficiently close to p, there exists an endowment perturbation which will reveal the sign of $p \cdot Z_\omega(p')$. However, in general, endowment changes need not be informative. A pair (p, ω) and (p', ω') is informative if, and essentially, only if, it is either not N-monotonic or strongly N-monotonic.

6. Comparative Statics of Production Economies

Consider a production economy with $l + m$ goods. The first l goods give utility to all consumers, while the other m goods (m could equal zero) do not and are useful only as factors of production. Production is chosen from a production possibility set Y contained in R^{l+m}. We denote the typical element in Y by (u, v) where u is in R^l and v is in R^m. A typical price vector will be written as (p, q) where p is the price of the first l goods and q the price of the m factors. We assume that there is a price vector (p, q) in R^{l+m}_{++} such that $\arg\max_{(u,v)\in Y}(p, q) \cdot (u, v)$ is non-empty and denote by P the set of such prices, which we assume is contained in R^{l+m}_{++}. For (p, q) in P, the *supply correspondence* S_Y has P as its domain and is defined by $S_Y(p, q) = \arg\max_{(u,v)\in Y}(p, q) \cdot (u, v)$. The *profit function* Π_Y is defined by $\Pi_Y(p, q) = (p, q) \cdot S_Y(p, q)$.

The consumer/household sector consists of a set A of agents. Agent a has a demand function f_a which we assume is nice. He has an endowment ω_a of consumer goods and μ_a of factors. He also has a claim on profits of α_a, so $\alpha_a \geq 0$ and $\sum_{a\in A} \alpha_a = 1$. At price (p, q) in P, agent a's income is $y_a = (p, q) \cdot (\omega_a, \mu_a) + \alpha_a \Pi_Y(p, q)$ and his demand is $f_a(p, y_a)$, which is in R^l_{++}. This completes our specification ot the production economy. Since we shall be studying the behavior of this economy when endowments and technology are varied , we will emphasize its dependence on these by denoting the economy by $\mathcal{E}(\omega, \mu, Y)$, where $\mu = (\mu_a)_{a \in A}$ and ω is similarly defined.

A price (p, q) in P is an *equilibrium price* of this economy if there is $s(p, q)$ in $S(p, q)$ such that $(F(p, y), 0) = s(p, q) + (\bar{\omega}, \bar{\mu})$. This needs a little explaining. The right hand side of this equation is just the economy's aggregate supply, with $\bar{\omega}$ and $\bar{\mu}$ being, respectively, the mean endowments of utility giving goods and factors. The left hand side is aggregate demand. Following the notation of the previous sections, $F(p, y)$ is the mean demand of the household sector at price p and at the income distribution generated by (p, q), which we denote by y. Note that $F(p, y)$ lives in R^l_{++}, so to make the equation formally correct, we write aggregate demand as $(F(p, y), 0)$, where 0 is the zero vector in R^m.

The type of technology changes we will consider are those of the factor augmenting variety, which are standard in macroeconomic models. For θ in R^m_{++}, we define the set Y_θ by $Y_\theta = \{(u, \theta \otimes v : (u, v) \in Y\}$, where $\theta \otimes v$ refers to the vector $(\theta^1 v^1, \theta^2 v^2, ..., \theta^m v^m)$.

Technology changes of this sort are easy to analyse since its equilibrium effects could be replicated by a suitable change in the economy's factor endowment.

LEMMA 6.1: *The price vector (p, q) is an equilibrium price of $\mathcal{E}(\omega, \mu, Y_\theta)$ if and only if $(p, \theta^{-1} \otimes q)$ is an equilibrum price of $\mathcal{E}(\omega, \theta \otimes \mu, Y)$.*

(Note that θ^{-1} refers to the vector whose ith entry is $1/\theta^i$, and $\theta \otimes \mu$ denotes $(\theta \otimes \mu_a)_{a \in A}$.)

Sketch of proof: The lemma follows immediately from the the following observations:

(a) the vector (u, v) is in $S_{Y_\theta}(p, q)$ if and only if $(u, \theta \otimes v)$ is in $S_Y(p, \theta^{-1} \otimes q)$;

(b) from (a), we know that $\Pi_{Y_\theta}(p, q) = \Pi_Y(p, \theta^{-1} \otimes q)$ and, furthermore, the income of agent a at price (p,q) in the economy $\mathcal{E}(\omega,\mu,Y)$ is equal to the income of the agent a at price $(p,\theta^{-1}\otimes q)$ in the economy $\mathcal{E}(\omega,\theta\otimes\mu,Y)$;

(c) consequently, household demand for consumer goods in the two situations is identical, while supply of consumer goods is also the same in the two situations. QED

Extending our results from exchange to production economies is in fact a rather straightforward exercise, since the profit maximization hypothesis guarantees that the supply correspondence has nice monotonic properties. In particular, profit maximization guarantees that for (u,v) in $S(p,q)$ and (u^*,v^*) in $S(p^*,q^*)$, we have $(p,q)\cdot((u,v)-(u^*,v^*))\geq 0$. We say that S has the *strong supply property* if this condition holds and in addition, the inequality is strict when there exists t such that$p=tp^*$ and $q\neq tq^*$. This slight strengthening is a mild curvature condition on the production set, and ensures that the result in Theorem 6.2 takes the form of a strict, rather than weak, inequality.

THEOREM 6.2: *Let (p,q) and (p',q') be equilibrium prices of $\mathcal{E}(\omega,\mu,Y)$ and $\mathcal{E}(\omega',\mu',Y_\theta)$ respectively and denote by y and y' the income distributions generated at the two equilibria. Suppose that the pair $(p,F(p,y))$ and $(p',F(p',y'))$ is N-monotonic and that S_Y has the strong supply property. Then the pair $((p,q),(\bar{\omega},\bar{\mu}))$ and $((p',\theta^{-1}\otimes q'),(\bar{\omega}',\theta\otimes\bar{\mu}'))$ satisfies N-monotonicity; when (p,q) and $(p',\theta^{-1}\otimes q')$ are not collinear, the normalizing vector can be chosen in the form $(a,0)$, where $a\in R^l_+\setminus\{0\}$ and $0\in R^m$.*

Proof: We first assume that (p,q) and $(p',\theta^{-1}\otimes q')$ are not collinear. From Lemma 6.1 and its proof, we know that $(p',\theta^{-1}\otimes q')$ is an equilibrium price in $\mathcal{E}(\omega',\theta\otimes\mu,Y)$ and the income distribution generated by this price in this economy is the same as that generated by (p',q') in $\mathcal{E}(\omega',\mu',Y_\theta)$, which is y. Therefore, by definition of N-monotonicity, there is

$a > 0$ in R^l such that $(p' - \lambda p) \cdot (F(p, y) - F(p', y')) \leq 0$, for λ satisfying $p' \cdot a = \lambda p \cdot a$, with equality only if p and p' are collinear. (In the case where p and p' are not collinear, simply choose λ to satisfy $p' = \lambda p$ and a to be any vector in $R^l_+ \setminus \{0\}$.)

By the definition of S_Y, $(p', \theta^{-1} \otimes q') \cdot ((u', \theta \otimes v') - (u, v)) \geq 0$ and $(p, q) \cdot ((u', \theta \otimes v') - (u, v)) \leq 0$, where $(u', \theta \otimes v')$ is in $S_Y(p', \theta^{-1} \otimes q')$ and (u, v) is in $S_Y(p, q)$. If $(p', \theta^{-1} \otimes q')$ and (p, q) are not collinear, but p and p' are collinear, then by the strong supply property, both inequalities must be strict. Therefore we obtain,

$$((p', \theta^{-1} \otimes q') - \lambda(p, q)) \cdot ((u', \theta \otimes v') - (u, v)) \geq 0,$$

with the inequality being strict if p and p' are collinear. Since $(\bar{\omega}, \bar{\mu}) = (F(p, y), 0) - (u, v)$ and $(\bar{\omega}', \bar{\mu}') = (F(p', y'), 0) - (u', \theta \otimes v')$ we find that

$$\begin{aligned} & \Big((p', \theta^{-1} \otimes q') - \lambda(p, q)\Big) \cdot ((\bar{\omega}, \bar{\mu}) - (\bar{\omega}', \bar{\mu}')) \\ = \; & \Big((p', \theta^{-1} \otimes q') - \lambda(p, q)\Big) \cdot ((F(p', y'), 0) - (F(p, y), 0) - (u', \theta \otimes v') + (u, v)) < 0. \end{aligned}$$

We now consider the case where (p, q) and $(p', p', \theta^{-1} \otimes q')$ are collinear. Since the pair $(p, F(P, y))$ and $(p', F(p', y'))$ is N-monotonic, it also satisfies the weak axiom, and therefore we can find λ such that $(p' - \lambda p) \cdot (F(p, y) - F(p', y')) < 0$. Since (p, q) and $(p', \theta^{-1} \otimes q')$ are collinear, $S_Y(p, q) = S_Y(p', \theta^{-1} \otimes q')$, which means that $(p', \theta^{-1} \otimes q') \cdot (u', \theta \otimes v') - (u, v)) = (p, q) \cdot (u', \theta \otimes v') - (u, v)) = 0$. In this way,

$$(p', \theta^{-1} \otimes q') - \lambda(p, q)) \cdot ((\bar{\omega}, \bar{\mu}) - (\bar{\omega}', \bar{\mu}')) = (p', \theta^{-1} \otimes q') - \lambda(p, q)) \cdot [(F(p', y'), 0) - (F(p, y), 0)]$$

must be strictly negative. QED

It is quite obvious that we could easily have replaced the condition in Theorem 6.2 that the pair $(p, F(p, y))$ and $(p', F(p', y'))$ satisfies N-monotonicity with conditions guaranteeing

N-monotonicity - for example, the conditions (B1) and (B2) developed in Section 3. To appreciate the significance of this theorem imagine that there is an increase in the endowment of the first factor, i.e., good $m+1$, causing prices to move from (p,q) to (p',q'). The result says that provided N-monotonicity is satisfied in the household sector, the price q^1 will be greater than $(q')^1$, using the normalization, $p \cdot a = p' \cdot a$. This normalization means that factor prices are measured in 'real' terms, i.e., in terms of the number of units of some representative bundle of consumer goods. In short, when a factor's endowment is increased, its real factor price will fall.

What if endowments remain the same, but there is a technology change which augments the first factor, i.e., $\theta = (k, 1, 1, ..., 1)$, with $k > 1$? This change is equivalent to an increase in the endowment of factor 1 by the multiple k. If the conditions of Theorem 5.1 are satisfied, we may conclude that q^1 is greater than $(q')^1/k$. In other words, we obtain the intuitive conclusion that the price of factor 1 will not rise by a multiple of more than k.

We may view Theorem 6.2 as a statement on the 'observable restrictions' of Walrasian equilibria in the spirit of Brown and Matzkin (1996). Assume that an observer can observe all prices, mean household demand and mean endowment. He will then be in a position to determine if N-monotonicity is satisfied in the household sector; if it is, then the model predicts that the same property will be extended to the whole economy. If he observes that it is not, the model has been refuted.

7. Welfare Theorems for Production Economies

We show in this section that the line of thought we began exploring in Section 6 could be used to extract some interesting welfare properties in production economies. It is appropriate for this purpose to have a somewhat broader conception of an agent's demand function. We assume that each agent's demand is a function of the prevailing price p and the whole income distribution y. Formally, we write his demand as $f_a(p, y)$, satisfying $p \cdot f_a(p, y) = y_a$. Essentially, we are allowing for demand in the household/consumer sector to be determined by some unmodelled social interaction among agents. The demand function f_a is said to be *individualistic* if it is independent of $y_{a'}$, $a' \neq a$. Mean household demand is given, as before, by $F(p, y) = \sum_{a \in A} f_a(p, y)/|A|$. The production/supply side of the economy remains as we have described it in Section 6.

An allocation which gives x_a to agent a in A (henceforth to be written as $(x_a)_{a \in A}$) is a *feasible allocation* if there is (u, v) in Y such that $(\bar{x}, 0) = (u, v) + (\bar{\omega}, \bar{\mu})$, where $\bar{x} = \sum_{a \in A} x_a/|A|$. A feasible allocation is *supportable* as a *Walrasian equlibrium with transfers* if there is (p, q) in P (called a *with-transfers equilibrium price*), an income distribution $y = (y_a)_{a \in A} \gg 0$ and $s(p, q)$ in $S(p, q)$ such that $x_a = f_a(p, y)$ and $(F(p, y), 0) = s(p, q) + (\bar{\omega}, \bar{\mu})$. The term "with transfers" alludes to the fact that we can think of the agent a as receiving a lump sum transfer of $t_a = y_a - [(p, q) \cdot (\omega_a, \mu_a) + \alpha_a \Pi_Y(p, q)]$, with $\sum_{a \in A} t_a = 0$.

A careful perusal of the proof of Theorem 6.2 will show that its conclusion is still true if (p, q) and (p', q') in that theorem were not equilibrium prices, but prices supporting Walrasian equilibria with transfers of two different economies. What is perhaps even more interesting is to consider what happens when we apply those arguments to two equilibria with transfers of the same economy. Before we do that, we need to introduce another notion of normality. We say that the function F satisfies *weak aggregate normality* if for all p, $F(p,y) > F(p,y')$ whenever $y > y'$. Note that this is a rather mild condition; in particular, it will hold if all agents have individualistic demand functions that individually obey normality.

THEOREM 7.1: *Consider two distinct allocations in $\mathcal{E}(\omega,\mu,Y)$, both supportable as Walrasian equilibria with transfers; the first supported with price (p,q) and income distribution y, and the second with price (p',q') and income distribution y'. Then $p'\cdot F(p',y') \geq p'\cdot F(p,y)$. Suppose, in addition, that F satisfies weak aggregate normality.*

(i) If F satisfies the limited weak axiom between $(p',F(p',y''))$ and $(p,F(p,y))$ where $y'' = (p'\cdot f_a(p,y))_{a\in A}$, then $y' \not\geq y''$.

(ii) If F satisfies the limited weak axiom between $(p',F(p',y))$ and $(p,F(p,y))$, and (p,q) is normalized to satisfy $p'\cdot F(p,y) = \bar{y}$, then $y' \not\geq y$.

Proof: Suppose that the supply at the Walrasian allocation supported by (p,q) is $s(p,q)$ and the supply at the other Walrasian allocation is $s(p',q')$. By the definition of the supply correspondence, $(p',q')\cdot s(p',q') \geq (p',q')\cdot s(p,q)$. Adding $(\bar{\omega},\bar{\mu})$ to both sides, we obtain $(p',q')\cdot(F(p',y'),0) \geq (p',q')\cdot(F(p,y),0)$, which is what we want.

We now prove (i) by contradiction. Since $p'\cdot F(p,y) = \bar{y}''$, the limited weak axiom guarantees that $p\cdot F(p',y'') > \bar{y}$. If $y' = y''$, we obtain $p\cdot F(p',y') > \bar{y}$ and so clearly the pair $(p',F(p',y'))$ and $(p,F(p,y))$ will satisfy the weak axiom. If $y' > y''$, then F will satisfy aggregate normality between (p',y') and (p',y''), which together with the limited weak axiom assumption, implies that the pair $(p',F(p',y'))$ and $(p,F(p,y))$ will satisfy N-monotonicity and therefore the weak axiom (conditions (A1) and (A2) in Section 3). So we know there must exist some positive λ such that $(p'-\lambda p)\cdot(F(p,',y')-F(p,y)) < 0$. The rest of the argument largely duplicates the proof of Theorem 5.2. By the supply property,

$$((p', q') - \lambda(p, q)) \cdot ((u', v') - (u, v)) \geq 0,$$

where (u', v') and (u, v) are the amounts supplied at prices (p', q') and (p, q) respectively. Taking the difference of these two inequalities, we obtain

$$((p', q') - \lambda(p, q)) \cdot ((\bar{\omega}, \bar{\mu}) - (\bar{\omega}, \bar{\mu})) < 0,$$

which is clearly not possible.

To show (ii), we first note that by the limited weak axiom $p \cdot F(p', y) > \bar{y}$ and if $y' = y$ we obtain $p \cdot F(p', y') > \bar{y}$. This means that the pair $(p', F(p', y'))$ and $(p, F(p, y))$ will satisfy the weak axiom. If $y' > y$, then F will satisfy aggregate normality between (p', y') and (p', y), so by conditions (B1) and (B2) in Section 3, the pair $(p', F(p', y'))$ and $(p, F(p, y))$ will satisfy N-monotonicity and hence the weak axiom. This is impossible, using exactly the same arguments as in (i) . QED

The intuition behind Theorem 7.1 is nicely captured in Figure 3. It shows the production possibility curve of an economy with two consumer goods, with $F(p, y)$ and $F(p', y')$ being the mean allocations at the two Walrasian equilibria with transfers. If the limited weak axiom holds, the point $F(p', y'')$ in part (i) of the theorem must be to the right of $F(p, y)$. The figure shows that F must violate aggregate normality between (p', y') and (p', y'') - which implies, since F obeys weak aggregate normality, that $y' \not\geq y''$. (The same intuitive argument will apply to part (ii) of the theorem, with $F(p', y)$ replacing $F(p', y'')$.)

Both parts of this corollary could be interpreted as welfare theorems for production economies. Just as the classical welfare theorems tell us that Walrasian allocations are not rankable in terms of the utility they achieve, so Theorem 7.1 tells us that they could not be ranked in terms of income either.

Since the income distribution can always be scaled up or down depending on the scaling of the equilibrium price, such a claim can only make sense for a particular normalization of prices. For the two allocations considered in the theorem, the mean household demands are $F(p, y)$ and $F(p', y')$. By multiplying (p', q') with a scalar if necessary, we can guarantee that $p' \cdot F(p, y) = \bar{y}$, i.e., the value of $F(p, y)$ at price p' exactly equals the mean income $\bar{y}$. Theorem 7.1 says that $\bar{y}' = p' \cdot F(p', y') \geq \bar{y} = p' \cdot F(p, y)$. In other words, the price p' always gives a higher *mean* value to the allocation it supports compared to any other equilibrium allocation. The issue is whether it is also possible that $y' \geq y$. Theorem 7.1(ii) says that this not possible provided aggregate demand satisfies the fixed income version of the weak axiom and weak aggregate normality.

Suppose that we are in the classical environment where each agent has an individualistic demand function derived from utility maximization. In that case, the classical welfare theorems are valid. The first welfare theorem says that equilibrium allocations are Pareto optimal. Theorem 7.1(ii) gives us another welfare property of the equilibrium allocation: subject to a natural normalization, the income distribution it generates will not be dominated by the income distribution induced at any equilibrium with transfers and hence, by the second welfare thoerem, at any other Pareto optimal allocation.

We now consider Theorem 7.1(i). Since $p' \cdot F(p', y') \geq p' \cdot F(p, y)$, we obtain $\bar{y}' \geq \bar{y}''$, and yet Theorem 7.1(i) says that y' will never dominate y''. In other words, while the allocation $(f_a(p', y'))_{a \in A}$ has a higher mean value than the allocation $(f_a(p, y))_{a \in A}$ when both are evaluated at the price p', at least one agent will end up with a bundle that is valued less, i.e., for this agent, $p' \cdot f_a(p,y) > p' \cdot f_a(p',y')$. It is important to note that Theorem 7.1(i) (but not Theorem 7.1(ii)) tells us nothing new if we assume that all agents have individualistic demand functions derived from utility maximization. In that case, the pair $(p', F(p', y''))$ and $(p, F(p,y))$ will certainly satisfy the limited weak axiom because each agent individually satisfies the limited weak axiom. However, essentially the same assumptions will guarantee that the first welfare theorem holds and the allocation $(f_a(p,y)_{a\in A}$ is Pareto optimal, in which case y' cannot dominate y'' and our conclusion is trivial.

What is interesting about Theorem 7.1 is that, unlike the classical welfare theorems, neither the assumptions nor the conclusions make any reference to preferences. The theorem imposes restrictions on *aggregate* demand while keeping open the possibility that this structure is achieved through ways other than the classical assumptions. For example, this aggregate structure could be attained through distributional regularities in demand behavior rather than from any property of individual demand derived from utility maximization (see Becker (1962) and Grandmont (1992) for theoretical models of this type).

In situations like these, an equilibrium is still a meaningful concept, while Pareto optimality may no longer make sense. However, there is a natural analog to Pareto optimal allocations, namely, those allocations which are achievable as equilibrium allocations after transfers. Theorem 7.1 provides a welfare theorem in this context. It considers (in cases (i) and (ii)) two ways of measuring the income distribution at two allocations supportable as Walrasian equilibria with transfers. With both ways, it shows that the income distributions at the two allocations are such that neither distribution dominates the other.

8. Comparative Statics when Markets are Incomplete

In this section, we show that the comparatvie statics we have developed in a complete markets context can be extended to financial markets with an incomplete asset structure. We assume that there are l states of the world, and m securities, with $m \leq l$. The $m \times l$ matrix D gives the payoffs of these securities, with the ijth entry being the payoff of the ith security in state j. We assume that there is some portfolio θ which guarantees positive consumption in all states, i.e., $D^T\theta \gg 0$. An agent a in this economy seeks to maximize his utility, given by the function $U_a : R^l_{++} \to R$. The agent has an endowment of securities ω_a in R^m satisfying $D^T\omega_a > 0$.

All the assumptions we have made so far, with the possible exception of the last are completely standard. A weaker alternative to the last assumption is to assume that the agent has an endowment of consumption in different states of nature, i.e., in R^l_+, which may or may not be achievable by some security portfolio. It is not completely clear the extent to which our results in this section will carry over to this more general case; the proofs in any case will have to more complicated, so we will stick with our stronger assumption.

Given a security price vector q in R^m, the agent's budget set in securities is given by $B(q, \omega_a) = \{\theta \in R^m : q^T\theta \leq q^T\omega_a\}$. The agent's utility maximization problem is to maximize $U_a(D^T\theta)$ subject to $\theta \in B(q, \omega_a)$. We assume that this problem has a solution if and only if q is *arbitrage free*, which, for our purpose, will mean that there is a vector $p \gg 0$ such that $Dp = q$. We refer to p as a state price vector of q. We let Q be the set of arbitrarge free prices, which, by its definition, must be a convex cone in R^m. For each q in Q, we further assume that the solution to the utility maximizaton problem is unique. This solution is the agent a's *demand for securities* and will be denoted by $g_a(q, \omega_a)$; so g_a is a function from $Q \times \Omega$ to R^m, where $\Omega = \{\omega \in R^m : D^T\omega > 0\}$. We assume also that the budget identity is satisfied, i.e., $q \cdot g_a(q, \omega_a) = q \cdot \omega_a$. The properties we have imposed on g_a are completely standard and could be justified by more basic assumptions on the utility function if we so wish (see, for example, Duffie (1992)).

It is helpful and common in this context to examine a related demand concept. For p in R^l_{++} and y in R_+, we define $B^*(p, y) = \{x \in R^l_{++} : x = D^T\theta \text{ for some } \theta \in R^m, \text{and } p^T x \leq y\}$. So $B^*(p, y)$ is the familiar budget set of classical demand theory, with the additional restriction that the consumption bundle x must be achievable by some securities portfolio. It is not hard to check that the existence of g_a guarantees that the problem of maximizing $U_a(x)$ subject to x being in $B^*(p, y)$ always has a unique solution, which we denote by $f_a(p, y)$. This is the agent a's *demand for consumption* across different states of the world at (p, y). The fact that g_a satisfies the budget identity will guarantee that f_a satisfies the budget identity, i.e., $p \cdot f_a(p, y) = y$.

Clearly the two concepts of demand are closely related. The next lemma lists some easy to check properties linking g_a and f_a. The proofs are omitted.

LEMMA 8.1: *(i)* $D^T B(q, \omega_a) = B^*(p, q \cdot \omega_a)$, *where* p *satisfies* $Dp = q$;

(ii) $D^T g_a(q, \omega_a) = f_a(p, q \cdot \omega_a)$ *for any* p *satisfying* $Dp = q$; *and*

(iii) $f_a(p, y) = D^T g_a(Dp, \hat{\theta})$ *where* $\hat{\theta}$ *is any vector which satisfies* $p \cdot D^T\hat{\theta} = y$.

Using these observations, we can translate structural properties on f_a like N-monotonicity into similar properties on g_a. We will show this formally at a later stage; for now we will establish the conditions for f_a to be N-monotonic. Of course, this problem has, in essence, already been solved. Proposition 2.2 says that two conditions need to be satisfied by f_a: it must satisfy the limited weak axiom and it must satisfy normality. The only issue is whether these conditions are reasonable in this context.

Normality means that $f_a(p, y) > f_a(p, y')$ if $y > y'$, and while not guaranteed by utility maximization alone, remains a reasonable assumption here. On the other hand, the definition of the limited weak axiom for individual demand given in Section 2 needs a minor adjustment. We assume that f_a satisfies *the limited weak axiom* if the following is true: when $B^*(p, y) \neq B^*(p', y')$ and $p' \cdot f_a(p, y) = y'$, we obtain $p \cdot f_a(p', y') > y$. This definition differs from the one in Section 2 in its requirement that $B^*(p, y) \neq B^*(p', y')$. It is easy to check that this condition holds if Dp and Dp' are not collinear. The reason for this stronger restriction is fairly obvious: because all consumption bundles in the budget set must be generated by some portfolio of securities, the mere fact that p and p' are not collinear does not guarantee that the budget sets are different, and when the two budget sets are the same, $f_a(p, y) = f_a(p', y')$ and we clearly cannot have $p \cdot f_a(p', y') > y$. Our modified version of the limited weak axiom will be valid provided the agent's utility function satisfy standard properties, for example, if it is differentiably strictly convex in the sense of Mas-Colell (1985). We will omit the proof of the next result, which can be proven by essentially duplicating the proof of Proposition 2.2.

PROPOSITION 8.2: *Suppose f_a satisfies normality and the limited weak axiom. Then the pair $(p, f_a(p, y))$ and $(p', f_a(p', y'))$ is N-monotonic whenever Dp and Dp' are not collinear, with a normalizing vector $b = D^T\hat{\theta} > 0$ for some $\hat{\theta}$ in R^m.*

Proposition 8.2 has identified the conditions needed for f_a to satisfy N-monotonicity.

We can define analogous conditions for g_a. We say that g_a satisfies normality if $q \cdot \omega_a > q \cdot \omega'_a$ implies that $D^T g_a(q, \omega_a) > D^T g_a(q, \omega'_a)$. We say that g_a satisfies the limited weak axiom if the following holds: whenever q and q' are not collinear, $q \cdot g_a(q', g_a(q, \omega_a)) > q \cdot g_a(q, \omega_a)$. It is not difficult to check, using Lemma 8.1, that f_a satisfies the limited weak axiom if and only if g_a satisfies the limited weak axiom and f_a satisfies normality if and only if g_a satisfies normality.

A pair (q, θ) and (q', θ'), both in $Q \times R^m$ is N-monotonic if there is a positive number λ such that $(q' - \lambda q) \cdot (\theta' - \theta) < 0$ and when q and q' are not collinear, λ can be chosen such that there is $\hat{\theta}$, with $D^T \hat{\theta} > 0$ and $q' \cdot \hat{\theta} = \lambda q \cdot \hat{\theta}$. We say that g_a satisfies N-monotonicity if for any (q, ω_a) and (q', ω'_a) where q and q' are not collinear, the pair $(q, g_a(q, \omega_a))$ and $(q', g_a(q', \omega'_a))$ is N-monotonic. N-monotonicity says that the price and demand for securities move in opposite directions, provided a suitable portfolio ($\hat{\theta}$ in our definition) is chosen as the *numeraire securities portfolio*. Furthermore, this numeraire portfolio generates weakly positive consumption in all states of the world. The next result is a straightforward consequence of Proposition 8.2.

PROPOSITION 8.3: *Suppose that g_a satisfies normality and the limited weak axiom. Then it is N-monotonic.*

Proof: Let (q, ω_a) and (q', ω'_a) be two elements in $Q \times \Omega$, with q and q' non-collinear, and let p and p' be state price vectors of q and q' respectively. Since g_a satisfies normality and the limited weak axiom, f_a will also satisfy these properties, and so by Proposition 8.2 $(p, f_a(p, q \cdot \omega_a))$ and $(p', f_a(p', q' \cdot \omega'_a))$ is an N-monotonic pair, which means that there is $\hat{\theta}$

satisfying $D^T\hat{\theta} > 0$ with

$$(p' - \lambda p) \cdot (f_a(p', q' \cdot \omega_a) - f_a(p, q \cdot \omega_a)) < 0$$

where λ satisfies $p' \cdot D^T\hat{\theta} = \lambda p \cdot D^T\hat{\theta}$. Since $f_a(p, q \cdot \omega_a) = D^T g_a(q, \omega_a)$ and $f_a(p', q' \cdot \omega_a') = D^T g_a(q', \omega_a')$ we find that $(q, g_a(q, \omega_a))$ and $(q', g_a(q', \omega_a'))$ is N-monotonic. QED

This completes our study of an individual's demand for securities. N-monotonicity is a property that one could expect of individual demand, since the conditions we have found to guarantee it are very mild. An extension of our results to the multi-agent setting can be constructed along essentially the same lines as those laid out in Sections 3 and 4. We confine ourselves to a brief description.

A *financial economy* consists of the set A of agents, each of whom has an endowment of securities ω_a in Ω and a demand function for securities g_a, generated by a utility function U_a. We denote this economy by $\mathcal{F}(\omega)$, where $\omega = (\omega_a)_{a \in A}$. A price q in Q is an equilibrium price if $\sum_{a \in A} g_a(q, \omega_a)/|A| = \bar{\omega}$, where $\bar{\omega}$ is the economy's mean endowment of securities. Related to this financial economy is the exchange economy $\mathcal{E}(D^T\omega)$ where by $D^T\omega$ we mean $(D^T\omega_a)_{a \in A}$. In this economy, the agent a has an endowment of $D^T\omega_a$ and the demand function f_a, generated by U_a. Clearly, q is an equilibrium price in $\mathcal{F}(\omega)$ if and only if p is an equilibrium price of $\mathcal{E}(D^T\omega)$, where p is any state price vector of q.

Now suppose that q is an equilibrium price of $\mathcal{F}(\omega)$ and q' an equilibrium price of $\mathcal{F}(\omega')$ and let p and p' be two corresponding state price vectors. Then p is an equilibrium price of $\mathcal{E}(D^T\omega)$ and p' an equilibrium price of $\mathcal{E}(D^T\omega')$. Quite clearly, the pair $(p, D^T\bar{\omega})$ and $(p', D^T\bar{\omega}')$ is N-monotonic, with a normalizing vector $b = D^T\hat{\theta}$ for some $\hat{\theta}$ in R^m, if and only if $(q, \bar{\omega})$ and $(q', \bar{\omega}')$ is N-monotonic with the normalizing vector $\hat{\theta}$. The latter means, among other things, that when the endowment for a security increases, the security price, subject to the normalization specified here, will fall.

The conditions which guarantee that the pair $(p, D^T\bar{\omega})$ and $(p', D^T\bar{\omega}')$ is N-monotonic include all the conditions identified in Section 4, and in particular, the conditions of Corollary 4.1. In this context they say that, normalizing p' to satisfy $p' \cdot D^T\omega = p \cdot D^T\omega$, the following holds:

(B1) $p' \cdot D^T\bar{\omega}' \neq p' \cdot D^T\bar{\omega}$ and F satisfies aggregate normality between (p', y) and (p', y') where $y = (p \cdot D^T\omega_a)_{a\in A}$ and $y' = (p' \cdot D^T\omega_a')_{a\in A}$; and

(B2) F satisfies the limited weak axiom between $(p', F(p', y))$ and $(p, F(p, y))$.

At least some of the results which support the fixed income version of the weak axiom for market demand remain applicable in this context, so (B2) remains a fairly reasonable assumption. In particular it is known that if each agent a has a utility function U_a satisfying the Milleron-Mitjuschin-Polterovich conditions, then f_a will be monotonic and F will satisfy (B2). These conditions will be satisfied if U_a has the von Neumann-Morgenstern form, i.e., $U_a(x) = \sum_{i=1}^{l} u_a^i(x^i)$ for concave functions u_a^i, and the coefficient of relative risk aversion has a variation of less than 4 across different levels of consumption in different states of the world, i.e.,

$$\left| \frac{x^i(u_a^i)''(x^i)}{(u_a^i)'(x^i)} - \frac{x^j(u_a^j)''(x^j)}{(u_a^j)'(x^j)} \right| < 4$$

for any i and j and positive numbers x^i and x^j (see Quah (personal manuscript) for details).

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Appendix

The proof of Proposition 2.3 relies on the following lemma:

Lemma A: *Suppose that the function f is regular and satisfies the limited weak axiom. Then for all x in R^l_{++}, there is a unique p such that $f(p,1) = x$. The map from x to p, which we denote by P, is continuous and satisfies the following boundary condition: if x_n tends to x on the boundary of R^l_+, $|P(x_n)|$ must tend to infinity.*

Proof: The problem of the existence of p is just a special case of the standard general equilibrium existence problem, and its existence is guaranteed by continuity and the boundary conditions required for the regularity of f (see Debreu (1982)). The uniqueness of p is guaranteed by the limited weak axiom property: the property says that if $p' \cdot f(p,1) = 1$ then $p \cdot f(p',1) > 1$. In particular, $f(p,1) \neq f(p',1)$. So the map P is well-defined.

To see that P is continuous, consider x_n tending to x in R^l_{++}. Since $P(x_n)$ is bounded, it must admit convergent subsequences. We claim that all of them have $P(x)$ as the limit. Suppose $P(x_{n_k})$ tends to $\bar{p}$; then by the continuity of f, $f(P(x_{n_k}),1) = x_{n_k}$ must have a limit of $f(\bar{p},1)$. Therefore, $f(\bar{p},1) = x$, or $\bar{p} = P(x)$.

For the boundary property, we assume to the contrary that for some sequence x_n tending to the boundary, $P(x_n)$ tends to $\bar{p}$. If $\bar{p}$ is in R^l_{++}, by the continuity of f, $f(P(x_n),1) = x_n$ tends to $f(\bar{p},1)$, which is not on the boundary of R^l_+. On the other hand, if $\bar{p}$ is on the boundary of R^l_+, then by the boundary property on f, $|f(P(x_n),1)| = |x_n|$ tends to infinity, which is a contradiction. QED

Proof of Proposition 2.3: We assume that there is $y < y'$ such that f violates normality between (p',y) and (p',y'). (The case of $y > y'$ can be treated in exactly the same way.)

Then $f(p', y)$ is not contained in the set $C = \{x \in R^l_{++} : p' \cdot x = y \text{ and } x < f(p', y')\}$. Choose $\bar{x}$ satisfying $p' \cdot \bar{x} = y$, with $\bar{x}$ not equal to $f(p', y)$ and not contained in C. By the separating hyperplane theorem, there is q in R^l such that $q \cdot \bar{x} = 1$, $q \cdot f(p', y) > 1$ and $q \cdot b < 1$ for all c in C. Define $S = \{x \in R^l_{++} : p' \cdot x = y \text{ and } q \cdot x = 1\}$ and the map $E : S \to R^l$ by $E(x) = P(x) - p'$, where P follows the definition in Lemma A. Note that $x \cdot E(x) = 0$ for all x in S.

Claim 1: *There is x^* in S with $t \cdot E(x^*) = 0$ for all t in $T = \{t \in R^l : p' \cdot t = q \cdot t = 0\}$.*

To see this, we first choose some $\hat{x}$ in S. By the boundary property on P (and therefore E), we know that there is a compact set J, such that $\hat{x} \cdot E(x) > 0$ for x not in J. Given this, we can choose a compact and convex set in S, call it K, which contains J such that $\hat{x} \cdot E(x) > 0$ whenever x is on the boundary of K (with respect to the relative topology). By the Gale-Nikaido-Debreu Lemma (see Debreu (1982)), there is x^* in K such that $x \cdot E(x^*) \leq 0$ for all x in K. Since $\hat{x}$ is in K, we know that x^* is not on the boundary of K.

We claim that $t \cdot E(x^*) = 0$ for all t in T. Without loss of generality, assume that $t \cdot E(x^*) > 0$ for some t. For δ positive and sufficiently close to zero, $x = x^* + \delta t$ must be in K (since x^* is not on the boundary) and so $x \cdot E(x^*) = \delta t \cdot E(x^*) \leq 0$, which is a contradiction. This establishes our claim.

Claim 2: *Suppose x^* satisfies the conditions in Claim 1. Then $c \cdot P(x^*) < y$ for all c in C.*

By our choice of q, for any c in C, $q \cdot c < 1$ while $q \cdot f(p', y) > 1$, so there is α strictly between 0 and 1 such $q \cdot [\alpha f(p', y) + (1 - \alpha) c] = 1$. Clearly then $[\alpha f(p', y) + (1 - \alpha) c] - x^*$ is in T (as defined in Claim 1) so that $([\alpha f(p', y) + (1 - \alpha) c] - x^*) \cdot E(x^*) = 0$. Since $x^* \cdot E(x^*) = 0$, and $E(x^*) = P(x^*) - p'$, we may re-write this equation as $[\alpha f(p', y) + (1 - \alpha) c] \cdot P(x^*) = y$.

By the limited weak axiom, $P(x^*) \cdot f(p', y) > y$, so we may conclude that $c \cdot P(x^*) < y$.

We have shown that there is x^* satisfying $p' \cdot x^* = y$ such that a vector c satisfying the conditions of Lemma 2.1 does not exist. This implies that the pair $(p', f(p', y'))$ and $(P(x^*), x^*)$ violates N-monotonicity. QED

Proof of Proposition 3.3: We first note that by the niceness of $F(\cdot, y)$, for any x satisfying $p \cdot x = \bar{y}$, there is a price p such that $F(p, y) = x$. Since F satisfies the limited weak axiom, p is also unique. The map from x to p, which we denote by P, is continuous and satisfies the boundary property that $|P(x_n)|$ will tend to infinity if x_n tends to x on the boundary of R^l_+. (These claims are just another version of Lemma A.)

Assume that $\bar{y} < \bar{y}'$ (the other case can be treated similarly). Since aggregate normality is violated, $F(p', y)$ is not contained in $C = \{x \in R^l_{++} : p' \cdot x = \bar{y} \text{ and } x < F(p', y')\}$. Following exactly the proof of Proposition 2.3, we can construct q, S, and the continuous map E, defined by $E(x) = P(x) - p'$. The same arguments then lead us to conclude that there is x^* such that $c \cdot P(x^*) < \bar{y}$ for all c in C. By Lemma 2.1, the pair $(p', F(p', y'))$ and $(P(x^*), x^*) = (P(x^*), F(P(x^*), y)$ violates N-monotonicity. QED

Proof of Proposition 4.2: Choose H sufficiently large so that $y'_a = Hy_a > p' \cdot \omega_a$ for all a in A. Note that $\bar{y}' > p' \cdot \bar{\omega}$ and $y' = Hy$. Define $\theta = F(p', y') - \bar{\omega}$ and consider the economy $\mathcal{E}(\omega')$ where $\omega'_a = \omega_a + k_a\theta$, where $k_a = (y'_a - p' \cdot \omega_a) / p' \cdot \theta$ is positive and sums up to one. The economy $\mathcal{E}(\omega)$ has p' as an equilibrium price and a mean endowment $\bar{\omega}' = \bar{\omega} + \theta$. We also have $F(p', y') = F(p'/H, y) = \bar{\omega}'$ and $F(p, y) = \bar{\omega}$. Since all agents have monotonic demand functions, $F(\cdot, y)$ must satisfy the limited weak axiom; since all agents have normal demand functions and y and y' are collinear, F must satisfy aggregate normality between

(p', y) and (p', y'). In short, conditions (B1) and (B2) are both satisfied, so by Corollary 4.1, (p, ω) and (p', ω') is an N-monotonic pair. It is also clear from our construction that if p' is close to p, and F is continuous, we can choose H such that $F(p', y')$ is close to $\bar{\omega}$ and θ is close to zero, in which case ω' will be close to ω. QED

Proof of Proposition 5.2: Suppose firstly, that $(p' - \lambda p) \cdot (\bar{\omega} - \bar{\omega}') < 0$. Then there is c satisfying conditions (i) to (iii) of Lemma 2.1. In particular, (iii) says that $p \cdot (c - \bar{\omega}) > 0$. So the issue is whether there exists x_a, x'_a and x''_a for all a such that conditions (a) to (c) along with the condition that aggregate demand at price p' and income distribution $(p' \cdot \omega_a)_{a \in A}$ is c, i.e., $\sum_{a \in A} x''_a = c$. By Andreu (1982), the answer is 'yes'. On the other hand if $(p' - \lambda p) \cdot (\bar{\omega} - \bar{\omega}') > 0$ for some b, then, by Lemma 2.1, there is c satisfying (i) and (ii) but not (iii), so $p \cdot (b - \bar{\omega}) < 0$. The rest of the proof follows the first case. QED

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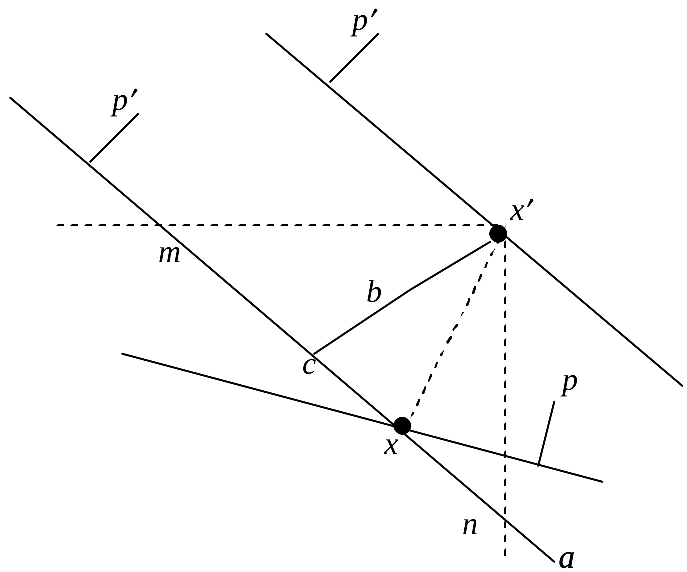

FIGURE 1

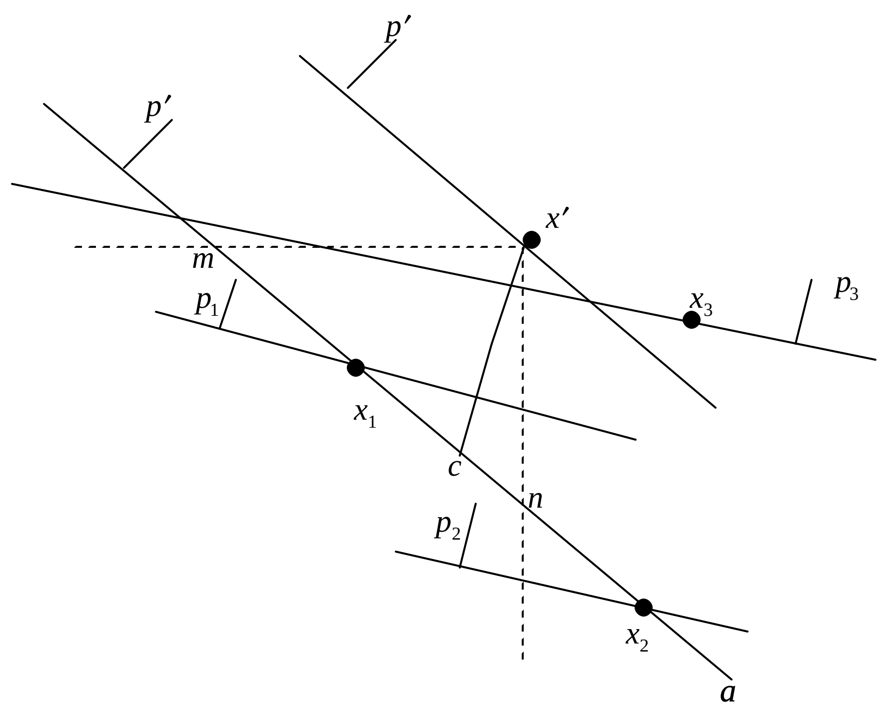

FIGURE 2

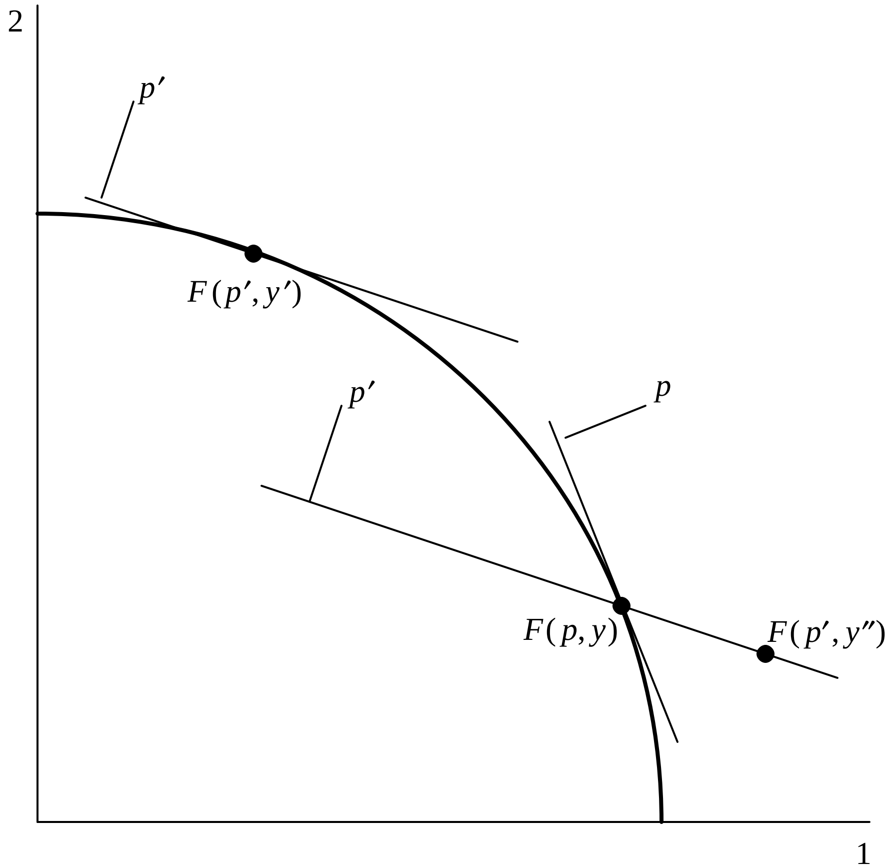

FIGURE 3

FIGURE 2