A MODEL OF PATH-DEPENDENCE IN DECISIONS OVER MULTIPLE PROPOSITIONS[1]

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I develop a new model of sequential decision processes over multiple interconnected propositions, combining social choice theory and propositional logic. I focus on so-called *priority-to-the-past* decision processes. A sequential decision over multiple propositions is path-dependent if its outcome depends on the order in which the propositions are considered. I prove three main results. (1) Path-dependence of a strong kind occurs if and only if the propositional attitudes of an agent (individual or group) on a set of propositions violate strong consistency. Path-dependence of a weaker kind occurs if and only if these propositional attitudes are strongly consistent, but violate deductive closure. (2) If we impose universal domain, anonymity and completeness on a collective *priority-to-the-past* decision process, path-dependencies are unavoidable. (3) Path-dependence makes sequential decision processes vulnerable to manipulation by changes of the decision-path and to manipulation by expression of untruthful views on the propositions. I discuss three escape-routes from the problem of path-dependence: the special support, dictatorship and domain restriction approaches.

1. INTRODUCTION

In classical models of individual and social choice the input to a decision process is typically each agent's *preference ordering*, or *utility function*, *over a set of alternatives*. The *beliefs* of an agent enter these models mostly in the form of the (subjective) probabilities the agent assigns to different states of the world. This paper is concerned with a new model, combining social choice theory and propositional logic. In the new model, the input to a decision process is each agent's *set of views over multiple interconnected propositions*, where the views on some propositions logically constrain the views on others. The propositions are formalized in terms of the propositional calculus, and each agent's views over the propositions are formalized in terms of the agent's propositional attitudes, or truth-value assignments, to these propositions.

Interest in decision problems over multiple interconnected propositions was first sparked by the identification of the "doctrinal paradox" in law and economics (Kornhauser and Sager 1986, 1993; Kornhauser 1992; Chapman 1998, 2002; Brennan

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2001; Pettit 2001a,b; Bovens and Rabinowicz 2003). A three-member court has to make a collective decision on a conclusion, R, on the basis of two *jointly necessary and sufficient* premises, P and Q. Suppose the first judge accepts both P and Q, the second accepts P but not Q, and the third accepts Q but not P. Since the conjunction of P and Q is held to be necessary and sufficient for R, only the first judge individually accepts R, while the other two reject R. We then face the apparently paradoxical situation that there is a majority on each of P and Q and yet there is a majority against R, even though all judges have consistent individual views over the propositions. General examples of sets of multiple interconnected propositions are political programmes, legal doctrines, ideologies, and scientific theories.

Although the "doctrinal paradox" and related decision problems have attracted the attention of lawyers and philosophers, they have only recently begun to be investigated from a more formal social-choice-theoretic perspective. The first formal model of aggregation over multiple interconnected propositions inspired by the "doctrinal paradox" is presented in List and Pettit (2002) and List (2003). A model by Nehring and Puppe (2002) can also represent decision problems over multiple interconnected propositions, but, as discussed below, the types of logical interconnections representable in it are less general than in the List and Pettit model. These models address *simultaneous* decisions over multiple propositions. In this paper, I extend and generalize the List and Pettit model. Decisions over multiple propositions, both at individual and collective levels, are often made not simultaneously, but sequentially. Earlier decisions may affect, and constrain, later ones. The present model addresses *sequential* decision processes over multiple interconnected propositions.

While the methodological goal of the paper is to advance the model itself, the substantive goal is to present some new social-choice-theoretic results. The results concern the phenomenon of *path-dependence*, i.e. the phenomenon that decisions over multiple propositions may depend on the order in which the propositions are considered. The phenomenon may occur at both individual and collective levels. At the level of an individual agent, we sometimes refer to the somewhat puzzling stylized fact that there can be two alternative "arguments", or individual decision processes, with the following properties: both begin with premises the agent is immediately inclined to accept, both use only valid logical inferences, and yet the two arguments lead to opposite conclusions. At the level of a collective decision-making body, it is also sometimes held that the order of decisions matters. Suppose a government decides first to commit itself to keeping taxes low. If it honours this prior commitment, it will have to reject subsequent proposals to increase expenditure, although it would have accepted the same proposals if it had considered them *before* making a decision on taxation.

Path-dependence has been studied in classical choice-theoretic models, where the decision concerns the choice of a winning alternative on the basis of one or multiple preference orderings (for example, Plott 1973). But so far there is no formal work on path-dependence in sequential decisions over multiple interconnected propositions (for a discussion of path-dependence in a legal context, see Stone Sweet 2002).

I provide a formal analysis of path-dependence in such decision processes, distinguishing between *weak* and *strong* forms of path-dependence. The analysis suggests that path-dependencies may occur even when the propositional attitudes of the agent(s) on all propositions are held fixed. The analysis further suggests that, while at the *individual* level path-dependencies are associated with certain violations of perfect rationality, path-dependencies can occur at the *collective* level even when all individual agents are perfectly rational.

Why is the problem of path-dependence relevant? Path-dependence opens up two types of strategic manipulation: manipulation by agenda setting, and manipulation by expression of untruthful views. Given path-dependence, an agenda-setter – whoever determines the decision-path – can influence the outcome of a decision process, even when the propositional attitudes of the agent(s) remain unchanged. This is somewhat similar in spirit to recent work by Aragones, Gilboa, Postlewaite and Schmeidler (2002) on rhetoric, suggesting that it is possible to change an agent's beliefs without giving them new information, just by organizing existing information differently. While Aragones et al. analyse the effects of *rhetorical analogies*, this paper suggests that similar effects may be achieved by suitably choosing the order in which propositions are presented to the agent(s) for consideration.

But suppose the decision-path is held fixed. Why should we *then* be concerned about path-dependence? As we will see, the conditions under which there *exists* an alternative decision-path that *would* change the decision outcome (even if that alternative path is never adopted) may entail the existence of incentives for individuals to strategically express untruthful views. Thus the *possibility* of path-dependence may have adverse effects even if we always stick to the same decision-path. Conditions for avoiding path-dependencies are therefore also relevant for avoiding strategic manipulation.

In Section 2, I introduce the model of sequential decisions over multiple interconnected propositions. In Section 3, I give examples of path-dependencies at individual and collective levels. In Section 4, I present the first main result, stating necessary and sufficient conditions for path-dependence. In Section 5, I present the second main result, showing that, under certain conditions, path-dependence is unavoidable at the collective level. In Section 6, I present the third main result, addressing the two types of strategic manipulation opened up by path-dependence. In Section 7, I explore three escape-routes from path-dependence at the collective level: the special support, dictatorship, and domain restriction approaches. Section 8 contains some concluding remarks. Formal proofs are given in an Appendix.

The model abstracts from many details and complications of real world decision processes. But, on the methodological side, it aims to illustrate how a new formal framework can address a class of decision problems that are not straightforwardly captured by classical social-choice-theoretic models. And, on the substantive side, it aims to highlight, in a distilled form, central features of such decision problems that lead to path-dependencies, and what the implications of these path-dependencies are.

2. A MODEL OF DECISIONS OVER MULTIPLE PROPOSITIONS

2.1. *A Propositional Language*

We define a simple propositional language $\boldsymbol{L}$. Let P, Q, R, S, … be finitely many *propositional letters*. Let $\neg$ ("not"), $\wedge$ ("and"), $\vee$ ("or"), $\rightarrow$ ("implies"), $\leftrightarrow$ ("if and only if") be *logical connectives*. The set of all propositions from $\boldsymbol{L}$ is defined by the following rules:

- Each of the propositional letters P, Q, R, S, … is a proposition.
- If ϕ and ψ are propositions, then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$.
- There are no other propositions.

Propositions without logical connectives are called *atomic propositions*; propositions with logical connectives are called *compound propositions*. A *truth-function* on $\boldsymbol{L}$ is a function $v : \boldsymbol{L} \rightarrow \{1, 0\}$, with the following properties: for any ϕ, $\psi \in \boldsymbol{L}$,

- $v(\neg\phi) = 1$ if and only if $v(\phi) = 0$;
- $v((\phi \wedge \psi)) = 1$ if and only if *both* $v(\phi) = 1$ *and* $v(\psi) = 1$;
- $v((\phi \vee \psi)) = 1$ if and only if *at least one of* $v(\phi) = 1$ *or* $v(\psi) = 1$;
- $v((\phi \rightarrow \psi)) = 1$ if and only if *at least one of* $v(\phi) = 0$ *or* $v(\psi) = 1$;
- $v((\phi \leftrightarrow \psi)) = 1$ if and only if $v(\phi) = v(\psi)$.

Any truth-function v represents one possible assignment of truth-values (1="true" or 0="false") to the propositions in $\boldsymbol{L}$. A proposition $\phi \in \boldsymbol{L}$ is a *tautology* if, for all truth-functions v, $v(\phi) = 1$. A proposition $\phi \in \boldsymbol{L}$ is a *contradiction* if, for all truth-functions v, $v(\phi) = 0$. A set of propositions $\boldsymbol{\Phi} \subseteq \boldsymbol{L}$ *logically entails* a proposition $\psi \in \boldsymbol{L}$ if, for all truth-functions v, [if $v(\phi) = 1$ for all $\phi \in \boldsymbol{\Phi}$, then $v(\psi)=1$]. If $\boldsymbol{\Phi}$ logically entails ψ, we write $\Phi \models \psi$. For any $\phi, \psi \in \boldsymbol{L}$, if $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$, we say that ϕ and ψ are *logically equivalent*. A set of propositions $\Phi \subseteq \boldsymbol{L}$ is *semantically consistent* if there exists a truth-function v such that $v(\phi) = 1$ for all $\phi \in \Phi$; the set Φ is *semantically inconsistent* if there exists no such truth-function.

The set of propositions on which decisions are to be made is defined to be a finite set $X \subseteq \boldsymbol{L}$. We require that the propositions in X are neither tautologies nor contradictions. We assume that X contains at least two distinct atomic propositions, P and Q, and the compound proposition $(P \wedge Q)$. The use of conjunction ($\wedge$) does not imply a loss of generality here: using other logical connectives would yield similar results.[2] For simplicity, for every $\phi \in X$, we identify $\neg\neg\phi$ with ϕ. We further assume that X contains proposition-negation pairs: specifically, whenever $\phi \in X$, we also have $\neg\phi \in X$.

2.2. *The Propositional Attitudes of an Agent*

We first assume that there is only a single agent. The agent can be an individual or a group of individuals acting collectively. We explicitly turn to collective decision problems in Section 5.

We assume that, for each proposition $\phi \in X$, the agent has a *propositional attitude* towards ϕ. That propositional attitude can be interpreted in (at least) two different ways. On one interpretation, it is the agent's *fully endorsed view* on ϕ, in the form of acceptance or non-acceptance of ϕ. On another interpretation, it is the agent's *initial disposition* on ϕ, i.e. the verdict (acceptance or non-acceptance) the agent *would* give on ϕ if they *were* to consider ϕ in isolation, with no reference to other propositions.

If an agent has given some consideration to ϕ, then it may make sense to say that they have a fully endorsed view on ϕ. But if an agent has never thought about ϕ, then they may not have such a fully endorsed view, but they may nonetheless have an *initial disposition* to accept or not to accept ϕ if they *were* to consider ϕ by itself. The agent's initial disposition on ϕ might, for example, be a function of the *prima facie* plausibility of ϕ from the agent's perspective.

The agent's propositional attitudes to the propositions in X are represented by an acceptance/rejection function.

DEFINITION 2.1: An *acceptance/rejection function* (hereafter *AR-function*) is a mapping $\delta : X \to \{1, 0\}$.

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[2] The set of connectives $\{\neg, \wedge\}$ is *expressively adequate*: for any proposition from $\boldsymbol{L}$, there exists a logically equivalent proposition whose only connectives are $\neg$ and $\wedge$.

For each proposition $\phi \in X$, $\delta(\phi)=1$ means that the agent accepts ϕ (or has an initial disposition to accept ϕ), and $\delta(\phi)=0$ means that the agent does not accept ϕ (or has an initial disposition not to accept ϕ). Note that $\delta(\phi)=0$ does not *by itself* mean that the agent accepts the negation of ϕ (or has such an initial disposition). The agent accepts that negation (or has such an initial disposition) if and only if $\delta(\neg\phi)=1$. To simplify the presentation, we will usually refer to $\delta(\phi)=1$ as 'acceptance' of ϕ, and to $\delta(\phi)=0$ as 'rejection' of ϕ, bearing in mind the more precise interpretation offered here.[3]

We consider the properties of δ in Section 4.1 below. If the agent is perfectly rational, then δ is (or can be extended to) a truth-function. We also consider cases where δ falls short of being a truth-function.

It is important to note that, if the agent is a group, assigning propositional attitudes to that group does not require us to make any metaphysical assumptions about 'group minds'. The group's propositional attitude on each proposition might simply be its majority verdict on that proposition, or the outcome of some other aggregation over the propositional attitudes of the group members. We address that issue in Sections 5 and 7.

2.3. *A Sequential Decision Process*

We are concerned with decisions over multiple propositions that are made sequentially. The order in which the propositions in X are considered is represented by a decision-path.

DEFINITION 2.2: A *decision-path* on X is a bijective mapping $\Omega : \{1, 2, ..., k\} \rightarrow X$, where $k = |X|$.

Here $\Omega(1)$, $\Omega(2)$, ..., $\Omega(k)$ are, respectively, the first, second, ..., kth propositions to be considered. There are (at least) two possible interpretations of a decision-path. On one interpretation, a decision-path is the *temporal order* in which the propositions are considered, i.e. earlier propositions in the decision-path come up earlier *in time* than later ones. On another interpretation, a decision-path is the *order of epistemic or logical priority* that is assigned to the propositions, i.e. earlier propositions in the decision-path are considered *epistemically* or *logically* "weightier than", or "prior to", later ones.

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[3] The agent's propositional attitudes are here modelled as acts of assent or dissent, acceptance or rejection, which do not allow of *degrees* of belief. This binary model of propositional attitudes seems more plausible in a context where propositions are *normative* than in a context where propositions are *factual*. An extension of the present approach would allow the use of credence functions taking values in the interval from 0 to 1, i.e. $\delta : X \rightarrow [0, 1]$.

We have distinguished two interpretations of an agent's propositional attitudes towards the propositions in X. In our definition of a sequential decision process, the interpretation of these propositional attitudes as *initial dispositions* is of particular interest. Under this interpretation, the agent enters the decision process without a fully endorsed view on each proposition – but with some initial dispositions. The agent now considers the propositions one by one in the order determined by a given decision-path Ω, and decides at each point in the sequence whether or not to accept the proposition under consideration, say ϕ. The agent has an initial disposition on ϕ and may *either* follow that initial disposition in making a decision on ϕ *or* overrule it for ensuring consistency with previously accepted propositions. The criteria for the acceptance or rejection of each proposition in the sequence are discussed below. The outcome of the decision process may then be interpreted as the *set of fully endorsed views* the agent has *formed* on the propositions *after* having considered them one by one in the order determined by Ω.

What are the criteria by which the agent accepts or rejects each proposition in the sequence? Suppose proposition ϕ is under consideration. There are two cases: *Either* the agent's initial disposition on ϕ is consistent with the propositions (if any) the agent has accepted at earlier points in the sequence: if so, the agent accepts or rejects ϕ according to that initial disposition. *Or* there is a logical inconsistency between the initial disposition on ϕ and some previously accepted propositions: if so, the agent requires a method of resolving this inconsistency. We call such a method a *conflict resolution rule*.

If the agent uses the *priority-to-the-past* rule – defined formally below –, then they resolve the inconsistency by accepting the logical implications of previously accepted propositions and overruling the initial disposition on the new proposition. Under that rule, the decisions constrain each other in an asymmetrical way: earlier decisions constrain later ones, but not vice-versa. This captures the notion that prior decisions ("prior commitments") are harder to overrule than the initial disposition on a new proposition. Precedent-based decision making might be interpreted as a form of priority-to-the-past decision making. Other conflict resolution rules are conceivable, for instance a *priority-to-the-present* rule, by which the inconsistency is resolved by accepting the initial disposition on the new proposition and revising previously accepted propositions in a suitable manner.

For simplicity, we only consider decision processes using the *priority-to-the-past* rule, but the model can be generalized. The priority-to-the-past rule seems most plausible in the following cases:

- when we adopt the epistemic or logical (as opposed to temporal) interpretation of a decision-path: the priority-to-the-past rule is then an implementation of the epistemic or logical priority of earlier propositions over later ones;
- when we adopt the temporal interpretation of a decision-path, *but* the temporal order in which the propositions occur coincides naturally with their order of epistemic or logical priority;
- when we model legal or political decision processes in which a temporal asymmetry is determined
  - by the law (e.g. by the demands of precedent), or
  - by external demands such as
    - political credibility (e.g. if the agent fails to honour prior commitments, the agent may risk losing credibility), or
    - feasibility (e.g. reversing previous decisions may be too costly);

  the priority-to-the-past rule is then an implementation of the requirement that prior decisions or commitments should be honoured.

The model is formally neutral with regard to different such interpretations. The reference to "time *t*" can easily be replaced with a reference to "step *t*".

DEFINITION 2.3: A *priority-to-the-past decision process* is the following procedure.

Consider the propositions in the order determined by a given decision-path Ω, i.e. proposition $\phi_1 := \Omega(1)$ at time 1, proposition $\phi_2 := \Omega(2)$ at time 2, …, proposition $\phi_k := \Omega(k)$ at time k, where $k = |X|$.

For each time $t \in \{1, 2, \ldots, k\}$, let Φ_t denote the set of all propositions accepted at times 1, 2, …, t. We define Φ_t inductively as follows (adding time 0):

$\underline{t = 0}$: $\Phi_0 := \emptyset$.

$\underline{t > 0}$: Proposition ϕ_t is being considered. There are two cases:

Case I: $\Phi_{t-1} \models \phi_t$ or $\Phi_{t-1} \models \neg\phi_t$ (i.e. previously accepted propositions have a logical implication for the acceptance or rejection of ϕ_t). Then

$$\Phi_t := \begin{cases} \Phi_{t-1} \cup \{\phi_t\} & \text{if } \Phi_{t-1} \models \phi_t \\ \Phi_{t-1} \cup \{\neg\phi_t\} & \text{if } \Phi_{t-1} \models \neg\phi_t. \end{cases}$$ [4]

Case II: *neither* $\Phi_{t-1} \models \phi_t$ *nor* $\Phi_{t-1} \models \neg\phi_t$ (i.e. previously accepted propositions have *no* logical implication for the acceptance or rejection of ϕ). Then

$$\Phi_t := \begin{cases} \Phi_{t-1} \cup \{\phi_t\} & \text{if } \delta(\phi_t) = 1 \\ \Phi_{t-1} & \text{if } \delta(\phi_t) = 0. \end{cases}$$ [5]

[4] If we already have $\phi_t \in \Phi_{t-1}$ or $\neg\phi_t \in \Phi_{t-1}$, then $\Phi_t = \Phi_{t-1}$ under this definition.

We define the *outcome set* of the decision process, using the AR-function δ and the decision-path Ω, to be $M(\delta, \Omega) := \Phi_k$.

By definition, a proposition that is inconsistent with previously accepted propositions is never accepted in a priority-to-the-past decision process. We therefore note the following proposition:

PROPOSITION 1: *For any δ and any Ω, $M(\delta, \Omega)$ is semantically consistent.*

Finally, we define the notion of completeness of a priority-to-the-past decision process.

DEFINITION 2.4: $M(\delta, \Omega)$ is *complete* if, for every $\phi \in X$, either $\phi \in M(\delta, \Omega)$ or $\neg\phi \in M(\delta, \Omega)$.

2.4. *The Notion of Path-Dependence*

We can now define invariance under changes of the decision-path, and weak and strong forms of path-dependence. Let δ be given.

DEFINITION 2.5: $M(\delta, \Omega)$ is *invariant under changes of the decision-path* if, for any two decision-paths Ω_1 and Ω_2, $M(\delta, \Omega_1) = M(\delta, \Omega_2)$.

DEFINITION 2.6: $M(\delta, \Omega)$ is *weakly path-dependent* if there exist two decision-paths Ω_1 and Ω_2 such that $M(\delta, \Omega_1) \neq M(\delta, \Omega_2)$.

DEFINITION 2.7: $M(\delta, \Omega)$ is *strongly path-dependent* if there exist two decision-paths Ω_1 and Ω_2 and a proposition $\phi \in X$ such that $\phi \in M(\delta, \Omega_1)$ and $\neg\phi \in M(\delta, \Omega_2)$.

Weak path-dependence is the existence of (at least) two decision-paths with *different* outcomes. Strong path-dependence is the existence of (at least) two decision-paths with *contradictory* outcomes. Strong path-dependence implies weak path-dependence. Also, if $M(\delta, \Omega)$ is complete for all decision-paths Ω, then $M(\delta, \Omega)$ is strongly path-dependent if and only if it is weakly path-dependent: if both $M(\delta, \Omega_1)$ and $M(\delta, \Omega_2)$ are complete, then $M(\delta, \Omega_1) \neq M(\delta, \Omega_2)$ implies the existence of a proposition $\phi \in X$ such that $\phi \in M(\delta, \Omega_1)$ and $\neg\phi \in M(\delta, \Omega_2)$.[6] The two notions of path-

[5] The reason that Φ_t is *not* defined to be $\Phi_{t-1} \cup \{\neg\phi_t\}$ if $\delta(\phi) = 0$ is to allow separate consideration of $\neg\phi_t$ at a different step from ϕ_t in the decision-path. If δ is incomplete, as defined formally below, the definition given here allows incomplete outcome sets (i.e. neither $\phi \notin M(\delta, \Omega)$ nor $\neg\phi \notin M(\delta, \Omega)$) when $\delta(\phi) = 0$, $\delta(\neg\phi) = 0$ and neither ϕ nor $\neg\phi$ is entailed by propositions that are considered earlier than ϕ or $\neg\phi$ in the decision-path.

[6] Bearing in mind the identification of $\neg\neg\phi$ with ϕ.

dependence may differ only if $M(\delta, \Omega)$ violates completeness for some decision-path Ω. In Section 4, I state necessary and sufficient conditions for both kinds of path-dependence, but I focus mainly on the (arguably more interesting) notion of strong path-dependence.

3. TWO EXAMPLES

3.1. *Path-Dependencies at an Individual Level*

Suppose an individual agent has an initial disposition to accept each of the following propositions; i.e. if he were to consider each proposition *in isolation*, he would find the proposition sufficiently plausible to accept.

P : Young people are to be free to decide their own life plans after school.
Q : Compulsory national service reduces the number of crimes committed by young people.
R : Compulsory national service is justifiable.
$(P \rightarrow \neg R)$: If young people are to be free to decide their own life plans after school, then compulsory national service is not justifiable.
$(Q \rightarrow R)$: If compulsory national service reduces the number of crimes committed by young people, then compulsory national service is justifiable.

Although we may notice an inconsistency in the initial dispositions to *accept* each of these propositions, we leave this point aside until the next section, assuming for the moment that there may plausibly exist an agent with these initial dispositions. Suppose the agent engages in a priority-to-the-past decision process over the five propositions.[7]

Case 1. The propositions are considered in the order: P at time 1, $(P\rightarrow\neg R)$ at time 2, R at time 3, Q at time 4, $(Q\rightarrow R)$ at time 5. At times 1 and 2, the agent accepts P and $(P\rightarrow\neg R)$, respectively, following his initial dispositions. At time 3, his initial disposition to accept R is inconsistent with the two previously accepted propositions. By the priority-to-the-past rule, he overrules that initial disposition, and accepts $\neg R$, as implied by the previously accepted propositions. At time 4, he accepts Q, following his initial disposition. At time 5, he faces another logical conflict. Previously accepted

[7] To simplify the presentation of the example, we here assume that each proposition-negation pair is considered at the same time in the decision-path, and that the individual rejects a proposition, ϕ, if and only if he or she accepts the negation, $\neg\phi$, of that proposition. The model introduced in Section 2 above is more general, in that it allows $\neg\phi$ to occur at a different (possibly even non-adjacent) step from ϕ in the decision-path.

propositions are inconsistent with his initial disposition to accept $(Q \rightarrow R)$. By the priority-to-the-past rule, he overrules that disposition, and accepts $\neg(Q \rightarrow R)$. The outcome set of the decision process is $\{P, (P \rightarrow \neg R), \neg R, Q, \neg(Q \rightarrow R)\}$.

Case 2. The propositions are considered in the order: Q at time 1, $(Q \rightarrow R)$ at time 2, R at time 3, $(P \rightarrow \neg R)$ at time 4, P at time 5. The agent follows his initial dispositions at times 1 to 4, accepting each of Q, $(Q \rightarrow R)$, R and $(P \rightarrow \neg R)$. At time 5, he faces an inconsistency between previously accepted propositions and his initial disposition to accept P. By the priority-to-the-past rule, he overrules that disposition, and accepts $\neg P$. The outcome set of the decision process is $\{Q, (Q \rightarrow R), R, (P \rightarrow \neg R), \neg P\}$.

The two cases lead to mutually contradictory outcome sets. Only a single agent is involved, and that agent's initial dispositions on the propositions are the same in both cases. In each case the agent uses only valid logical inferences. The only difference between the two cases is the decision-path. What has happened?

3.2. *Path-Dependencies at a Collective Level*

The second example is a version of an example given by Pettit (2001b). Suppose a multi-member government has to make decisions on the following propositions.

P : Spending on education shall be increased.
Q : Spending on health care shall be increased.
R : Spending on defence shall be increased.
S : Taxes shall be increased.

The government members unanimously agree that the propositions are interconnected as follows: P, Q and R can be accepted simultaneously *only if* S is also accepted, i.e. $((P \wedge Q \wedge R) \rightarrow S)$. The argument for $((P \wedge Q \wedge R) \rightarrow S)$ is that increasing spending on *all* of education, health care and defence necessitates a tax increase, while increasing spending on *two or fewer* of these items is possible without a tax increase.

For simplicity, we assume that the government consists of three individuals, with views on the propositions as shown in Table I. Each individual's set of views is consistent. The government determines its initial disposition on each proposition by majority voting over the individual views, and uses the priority-to-the-past rule for resolving conflict in the sequential decision process.

TABLE I

	P "increase spending on education"	Q "increase spending on health care"	R "increase spending on defence"	S "increase taxes"	$((P \wedge Q \wedge R) \rightarrow S)$
Individual 1	yes	yes	no	no	yes
Individual 2	yes	no	yes	no	yes
Individual 3	no	yes	yes	no	yes

Case 1. The propositions are considered in the order: S at time 1 (January), P at time 2 (February), Q at time 3 (March), R at time 4 (April). At time 1, S is unanimously rejected. At time 2, P is accepted by a 2/3 majority. At time 3, Q is accepted by a 2/3 majority. At time 4, the government faces a logical conflict. A 2/3 majority supports R, but the government is already committed to increasing spending on two items (by having accepted P and Q) and to not increasing taxes (by having rejected S). By the priority-to-the-past rule, the government accepts $\neg R$, overruling its positive majority verdict on R to ensure consistency (and to avoid losing credibility and re-election). The outcome set of the decision process is $\{P, Q, \neg R, \neg S\}$.

Case 2. The propositions are considered in the order: P at time 1, Q at time 2, R at time 3, S at time 4. At times 1, 2 and 3, respectively, P, Q and R are each accepted by 2/3 majorities. At time 4, the government faces a logical conflict. S is unanimously rejected, but the government is already committed to increasing spending on three items (by having accepted P, Q and R). By the priority-to-the-past rule, the government accepts S, overruling the negative majority verdict on S to ensure consistency. The outcome set of the decision process is $\{P, Q, R$ and $S\}$.

Cases 1 and 2 lead to mutually contradictory outcome sets. The views of the individual government members are identical in both cases, as is the priority-to-the-past decision method. The only difference between the two cases is the decision-path. This suggests that, if a group's propositional attitude on each proposition is defined by majority voting over the individual views on that proposition, then the resulting priority-to-the-past decision process may be path-dependent. Is there any other way of defining the group's propositional attitudes such that path-dependencies can be avoided?

4. RATIONALITY VIOLATIONS AND PATH-DEPENDENCE

4.1. *Rationality Conditions on Propositional Attitudes*

We first introduce four rationality conditions which an agent's propositional attitudes, represented by the AR-function δ, may or may not satisfy.

DEFINITION 4.1: The AR-function δ is *complete* if, for any $\phi \in X$, $\delta(\phi)=1$ or $\delta(\neg\phi)=1$.

Completeness is the requirement that the agent should accept *at least one* member of each proposition-negation pair.

DEFINITION 4.2: The AR-function δ is *weakly consistent* if there exists no $\phi \in X$ such that $\delta(\phi)=1$ and $\delta(\neg\phi)=1$.

Weak consistency is the requirement that the agent should accept *at most one* member of each proposition-negation pair.

DEFINITION 4.3: The AR-function δ is *strongly consistent* if the set $\{\phi \in X : \delta(\phi)=1\}$ is semantically consistent.

Strong consistency is the requirement that it should be possible for the propositions accepted by the agent to be simultaneously true. While weak consistency is a narrow (syntactic) notion of consistency, strong consistency is a broader (semantic) notion of consistency. The two notions do not coincide. If the agent accepts P, $(P \rightarrow Q)$ and $\neg Q$, the agent's AR-function is weakly consistent: the agent does not accept a proposition and its negation simultaneously. But the AR-function is not strongly consistent: P, $(P \rightarrow Q)$ and $\neg Q$ cannot be simultaneously true.

For any subset $\Phi \subseteq X$, we write $\delta(\Phi) = 1$ to mean [$\delta(\phi)=1$ for all $\phi \in \Phi$].

DEFINITION 4.4: The AR-function δ is *deductively closed* if the following holds: for any $\Phi \subseteq X$ and any $\phi \in X$, if $\delta(\Phi) = 1$ and $\Phi \models \phi$, then $\delta(\phi)=1$.

Deductive closure is the requirement that the agent should accept all implications of other propositions he or she accepts.

The four conditions are not all logically independent from each other. Strong consistency implies weak consistency. The conjunction of weak consistency and deductive closure implies strong consistency.[8]

It is easily seen that δ satisfies strong consistency if and only if δ can be extended to a truth-function $v : \boldsymbol{L} \rightarrow \{1, 0\}$ such that, for all $\phi \in X \subseteq \boldsymbol{L}$, $\delta(\phi) = v(\phi)$. By Lemma 1 in the Appendix, δ violates strong consistency if and only if there exist two semantically consistent subsets $\Psi_1, \Psi_2 \subseteq X$ and a proposition $\phi \in X$ such that [$\delta(\Psi_1) = 1$ and $\Psi_1 \models \phi$] and [$\delta(\Psi_2) = 1$ and $\Psi_2 \models \neg\phi$]. If the latter condition is met, we say that δ *violates strong consistency with respect to* ϕ. We say that δ *violates weak consistency*

[8] All these properties can easily be proved in the propositional calculus.

with respect to $\phi \in X$ if $\delta(\phi)=1$ *and* $\delta(\neg\phi)=1$. All violations of weak consistency are also violations of strong consistency, but not all violations of strong consistency are also violations of weak consistency. We say that δ is *not deductively closed with respect to* $\phi \in X$ if there exists a subset $\Phi \subseteq X$ such that $\delta(\Phi) = 1, \Phi \models \phi$ and $\delta(\phi)=0$.

4.2. *Necessary and Sufficient Conditions for Path-Dependence*

Let us revisit the examples in Section 3. Consider first the one in Section 3.1. Are the agent's propositional attitudes in that example inconsistent? They do not violate weak consistency, but they violate strong consistency: the agent has initial dispositions to accept each of Q, $(Q \rightarrow R)$, P, $(P \rightarrow \neg R)$, where the first pair of propositions implies R and the second implies $\neg R$. The agent's propositional attitudes also violate deductive closure: the agent has an initial disposition to accept P and $(P \rightarrow \neg R)$, but not $\neg R$, although P and $(P \rightarrow \neg R)$ imply $\neg R$.

Consider next the example of Section 3.2. The views of each individual government member satisfy completeness, weak and strong consistency and deductive closure. But what about the propositional attitudes of the multi-member government acting collectively, as determined by majority voting over the individual views? There are majorities on each of P, Q, R and $((P \wedge Q \wedge R) \rightarrow S)$. These propositions imply S; and yet there is no majority on S, a violation of deductive closure with respect to S. Further, if a vote is taken on $\neg S$, then $\neg S$ is unanimously accepted. Hence the majority views also violate strong consistency.

This suggests that violations of the rationality conditions by the agent, particularly of strong consistency, might be responsible for path-dependence. A general result confirms this suggestion. The result applies to any agent (individual or group) making decisions over multiple propositions, where the agent's propositional attitudes are represented by $\delta: X \rightarrow \{0,1\}$.

THEOREM 1: *For any* $\phi \in X$,

(i) there exist two decision-paths Ω_1 *and* Ω_2 *such that* $\phi \in M(\delta, \Omega_1)$ *and* $\neg\phi \in M(\delta, \Omega_2)$ (*i.e.* $M(\delta, \Omega)$ *is strongly path-dependent*)

if and only if

(ii) the AR-function δ *violates strong consistency with respect to* ϕ.

For a proof see the Appendix. Theorem 1 shows that, if (and only if) the agent's propositional attitudes violate strong consistency with respect to a proposition ϕ, there exist (at least) two decision-paths such that one leads to the acceptance of ϕ, whereas the other leads to the acceptance of $\neg\phi$. Violations of strong consistency occur when the agent's propositional attitudes violate weak consistency, or when they satisfy weak consistency but an inconsistency is "hidden" by a violation of deductive closure.

PROPOSITION 2: *Suppose the AR-function* $\delta: X \rightarrow \{0,1\}$ *is complete and weakly consistent. For any* $\phi \in X$,
(i) δ *violates strong consistency with respect to* ϕ
if and only if
(ii) δ *is not deductively closed with respect to one of* ϕ *or* $\neg\phi$.

The conjunction of Theorem 1 and Proposition 2 implies the following theorem.

THEOREM 2: *Suppose the AR-function* $\delta: X \rightarrow \{0,1\}$ *is complete and weakly consistent. For any* $\phi \in X$,
(i) there exist two decision-paths Ω_1 *and* Ω_2 *such that* $\phi \in M(\delta, \Omega_1)$ *and* $\neg\phi \in M(\delta, \Omega_2)$ (*i.e.* $M(\delta, \Omega)$ *is strongly path-dependent*)
if and only if
(ii) δ *is not deductively closed with respect to one of* ϕ *or* $\neg\phi$.

Suppose the agent's propositional attitudes are complete and weakly consistent. By Theorem 2, there exist (at least) two alternative decision-paths with mutually inconsistent outcomes on the proposition ϕ if and only if the agent's propositional attitudes violate deductive closure with respect to ϕ. While strong path-dependence is ruled out when the agent's propositional attitudes are strongly consistent, weak path-dependence is still possible.

THEOREM 3: *Suppose the AR-function* $\delta: X \rightarrow \{0,1\}$ *is strongly consistent. For any* $\phi \in X$,
(i) there exist two decision-paths Ω_1 *and* Ω_2 *such that* $\phi \in M(\delta, \Omega_1)$ *and* $\phi \notin M(\delta, \Omega_2)$ (*i.e.* $M(\delta, \Omega)$ *is weakly path-dependent*)
if and only if
(ii) δ *is not deductively closed with respect to* ϕ.

For a proof see the Appendix. Under the assumptions of Theorem 3, the AR-function δ violates deductive closure if and only if it violates completeness. Under the assumptions of Theorem 2, by contrast, δ violates deductive closure if and only if it violates strong consistency.

If the agent is an individual, the response to this path-dependence problem may seem fairly simple. If that individual is sufficiently rational – i.e. if his or her propositional attitudes satisfy weak consistency and deductive closure (and hence strong consistency) –, the individual is immune to path-dependencies in a priority-to-the-past decision process. Of course, it is an interesting empirical question whether individuals actually satisfy these rationality conditions, or whether violations of these conditions by individual agents are empirically frequent.

If the agent is a group, on the other hand, the response is less straightforward. It would be desirable if we could find an aggregation function δ which generates a collective AR-function satisfying completeness, weak consistency and deductive closure (and hence strong consistency). By Theorem 2, this would solve the problem of path-dependence. As we have seen, majority voting may violate these properties. We now see that this is not an accidental property of majority voting. If we demand some minimal conditions, no aggregation function with the desired properties exists.

5. PATH-DEPENDENCIES AT A COLLECTIVE LEVEL: A GENERAL RESULT

We now assume that the agent is a *group* of individuals. Let $N = \{1, 2, \ldots, n\}$ be a set of individuals ($n \geq 2$). Following the definition of propositional attitudes above, the views of each individual, $i \in N$, over the propositions in X are represented by an *AR-function* $\delta_i : X \rightarrow \{1, 0\}$. We may here interpret each individual's propositional attitudes as fully endorsed views, and assume, as a best-case scenario, that each individual's AR-function δ_i satisfies completeness, weak consistency and deductive closure (and thus strong consistency), as defined above.

DEFINITION 5.1: A *profile of individual AR-functions* (hereafter *profile*) is an assignment of one AR-function to each individual, $\{\delta_i\}_{i \in N} = \{\delta_1, \delta_2, \ldots, \delta_n\}$.

In this model different individuals can hold different views not only on atomic propositions, but also on compound propositions. Nehring and Puppe's model (2002) (N&P) can also represent social choice problems over multiple interconnected propositions, but the model is more restrictive. In N&P, agents do not have AR-functions over a set of propositions from propositional logic, but instead they have preferences over vectors of 'properties', $<a_1, a_2, \ldots, a_m>$, where each $a_j \in \{0,1\}$. If all individuals accept the *same* logical connection rule(s) over the atomic propositions, the logical structure of our model can be represented in terms of the property structure of the N&P model. We can then identify each property in the N&P model with an atomic proposition, and represent the unanimously accepted logical connection rule(s) by restricting the set of alternatives – i.e. the set of admissible $<a_1, a_2, \ldots, a_m>$ vectors – appropriately. For example, if all individuals accept the connection rule $(R \leftrightarrow (P \wedge Q))$ over the atomic propositions P, Q and R, then each vector $<a_1, a_2, a_3>$ can be interpreted as an assignment of truth-values to P, Q and R; the connection rule

$(R\leftrightarrow(P\wedge Q))$ can be captured by restricting the set of admissible alternatives to {<1, 1, 1>, <1, 0, 0>, <0, 1, 0>, <0, 0, 0>}. But as soon as some individuals disagree about compound propositions, this approach leads to the problem that different individuals will disagree about what the set of admissible alternatives is. Suppose individual 1 accepts the connection rule $(R\leftrightarrow(P\wedge Q))$ while individual 2 accepts $(\neg R\leftrightarrow(P\wedge Q))$. Then for individual 1 the set of admissible alternatives is {<1, 1, 1>, <1, 0, 0>, <0, 1, 0>, <0, 0, 0>}, while for individual 2 it is {<1, 1, 0>, <1, 0, 1>, <0, 1, 1>, <0, 0, 1>}. The N&P model requires a single set of alternatives. Therefore the types of logical connections that can be represented in our model are more general than those which can be represented in the N&P model. Let us return to our model.

To determine the group's propositional attitude to a proposition ϕ, interpreted as the *group's* initial disposition on ϕ, we require a way of *aggregating* the n individual views on ϕ into a single collective propositional attitude on ϕ. We thus need to aggregate the vector of 0s and 1s, $\{\delta_1(\phi), \delta_2(\phi), \ldots, \delta_n(\phi)\}$, into a single overall disposition of *either* 0 (non-acceptance) *or* 1 (acceptance).

DEFINITION 5.2: An *aggregation function* is a function $\delta: \{0,1\}^n \rightarrow \{0,1\}$.

Majority voting, as discussed above, is an example of an aggregation function.

DEFINITION 5.3: *Majority voting* is the aggregation function $\delta: \{0,1\}^n \rightarrow \{0,1\}$ defined as follows:

for any $(d_1, d_2, ..., d_n)\in \{0,1\}^n$,

$$\delta(d_1, d_2, ..., d_n) = \begin{cases} 1 \text{ if } \sum_{i\in N} d_i > {}^n/_2 \\ 0 \text{ otherwise.} \end{cases}$$

For each profile $\{\delta_i\}_{i\in N}$, an aggregation function δ induces a collective AR-function $\delta_{\{\delta_i\}_{i\in N}} : X \rightarrow \{0,1\}$ defined as follows:

for each $\phi\in X$, $\delta_{\{\delta_i\}_{i\in N}}(\phi) := \delta(\delta_1(\phi), \delta_2(\phi), \ldots, \delta_n(\phi))$.

To simplify the notation, once a profile $\{\delta_i\}_{i\in N}$ has been fixed, we write δ instead of $\delta_{\{\delta_i\}_{i\in N}}$, thereby identifying the aggregation function $\delta: \{0,1\}^n \rightarrow \{0,1\}$ with the collective AR-function $\delta_{\{\delta_i\}_{i\in N}} : X \rightarrow \{0,1\}$ induced by δ.[9]

[9] If the collective AR-function δ is the result of applying an aggregation function δ to a given profile $\{\delta_i\}_{i\in N}$, then $M(\delta, \Omega)$ is of course dependent on that profile. A more precise notation would express this profile-dependency by using the label $M(\delta_{\{\delta_i\}_{i\in N}}, \Omega)$ to denote the outcome set of the decision process, using the aggregation function δ and the decision-path Ω, for a given profile $\{\delta_i\}_{i\in N}$. To simplify the notation, we drop the subscript $\{\delta_i\}_{i\in N}$ and simply write $M(\delta, \Omega)$ for $M(\delta_{\{\delta_i\}_{i\in N}}, \Omega)$.

We can now restate the observation and the question raised in the previous section. First, the observation: Majority voting is an aggregation function which may fail, for some profiles, to induce a complete, weakly consistent and deductively closed collective AR-function. And, second, the question: Are there any alternative aggregation functions which always generate complete, weakly consistent and deductively closed collective AR-functions? We now see that, if we impose some seemingly undemanding conditions on an aggregation function, the answer to this question is negative. In Section 7, we address some escape-routes from that negative result. We consider an aggregation function δ: $\{0,1\}^n \to \{0,1\}$.

UNIVERSAL DOMAIN. Let U be the set of all logically possible profiles of individual AR-functions satisfying completeness, weak consistency and deductive closure.

ANONYMITY. For any $(d_1, d_2, ..., d_n) \in \{0,1\}^n$ and any permutation σ: $N \to N$, $\delta(d_1, d_2, ..., d_n) = \delta(d_{\sigma(1)}, d_{\sigma(2)}, ..., d_{\sigma(n)})$.

THEOREM 4 (Corollary of List and Pettit 2002): *There exists no aggregation function* δ : $\{0,1\}^n \to \{0,1\}$ (*satisfying anonymity*) *which induces*, *for every* $\{\delta_i\}_{i\in N}\in U$, *a complete*, *weakly consistent and deductively closed collective AR-function* $\delta: X \to \{0,1\}$.

For a proof see the Appendix. By Theorem 4, any aggregation function (satisfying universal domain and anonymity) which generates complete and weakly consistent collective AR-functions – such as majority voting – *necessarily* generates violations of deductive closure for some profiles. By Theorem 2, for these profiles, there exist different decision-paths with mutually inconsistent outcomes. This is the intuition underlying the following result, proved in the Appendix.

THEOREM 5: *There exists no aggregation function* δ: $\{0,1\}^n \to \{0,1\}$ (*satisfying anonymity*) *such that*, *for every* $\{\delta_i\}_{i\in N}\in U$, $M(\delta, \Omega)$ *is complete and invariant under changes of the decision-path*.

Theorem 5 is premised on three conditions. First, individuals are free to form any logically possible combination of views on the propositions, provided their views satisfy completeness, weak consistency and deductive closure (the universal domain condition). Second, an aggregation function should give equal weight to all individuals in determining the group's propositional attitude on any proposition (the anonymity condition). Third, a priority-to-the-past decision process should produce a determinate decision on every proposition (the completeness condition). Then, by Theorem 5, there exists no aggregation function for which a priority-to-the-past decision process is always invariant under changes of the decision-path. This does *not* mean that the outcome of a priority-to-the-past decision process is path-dependent for *every* profile. Rather, no aggregation function can *guarantee* the avoidance of path-dependencies. Under *any* aggregation function, there exist *some* profiles for which the outcome of a priority-to-the-past decision process is (strongly) path-dependent.[10]

Does this imply that, at a collective level, path-dependence is *in principle* unavoidable? And if path-dependence is unavoidable, what is the cost of this? We first address the second question, and show that path-dependence opens up two types of strategic manipulation. We then turn to the first question, and discuss some escape-routes from the problem of path-dependence.

6. THE COST OF PATH-DEPENDENCE: THE POSSIBILITY OF STRATEGIC MANIPULATION

6.1. *Manipulation by agenda setting*

Whenever the agent's propositional attitudes violate strong consistency with respect to ϕ, the agenda-setter – whoever chooses the decision-path – may have power to determine whether the outcome of the decision process will be ϕ or $\neg\phi$. By Theorem 1, given a violation of strong consistency with respect to ϕ, there exist alternative decision-paths leading to each of these two opposite outcomes on ϕ. Given sufficient information and computational power, the agenda-setter can thus determine the decision-path that is required to bring about the preferred outcome. In our example of the multi-member government in Section 3.2, an agenda-setter who cares a lot about increasing defence spending (proposition R) would advocate the decision-path of case 2, which results in the acceptance of R, whereas an agenda-setter who wants to avoid an increase in defence spending would advocate the decision-path of case 1, which results in the rejection of R.

6.2. *Manipulation by expression of untruthful views*

Suppose a decision process is (strongly) path-dependent, and suppose a particular decision-path has been chosen. If some individual (or group of individuals) cares particularly about some propositions that occur *later* in the decision-path, they might strategically express untruthful views on *earlier* propositions in that path, as the decision on the later propositions will be affected by the decisions on the earlier ones.

[10] Since we are considering a *complete* $M(\delta, \Omega)$ here, the notions of strong and weak path-dependence coincide.

Considering our example of the multi-member government in Section 3.2 again, suppose that individual 3 cares most about increasing defence spending (proposition R), and is willing to sacrifice her conviction that spending on health care should also be increased (proposition Q), in order to get her way on the defence issue. Suppose the decision-path is the one of case 1. At time 3, when proposition Q is considered, individual 3 might untruthfully vote against Q, thus bringing about a majority *rejection* of Q. At time 4, when proposition R is finally considered, the government would then no longer face a conflict between its prior commitments and the majority verdict on R; it would be able to follow that majority verdict, and accept R. Without individual 3's strategic intervention, the outcome set of the decision process would have been $\{P, Q, \neg R$ and $\neg S\}$. With the strategic intervention, the outcome set is $\{P, \neg Q, R$ and $\neg S\}$, a preferred outcome from individual 3's perspective. We say that individual 3 has an incentive to express an untruthful view on proposition Q.[11]

To give a formal definition of strategic manipulability by expression of untruthful views, a few definitions are due. Let Ξ be the set of all possible outcome sets $M(\delta, \Omega)$ of a priority-to-the-past decision process (each satisfying $M(\delta, \Omega) \subseteq X$). To each individual $i \in N$ with AR-function δ_i, we assign a weak preference ordering R_{i,δ_i} (reflexive, transitive and connected) over Ξ. For any $\Phi_1, \Phi_2 \in \Xi$, $\Phi_1 R_{i,\delta_i} \Phi_2$ means that individual i weakly prefers the outcome set Φ_1 to the outcome set Φ_2. We write $\Phi_1 P_{i,\delta_i} \Phi_2$ as an abbreviation for $[\Phi_1 R_{i,\delta_i} \Phi_2$ and not $\Phi_2 R_{i,\delta_i} \Phi_1]$. The preference ordering R_{i,δ_i} depends on individual i's AR-function δ_i. A weak form of such a dependency is captured by a weak monotonicity condition, as defined below.

DEFINITION 6.1: An outcome set $\Phi_1 \in \Xi$ is *at least as close* to individual's i's AR-function δ_i as another outcome set $\Phi_2 \in \Xi$ if, for every $\phi \in X$, we have $|\Delta_1(\phi) - \delta_i(\phi)| \leq |\Delta_2(\phi) - \delta_i(\phi)|$, where, for each $j \in \{1,2\}$, $\Delta_j(\phi) = 1$ if $\phi \in \Phi_j$ and $\Delta_j(\phi) = 0$ if $\phi \notin \Phi_j$.

DEFINITION 6.2: The individual preference ordering R_{i,δ_i} is *weakly monotonic* if $[\Phi_1$ is as least as close to δ_i as $\Phi_2]$ implies $\Phi_1 R_{i,\delta_i} \Phi_2$.

ASSUMPTION. For each $i \in N$ with AR-function δ_i, the associated preference ordering R_{i,δ_i} is *weakly monotonic*.

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[11] To formalize this example in terms of the definitions below, we need to assign to individual 3 a weakly monotonic preference ordering with respect to which individual 3 strictly prefers the outcome set $\{P, \neg Q, R$ and $\neg S\}$ to the outcome set $\{P, Q, \neg R$ and $\neg S\}$.

DEFINITION 6.3: In the decision process $M(\delta, \Omega)$, an individual $i \in N$ with AR-function δ_i and preference ordering R_{i,δ_i} has an *incentive to express an untruthful AR-function* at the profile $\{\delta_i\}_{i \in N}$ if there exists an AR-function δ^*_i ($\neq \delta_i$) such that $\Phi^* P_{i,\delta_i} \Phi$, where $\Phi^* = M(\delta_{\{\delta_1, \ldots, \delta^*_i, \ldots, \delta_n\}}, \Omega)$ and $\Phi = M(\delta_{\{\delta_i\}_{i \in N}}, \Omega)$.[12]

Informally, individual i has an incentive to express an untruthful AR-function at the profile $\{\delta_i\}_{i \in N}$ if the following three conditions hold: (i) If individual i expresses his or her truthful AR-function δ_i (holding the AR-functions of the other individuals fixed), the decision process leads to the outcome set Φ. (ii) If individual i expresses the alternative (untruthful) AR-function δ^*_i (holding the AR-functions of the other individuals fixed), the decision process leads to the outcome set Φ^*. (iii) Individual i strictly prefers the outcome set Φ^* to the outcome set Φ (on the basis of his or her truthful preference ordering R_{i,δ_i}).

The example of the multi-member government shows that path-dependent decision processes may give individuals incentives to express untruthful AR-functions. Whether or not an individual has such an incentive in a particular decision process depends on a number of factors: the individual's preference ordering, the decision-path, whether or not the individual's views are pivotal for the collective decisions on some propositions. Below we state a general result showing that (weak) path-dependence is a *necessary* condition for the existence of individuals with incentives to express untruthful views. It is not a *sufficient* condition. Even in cases of (strong) path-dependence there may not exist a single individual who is pivotal for the outcome on a relevant proposition; and hence there may not exist an individual who can single-handedly manipulate the outcome. A more technical analysis may be used to show that, under certain conditions, (strong) path-dependence implies that there exists a *coalition of individuals* that has an incentive to express untruthful views.

6.3. *Avoiding strategic manipulation*

Neither of the two types of strategic manipulation is possible when a decision process is invariant under changes of the decision-path. In the case of manipulation by agenda setting this is obvious. Agenda setting – i.e. determining the order in which the propositions are considered – has no effect when the decision process is invariant under changes of the decision-path. In the case of manipulation by expression of untruthful views, the following result holds.

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[12] Here Φ^* is the outcome of the decision process when individual i expresses the "untruthful" AR-function δ^*_i and every other individual j expresses the "truthful" AR-function δ_j, whereas Φ is the outcome when *every* individual $i \in N$ expresses the "truthful" AR-function δ_i.

DEFINITION 6.4: The aggregation function δ is *weakly monotonic* if, for any $(d_1, d_2, ..., d_n), (e_1, e_2, ..., e_n) \in \{0,1\}^n$, [for every $i \in N$, $d_i \geq e_i$] implies $\delta(d_1, d_2, ..., d_n) \geq \delta(e_1, e_2, ..., e_n)$.

Propositionwise majority voting is a weakly monotonic aggregation function.

THEOREM 6: *Suppose δ is a weakly monotonic aggregation function. For any $\{\delta_i\}_{i \in N}$, if $M(\delta, \Omega)$ is invariant under changes of the decision-path, there exists no individual $i \in N$ and no weakly monotonic preference ordering R_{i,δ_i} such that i has an incentive to express an untruthful AR-function at $\{\delta_i\}_{i \in N}$.*

For a proof see the Appendix. By Theorem 6, if a priority-to-the-past decision process is invariant under changes of the decision-path at some profile, then no individual has an incentive to express an untruthful AR-function at that profile. The result implies that, if we can find a domain of profiles such that $M(\delta, \Omega)$ is never path-dependent in that domain, then $M(\delta, \Omega)$ is strategy-proof in that domain.

STRATEGY-PROOFNESS IN D. There exists no profile $\{\delta_i\}_{i \in N} \in D$ such that, for some individual $i \in N$ and some weakly monotonic preference ordering R_{i,δ_i}, individual i has an incentive to express an untruthful AR-function at $\{\delta_i\}_{i \in N}$.

COROLLARY OF THEOREM 6. *Suppose δ is a weakly monotonic aggregation function. If $M(\delta, \Omega)$ is invariant under changes of the decision-path for every $\{\delta_i\}_{i \in N} \in D$, then $M(\delta, \Omega)$ is strategy-proof in D.*

Note that, in decision processes that violate strategy-proofness, the situation where all individuals express truthful views is *not* always a Nash equilibrium. In strategy-proof decision processes, by contrast, that situation is a Nash equilibrium.

Despite the impossibility result of Theorem 5, Theorem 6 and its Corollary are *not* vacuous. As we see in Section 7, there do exist profiles at which a suitably defined priority-to-the-past decision process is invariant under changes of the decision-path. The examples of path-independent priority-to-the-past decision processes discussed in Section 7 satisfy the conditions of Theorem 6.[13] The restricted domain R discussed in Section 7.3 satisfies the conditions of the Corollary of Theorem 6.

[13] With the exception of supermajority voting, which is sufficient for avoiding strong but not weak path-dependence.

7. ESCAPE-ROUTES FROM PATH-DEPENDENCIES AT A COLLECTIVE LEVEL

We suppose again that the views of all individuals are complete, weakly consistent and deductively closed. According to Theorem 5, there exists no aggregation function (satisfying universal domain and anonymity) for which a priority-to-the-past decision process (satisfying completeness) is invariant under changes of the decision-path.

To avoid path-dependencies it is therefore necessary to relax at least one of the conditions of Theorem 5, namely universal domain, anonymity or completeness. By Theorems 1 to 3, we know that we need to relax these conditions so as to find an aggregation function which generates a weakly consistent and deductively closed (and hence strongly consistent) AR-function for the group.

7.1. *Relaxing completeness: the special support approach*

Suppose we do not insist that the priority-to-the-past decision process should produce a determinate verdict on every proposition (acceptance of the proposition or of its negation), but we allow that it may fail to produce such a verdict on some propositions. We can define the group's propositional attitude on each proposition by the unanimity rule or by a supermajority rule. We first consider the unanimity rule.

DEFINITION 7.1: The *unanimity rule* is the aggregation function $\delta: \{0,1\}^n \rightarrow \{0,1\}$ defined as follows:

for any $(d_1, d_2, ..., d_n) \in \{0,1\}^n$,

$$\delta(d_1, d_2, ..., d_n) = \begin{cases} 1 \text{ if } d_i = 1 \text{ for every } i \in N \\ 0 \text{ otherwise.} \end{cases}$$

For each proposition ϕ, the group will have a disposition to accept ϕ if and only if *every* individual accepts ϕ. The group's propositional attitudes are then weakly consistent and deductively closed (and hence strongly consistent), but not necessarily complete.[14] They are incomplete to the extent that there is a lack of unanimity on some propositions *and* their negations. The resulting priority-to-the-past decision process is invariant under changes of the decision-path, but it may fail to produce a verdict on some propositions – namely on those propositions which are neither unanimously accepted nor unanimously rejected by the individuals. The approach would give veto power to every individual, and would thus be prone to 'stalemate'.

[14] So long as the views of the individuals are complete, weakly consistent and deductively closed.

Can we relax the rather demanding unanimity requirement and require only a suitable supermajority?

DEFINITION 7.2: *Supermajority voting* with parameter q is the aggregation function δ: $\{0,1\}^n \rightarrow \{0,1\}$ defined as follows:

for any $(d_1, d_2, ..., d_n) \in \{0,1\}^n$,

$$\delta(d_1, d_2, ..., d_n) = \begin{cases} 1 \text{ if } \sum_{i\in N} d_i > qn \\ 0 \text{ otherwise.} \end{cases}$$

If $q = ½$, supermajority voting reduces to simple majority voting.

To state a theorem on supermajority voting, note that the set of propositions X can be partitioned into $2m$ equivalence classes of logically equivalent propositions, where m is an integer greater than 1. The number of equivalence classes is even because, by assumption, X always contains proposition-negation pairs. It is greater than 1 because, by assumption, X contains more than one proposition-negation pair.

THEOREM 7: *Suppose X can be partitioned into $2m$ equivalence classes of logically equivalent propositions. Let δ : $\{0,1\}^n \rightarrow \{0,1\}$ be supermajority voting with parameter $(m\text{-}1)/m$. Then δ is an aggregation function (satisfying anonymity) which induces, for every $\{\delta_i\}_{i\in N} \in U$, a strongly consistent (but not necessarily complete and deductively closed) collective AR-function δ: $X \rightarrow \{0,1\}$.*

For a proof see the Appendix. For each proposition ϕ, the group will have a disposition to accept ϕ if and only if a proportion of *more than* $(m\text{-}1)/m$ of the individuals accept ϕ. The group's propositional attitudes are then strongly consistent, but not necessarily complete and deductively closed.[15] They are incomplete to the extent that some propositions *and* their negations lack the requisite supermajority. They violate deductive closure to the extent that there is no supermajority on some *implications* of propositions that each have the required supermajority support. The resulting priority-to-the-past decision process is not strongly path-dependent (as the group's propositional attitudes are strongly consistent), but it may be weakly path-dependent (as the propositional attitudes may violate deductive closure) and it may fail to produce a verdict on some propositions (as the propositional attitudes may violate completeness). Although the approach would not give veto power to every individual,[16] it would give veto power to every *group* consisting of a proportion of *at*

[15] See previous note.
[16] At least if $n > m$ (i.e. if $n/m > 1$).

least $1/m$ of the individuals, and it might thus still be prone to 'stalemate', albeit less so than the unanimity approach.

7.2. *Relaxing anonymity: the dictatorship approach*

Suppose we do not insist that all individuals should have equal weight in determining the group's propositional attitude on any proposition, but we allow that a fixed single individual may be dictatorial. Specifically, we choose one individual, the *dictator*, and define the group's propositional attitude on each proposition to be that individual's view.

DEFINITION 7.3: A *dictatorship of individual i* is the aggregation function δ : $\{0,1\}^n \rightarrow \{0,1\}$ defined as follows:

for any $(d_1, d_2, ..., d_n) \in \{0,1\}^n$, $\delta(d_1, d_2, ..., d_n) = d_i$,

where $i \in N$ is some fixed single individual.

Then, if the dictator's views are complete, weakly consistent and deductively closed, so are the propositional attitudes of the group. The resulting priority-to-the-past decision process is invariant under changes of the decision-path, but it may not reflect the views of any individuals other than the dictator.

7.3. *Relaxing universal domain*

Suppose not all logically possible profiles over the propositions will occur. There are two possible reasons for this. Either certain profiles are *explicitly ruled out* by restrictions on the views that individuals can express (the coercive solution); or certain profiles *happen*, as a matter of fact, *not to occur* in practice (the empirical solution). We can identify a structure condition on a profile such that, if the domain of admissible input to the decision process includes only profiles satisfying that condition, then the impossibility result of Theorem 4 and the path-dependency result of Theorem 5 can be avoided.

Fix a profile $\{\delta_i\}_{i \in N}$. For each $\phi \in X$, define $N_{accept\text{-}\phi} := \{i \in N : \delta_i(\phi)=1\}$ and $N_{reject\text{-}\phi} := \{i \in N : \delta_i(\phi)=0\}$. Further, given any linear ordering ω on N and any $N_1, N_2 \subseteq N$, we write $N_1\ \omega\ N_2$ as an abbreviation for [for all $i \in N_1$ and all $j \in N_2$, $i \omega j$].

DEFINITION 7.4. (List 2003) (i) A profile $\{\delta_i\}_{i \in N}$ satisfies *unidimensional alignment* if there exists a linear ordering ω on N such that, for every $\phi \in X$, either $N_{accept\text{-}\phi}\ \omega\ N_{reject\text{-}\phi}$ or $N_{reject\text{-}\phi}\ \omega\ N_{accept\text{-}\phi}$.[17]

[17] Note that this definition permits $N_{accept\text{-}\phi} = \emptyset$ or $N_{reject\text{-}\phi} = \emptyset$.

(ii) A *structuring ordering* of N for $\{\delta_i\}_{i\in N}$ is an ordering ω with property (a).

(iii) If n is odd, individual $m\in N$ is the *median individual* with respect to ω if $|\{i\in N : i\omega m\}| = |\{i\in N : m\omega i\}|$. If n is even, individuals m_1, $m_2\in N$ are the *median pair of individuals* with respect to ω if (i) $m_1\omega m_2$, (ii) there exists no $i\in N$ such that $m_1\omega i$ and $i\omega m_2$, and (iii) $|\{i\in N : i\omega m_1\}| = |\{i\in N : m_2\omega i\}|$.

Informally, a profile satisfies unidimensional alignment if there is a single ordering of the individuals from left to right such that, for *every* proposition in X, the individuals accepting that proposition are either all to the left, or all to the right, of those rejecting it.

TABLE II

	Individual 4	Individual 1	Individual 5	Individual 2	Individual 3
P	1	1	1	1	0
Q	1	1	1	0	0
R	1	1	0	0	0
S	1	1	0	0	0
$((P\wedge Q\wedge R)\rightarrow S))$	1	1	1	1	1

To illustrate, the profile shown in Table II satisfies unidimensional alignment. The corresponding structuring ordering (ω) is 4, 1, 5, 2, 3; and the median individual (m) is individual 5.

UNIDIMENSIONAL ALIGNMENT DOMAIN. Let $\mathcal{R}$ be the set of all logically possible profiles of individual AR-functions satisfying completeness, weak consistency, deductive closure and unidimensional alignment.

THEOREM 8 (List 2003): *Let δ be propositionwise majority voting. For any $\{\delta_i\}_{i\in N}\in\mathcal{R}$ with corresponding structuring ordering ω, the following holds*:

(i) If n is odd, $\delta_{\{\delta_i\}_{i\in N}} = \delta_m$, where m is the median individual with respect to ω.

(ii) If n is even, $\delta_{\{\delta_i\}_{i\in N}} = \delta_{m_1}\delta_{m_2}$, where m_1 and m_2 are the median pair of individuals with respect to ω.[18]

Suppose a profile $\{\delta_i\}_{i\in N}$ satisfies unidimensional alignment. Order the individuals along a corresponding structuring ordering. If n is odd, by unidimensional alignment, the median individual shares the majority view on every proposition. Hence the collective AR-function induced by majority voting coincides with the AR-function of the median individual. If n is even, there is no single median individual. But by unidimensional alignment, if (and only if) the median *pair* of individuals have

[18] Here $\delta_{m_1}\delta_{m_2}$ is simply the AR-function defined as follows: for any $\phi\in X$, $\delta_{m_1}\delta_{m_2}(\phi) = \delta_{m_1}(\phi)\delta_{m_2}(\phi)$.

the same view on a proposition, then that view is also the majority view on that proposition. Hence the collective AR-function induced by majority voting is the product (the "intersection") of the AR-functions of the median pair of individuals.

Now, provided that the AR-function of the median individual (or those of the median pair of individuals) satisfies weak consistency and deductive closure, so does the collective AR-function. If n is even, the collective AR-function may, however, violate completeness – namely when the median pair of individuals disagree on a proposition ϕ (i.e. when $\delta_{m_1}(\phi) \neq \delta_{m_2}(\phi)$), so that $\delta_{m_1}\delta_{m_2}(\phi) = 0 = \delta_{m_1}\delta_{m_2}(\neg\phi)$. But such violations of completeness occur *only* when there is a majority tie, i.e. when $|N_{accept\text{-}\phi}| = |N_{reject\text{-}\phi}|$. Thus the collective AR-function satisfies the following condition.

DEFINITION 7.5: Given a profile $\{\delta_i\}_{i\in N}$, the collective AR-function $\delta_{\{\delta_i\}_{i\in N}}$ is *almost complete* if, for any $\phi\in X$, $|N_{accept\text{-}\phi}| \neq |N_{reject\text{-}\phi}|$ implies that $\delta_{\{\delta_i\}_{i\in N}}(\phi)=1$ or $\delta_{\{\delta_i\}_{i\in N}}(\neg\phi)=1$.

When n is odd, we can never have $|N_{accept\text{-}\phi}| = |N_{reject\text{-}\phi}|$, and hence the notions of *completeness* and *almost completeness* coincide. When n is even, the two notions are distinct: completeness implies almost completeness, but not vice-versa.

THEOREM 9 (List 2003): *Let δ be an aggregation function. Then*

(i) δ satisfies anonymity and, for any $\{\delta_i\}_{i\in N} \in \mathcal{R}$, $\delta_{\{\delta_i\}_{i\in N}}$ is almost complete, weakly consistent and deductively closed

if and only if

(ii) δ is majority voting.

Given a profile satisfying unidimensional alignment, the group's propositional attitudes are almost complete, weakly consistent and deductively closed *if and only if* these propositional attitudes are determined by majority voting over the individual views. The resulting priority-to-the-past decision process is invariant under changes of the decision-path.

DEFINITION 7.6: Given a profile $\{\delta_i\}_{i\in N}$, $M(\delta, \Omega)$ is *almost complete* if, for every $\phi\in X$, $|N_{accept\text{-}\phi}| \neq |N_{reject\text{-}\phi}|$ implies that $\phi\in M(\delta, \Omega)$ or $\neg\phi\in M(\delta, \Omega)$.

THEOREM 10: *Let δ be an aggregation function. Then*

(i) δ satisfies anonymity and, for any $\{\delta_i\}_{i\in N} \in \mathcal{R}$, $M(\delta, \Omega)$ is almost complete and invariant under changes of the decision-path

if and only if

(ii) δ is majority voting.

For a proof see the Appendix. Theorem 10 states that majority voting is the *unique* aggregation function satisfying anonymity for which $M(\delta, \Omega)$ is almost complete and invariant under changes of the decision-path for every profile in the domain $\mathcal{R}$. Moreover, the outcome set of the priority-to-the-past decision process coincides with the set of propositions accepted by the median individual (if n is odd) or with the intersection of the sets of propositions accepted by the median *pair* of individuals (if n is even).

THEOREM 11: *Let δ be majority voting. For any $\{\delta_i\}_{i\in N} \in \mathcal{R}$ with corresponding structuring ordering ω (and any Ω), the following holds:*

(i) If n is odd, $M(\delta, \Omega) = \{\phi\in X : \delta_m(\phi)=1\}$, where m is the median individual with respect to ω.

(ii) If n is even, $M(\delta, \Omega) = \{\phi\in X : \delta_{m_1}(\phi)\delta_{m_2}(\phi)=1\}$, where m_1 and m_2 are the median pair of individuals with respect to ω.

Is unidimensional alignment just an artificial condition, or can we imagine plausible circumstances in which unidimensional alignment might hold? Suppose we have a situation in which different individuals may endorse different views on the propositions, but they reach “structural agreement”, as follows. Suppose, firstly, that they agree on a single linear dimension (such as from “most liberal” to “most conservative”) that characterizes the range of their disagreement; in particular, suppose that each individual takes a certain position on that dimension. For simplicity, we call it a left/right dimension, but a range of interpretations is possible. And suppose, secondly, that, for each proposition, the extreme positions on the left/right dimension correspond to either clear acceptance or clear rejection of the proposition and there exists an 'acceptance threshold' on the dimension (possibly different for different propositions) such that all the individuals to the left of the threshold accept the proposition and all the individuals to its right reject it (or vice-versa). These two conditions entail unidimensional alignment. In other words, “structural agreement” (as described) can induce unidimensional alignment. This suggests that, to the extent that the individuals reach such structural agreement, the escape-route from the path-dependency result opened up by Theorem 10 becomes available. For the present purposes, however, the specific interpretation of unidimensional alignment is less relevant than the more general insight that path-dependencies can be avoided if there is a sufficient level of structure among the individuals’ views.

8. CONCLUDING REMARKS

I have developed a model of sequential decision processes over multiple interconnected propositions, and I have tried to illustrate the usefulness of the model by investigating the phenomenon of path-dependence in *priority-to-the-past* decision processes.

Let me briefly summarize the main substantive results. Path-dependence is linked with the violation of certain rationality conditions by the agent. If (and only if) the agent's propositional attitudes violate strong consistency, a priority-to-the-past decision process is strongly path-dependent: there exist (at least) two alternative decision-paths which lead to mutually inconsistent outcomes (Theorem 1). But even if the agent has strongly consistent propositional attitudes, a priority-to-the-past decision process may be weakly path-dependent, namely if and only if the propositional attitudes violate deductive closure (Theorem 3): then there exist (at least) two alternative decision-paths with different, though not mutually inconsistent, outcomes.

The problem is particularly serious at a collective level. While individuals might try to avoid path-dependence through a self-imposed 'discipline' of rationality, no such option is generally available to groups. Under certain conditions, any group that determines its propositional attitudes by *aggregation* over individual views will *necessarily* run the risk of violations of completeness, weak consistency and deductive closure (Theorem 4). Such violations in turn lead to strong path-dependence (Theorem 2). The implication is that there exists no aggregation function (satisfying some conditions) that guarantees invariance under changes of the decision-path in a priority-to-the-past decision process (Theorem 5). Path-dependence makes the decision process vulnerable to two types of strategic manipulation: manipulation by agenda setting, and manipulation by expression of untruthful views. If a decision process is invariant under changes of the decision-path, both types of strategic manipulation can be avoided (Section 6.3 and Theorem 6).

The escape-routes from path-dependence available to groups are limited: we have discussed relaxations of anonymity, completeness and universal domain. Relaxing anonymity may involve the rather undemocratic solution of a dictatorship. Relaxing completeness may involve the risk of 'stalemate' in that the decision process would not generally produce a determinate verdict on every proposition. The risk of 'stalemate' is greater if the unanimity rule is used for determining the group's propositional attitudes than if a suitable supermajority rule is used. But the use of the unanimity rule guarantees the avoidance of strong *and* weak path-dependence, whereas the use of a supermajority rule guarantees the avoidance of only strong but not weak path-dependence. Relaxing universal domain, finally, may be a promising escape-route from path-dependence (Theorems 10 and 11). Future research will have to address the question of how plausible it is (theoretically and empirically) that the relevant structure conditions on profiles are satisfied and how they can be brought about.

There might be factors that make path-dependence less of a threat. In some decision problems the subject-matter of the decision might by itself single out a specific decision-path as the appropriate one. Some propositions might, for instance, be regarded as "weightier than", or "prior to", others. In that case, the order in which the propositions are to be considered might be uncontroversial. But even in those cases it is of interest to ask whether or not such a privileged decision-path makes a difference to the outcome. If there is no path-dependence, regardless of whether there is a dispute about the decision-path, the perceived legitimacy of a particular outcome would be under no threat – the choice of a decision-path would be irrelevant to the outcome. If there is path-dependence, on the other hand, then a justification of the chosen decision-path becomes crucial. Moreover, in such cases, even if agreement can be reached on a privileged decision-path, this would solve only one of the two identified problems of strategic manipulation. It would solve the problem of agenda manipulation, as an agenda-setter would be constrained by the agreed choice of a decision-path. But it would not solve the problem of manipulation by the expression of untruthful views. As we have seen in Section 6, when a decision process merely has the *property* of being path-dependent (even if the decision-path is fixed), individuals may have incentives to express untruthful views. In our example in Section 6.2, individual 3 has an incentive to express untruthful views, even when the decision-path remains fixed. So, curiously, path-dependence matters even when the decision-path is not up for grabs.

Improving our understanding of path-dependence is an important challenge in the theory of democracy. Many democratic decision processes are sequential, and hence it is important to learn whether, and how, the decision-path matters, and what the implications of path-dependence are.

APPENDIX

LEMMA 1:

(i) The AR-function δ violates strong consistency

if and only if

(ii) there exist two semantically consistent subsets $\Psi_1, \Psi_2 \subseteq X$ and a proposition $\phi \in X$ such that [$\delta(\Psi_1) = 1$ *and* $\Psi_1 \models \phi$] *and* [$\delta(\Psi_2) = 1$ *and* $\Psi_2 \models \neg\phi$].

PROOF OF LEMMA 1:

(i) implies (ii): Suppose (i) holds. Then $\{\phi \in X : \delta(\phi)=1\}$ is semantically inconsistent. Let Ψ_2 be a maximal semantically consistent subset of $\{\phi \in X : \delta(\phi)=1\}$. First, Ψ_2 is non-empty, as, by assumption, X contains no contradictions. Second, Ψ_2 is a proper subset of $\{\phi \in X : \delta(\phi)=1\}$, as $\{\phi \in X : \delta(\phi)=1\}$ itself is not semantically consistent. Choose any $\psi \in \{\phi \in X : \delta(\phi)=1\} \backslash \Psi_2$. Since Ψ_2 is a *maximal* semantically consistent subset of $\{\phi \in X : \delta(\phi)=1\}$, $\Psi_2 \cup \{\psi\}$ is not semantically consistent (otherwise Ψ_2 would not be maximal); hence $\Psi_2 \models \neg\psi$. Let $\Psi_1 = \{\psi\}$; Ψ_1 is semantically consistent as, by assumption, ψ is not a contradiction. Then Ψ_1 and Ψ_2 have the properties required by (ii).

(ii) implies (i): Suppose (ii) holds. Since $\Psi_1 \models \phi$ and $\Psi_2 \models \neg\phi$, the set $\Psi_1 \cup \Psi_2$ is semantically inconsistent. But $\{\phi \in X : \delta(\phi)=1\}$ is a superset of $\Psi_1 \cup \Psi_2$. Therefore $\{\phi \in X : \delta(\phi)=1\}$ is also semantically inconsistent, and (i) holds. ■

LEMMA 2: *For any $\phi \in X$,*

(i) there exists a decision-path Ω such that $\phi \in M(\delta, \Omega)$

if and only if

(ii) there exists a semantically consistent subset $\Psi \subseteq X$ such that $\delta(\Psi)=1$ and $\Psi \models \phi$.

PROOF OF LEMMA 2:

(i) implies (ii): Suppose (i) holds. Let Ω be a decision-path such that $\phi \in M(\delta, \Omega)$. Choose t such that ϕ is accepted at time t in the priority-to-the-past decision process under the decision-path Ω. Let $\Psi = \{\psi \in X : \delta(\psi)=1$ *and ψ is accepted at some time $m<t$ in the decision-process under Ω*$\}$. As ϕ is accepted at time t, we have *either* $\delta(\phi)=1$ *or* $\Psi \models \phi$. If $\delta(\phi)=1$, then $\{\phi\}$ has the properties required by (ii); $\{\phi\}$ is semantically consistent, as, by assumption, ϕ is not a contradiction. If $\Psi \models \phi$, then Ψ has the properties required by (ii); Ψ is semantically consistent, as $\Psi \subseteq M(\delta, \Omega)$, which is semantically consistent.

(ii) implies (i): Suppose (ii) holds. Define Ω as follows. Let $t = |\Psi \cup \{\phi\}|$. On $\{1, 2, \ldots, t\}$, let Ω be any bijective mapping from $\{1, 2, \ldots, t\}$ onto $\Psi \cup \{\phi\}$ such that $\Omega(t) = \phi$. To make Ω complete, we add the following definition. On $\{t+1, ..., k\}$ (where $k = |X|$), let Ω be any bijective mapping from $\{t+1, ..., k\}$ onto $X \backslash (\Psi \cup \{\phi\})$. Then Ω has the properties required by (i). ■

PROOF OF THEOREM 1: Theorem 1 follows immediately from Lemmas 1 and 2. ■

LEMMA 3: *If* $M(\delta, \Omega)$ *is invariant under changes of* Ω, *then, for every* $\phi \in X$ (*and any decision-path* Ω), $\phi \in M(\delta, \Omega)$ *if and only if* $\delta(\phi)=1$.

PROOF OF LEMMA 3: Let $\phi \in X$. As $M(\delta, \Omega)$ is invariant under changes of Ω, we may consider any path Ω. For each $\phi \in X$, define a specific path Ω_ϕ as follows: let $\Omega_\phi(1) := \phi$, and $\Omega_\phi(2) := \neg\phi$; on $\{3, \ldots, k\}$, let Ω_ϕ be any bijective mapping from $\{3, \ldots, k\}$ into $X \backslash \{\phi, \neg\phi\}$. Suppose $\delta(\phi)=1$. Then, by the definition of a priority-to-the-past decision process, we have $\phi \in M(\delta, \Omega_\phi)$. Suppose, conversely, $\phi \in M(\delta, \Omega_\phi)$. If $\delta(\phi)=0$, then, as it is not the case that $\Phi(0) \models \phi$, we have $\phi \notin \Phi(1)$ and also $\phi \notin \Phi(2)$. But since ϕ or $\neg\phi$ do not occur elsewhere in the decision-path Ω_ϕ, we have $\phi \notin M(\delta, \Omega_\phi)$, contrary to the assumption that $\phi \in M(\delta, \Omega_\phi)$. Hence $\delta(\phi)=1$, as required. ■

PROOF OF THEOREM 3: Suppose $\delta: X \to \{0,1\}$ is strongly consistent.

(i) implies (ii): Suppose there exist two decision-paths Ω_1 and Ω_2 such that $\phi \in M(\delta, \Omega_1)$ and $\phi \notin M(\delta, \Omega_2)$, and assume, for a contradiction, that δ is deductively closed. As δ is strongly consistent, $M(\delta, \Omega)$ can be only weakly but not strongly path-dependent. As ϕ and Ω_1 satisfy condition (i) in Lemma 2, there exists a semantically consistent subset $\Phi \subseteq X$ such that $\delta(\Phi)=1$ and $\Phi \models \phi$. Then $\delta(\phi)=1$ by deductive closure of δ. Choose t such that $\Omega_2(t) = \phi$. Under path Ω_2, at time t, ϕ is *not* accepted. As $(\phi)=1$, this can only be because $\Phi_{t-1} \models \neg\phi$. Then, by the definition of a priority-to-the-past decision process, we have $\neg\phi \in M(\delta, \Omega_2)$, and hence $M(\delta, \Omega)$ is strongly path-dependent. This contradicts the strong consistency of δ. Hence δ is not deductively closed.

(ii) implies (i): Suppose that δ is not deductively closed with respect to $\phi \in X$, and assume, for a contradiction, that $M(\delta, \Omega)$ is invariant under changes of the decision-path. There exist $\Phi \subseteq X$ and $\phi \in X$ such that $\delta(\Phi) = 1$ and $\Phi \models \phi$ but $\delta(\phi)=0$. As δ is strongly consistent, $\delta(\Phi) = 1$ implies that Φ is semantically consistent. By Lemma 2, there exists a decision-path Ω such that $\phi \in M(\delta, \Omega)$. As, by assumption, $M(\delta, \Omega)$ is invariant under changes of the decision-path, we have $\phi \in M(\delta, \Omega)$ for *any* decision-path Ω. Consider in particular the path Ω_ϕ defined as in the proof of Lemma 3. We know that $\phi \in M(\delta, \Omega_\phi)$. As $\delta(\phi)=0$ (and it is not the case that $\Phi(0) \models \phi$), we have $\phi \notin \Phi(1)$ and also $\phi \notin \Phi(2)$. But as ϕ or $\neg\phi$ do not occur elsewhere in the decision-path Ω_ϕ, we have $\phi \notin M(\delta, \Omega_\phi)$, a contradiction. Hence $M(\delta, \Omega)$ is not invariant under changes of the decision-path. ■

PROOF OF THEOREM 4 (see also List and Pettit 2002): Suppose, for a contradiction, that δ is an aggregation function satisfying the conditions of Theorem 4. By assumption, we have P, Q, $(P \wedge Q)$, $\neg(P \wedge Q) \in X$.

Step 1. Since δ satisfies anonymity, we have: for any $(d_1, d_2, ..., d_n)$, $(e_1, e_2, ..., e_n) \in \{0, 1\}^n$, $\delta(d_1, d_2, ..., d_n) = \delta(e_1, e_2, ..., e_n)$ if $|\{i \in N : d_i = 1\}| = |\{i \in N : e_i = 1\}|$. For each $\phi \in X$, define $N_{accept\text{-}\phi} := \{i \in N : \delta_i(\phi)=1\}$. Then, for any ϕ, $\psi \in X$, if $|N_{accept\text{-}\phi}| = |N_{accept\text{-}\psi}|$, then $\delta(\phi) = \delta(\psi)$.

Step 2. By assumption, $n \geq 2$. Consider a profile $\{\delta_i\}_{i \in N} \in U$ with properties as shown in Table III.

TABLE III

	$\delta_i(P)$	$\delta_i(Q)$	$\delta_i((P \wedge Q))$	$\delta_i(\neg(P \wedge Q))$
$i = 1$	1	1	1	0
$i = 2$	1	0	0	1
$i = 3$	0	1	0	1
$i > 3$ and i is even	1	1	1	0
$i > 3$ and i is odd	0	0	0	1

Since δ satisfies the conditions of Theorem 4, it induces a complete, weakly consistent and deductively closed collective AR-function.

Case (i). n is even. We have $|N_{accept\text{-}(P \wedge Q)}| = |N_{accept\text{-}\neg(P \wedge Q)}|$, whence $\delta((P \wedge Q)) = \delta(\neg(P \wedge Q))$. By the completeness of δ, at least one of $\delta((P \wedge Q)) = 1$ or $\delta(\neg(P \wedge Q)) = 1$ must hold. But then we must have both $\delta((P \wedge Q)) = 1$ and $\delta(\neg(P \wedge Q)) = 1$, which contradicts the weak consistency of δ.

Case (ii). n is odd. We have $|N_{accept\text{-}P}| = |N_{accept\text{-}Q}| = |N_{accept\text{-}\neg(P \wedge Q)}|$, whence $\delta(P) = \delta(Q) = \delta(\neg(P \wedge Q))$.

If $\delta(P)=\delta(Q)=\delta(\neg(P \wedge Q))=1$, then, as $\delta(P)=\delta(Q)=1$ and $\{P, Q\} \models (P \wedge Q)$, we must have $\delta((P \wedge Q)) = 1$, by deductive closure. But then both $\delta(\neg(P \wedge Q)) = 1$ and $\delta((P \wedge Q)) = 1$, which contradicts the weak consistency of δ.

If $\delta(P)=\delta(Q)=\delta(\neg(P \wedge Q))=0$, then, by the completeness of δ, $\delta((P \wedge Q))=1$. But, as $\{(P \wedge Q)\} \models P$ and δ is deductively closed, we must have $\delta(P)=1$, a contradiction. ■

PROOF OF THEOREM 5: Suppose, for a contradiction, that δ is an aggregation function which satisfies the conditions of Theorem 5. Lemma 3 implies that, for every $\phi \in X$, $\phi \in M(\delta, \Omega)$ if and only if $\delta(\phi)=1$. As $M(\delta, \Omega)$ is complete and semantically consistent, δ must then be complete and strongly consistent, and hence also weakly consistent. By Theorem 4, as δ also satisfies anonymity, there must exist at least one profile $\{\delta_i\}_{i \in N} \in U$ such that δ is not deductively closed with respect to some proposition ϕ. But then, by Theorem 2, there exist two decision-paths Ω_1 and Ω_2 such that $\phi \in M(\delta, \Omega_1)$ and $\neg\phi \in M(\delta, \Omega_2)$. This contradicts the assumption that $M(\delta, \Omega)$ is invariant under changes of Ω. ■

PROOF OF THEOREM 6: Suppose δ is a weakly monotonic aggregation function, and for $\{\delta_i\}_{i\in N}$, $M(\delta, \Omega)$ is invariant under changes of the decision-path. Assume, for a contradiction, that there exists some individual $i\in N$ and some weakly monotonic preference ordering R_{i,δ_i} such that i has an incentive to express an untruthful AR-function δ^*_i ($\neq\delta_i$) at $\{\delta_i\}_{i\in N}$. Then $\Phi^* P_{i,\delta_i} \Phi$, where $\Phi^* = M(\delta_{\{\delta_1, ..., \delta^*_i, ..., \delta_n\}}, \Omega)$ and $\Phi = M(\delta_{\{\delta_i\}_{i\in N}}, \Omega)$. To deduce a contradiction, we show that the outcome set Φ is at least as close to δ_i as the outcome set Φ^*. As R_{i,δ_i} is weakly monotonic, this implies that $\Phi R_{i,\delta_i} \Phi^*$, contradicting $\Phi^* P_{i,\delta_i} \Phi$. To show that the outcome set Φ is at least as close to δ_i as the outcome set Φ^*, note first that, by Lemma 3, for every $\phi\in X$, $\phi\in M(\delta, \Omega)$ if and only if $\delta(\phi)=1$. Let δ and δ^* be the collective AR-functions induced by δ for profiles $\{\delta_i\}_{i\in N}$ and $\{\delta_1, \ldots, \delta^*_i, \ldots, \delta_n\}$, respectively. Then the outcome set Φ is at least as close to δ_i as the outcome set Φ^* if, for every $\phi\in X$, $|\delta(\phi) - \delta_i(\phi)| \leq |\delta^*(\phi) - \delta_i(\phi)|$. Now take any $\phi\in X$. Let $(d_1, d_2, ..., d_n) = (\delta_1(\phi), \delta_2(\phi), \ldots, \delta_n(\phi))$, and let $d^*_i = \delta^*_i(\phi)$. We have $\delta(\phi) = \delta(\delta_1(\phi), \ldots, \delta_i(\phi), \ldots, \delta_n(\phi)) = \delta(d_1, \ldots, d_i, ..., d_n)$ and $\delta^*(\phi) = \delta(\delta_1(\phi), \ldots, \delta^*_i(\phi), \ldots, \delta_n(\phi)) = \delta(d_1, \ldots, d^*_i, ..., d_n)$. As δ is weakly monotonic, we have $|\delta(d_1, \ldots, d_i, ..., d_n) - d_i| \leq |\delta(d_1, \ldots, d^*_i, ..., d_n) - d_i|$, and thus $|\delta(\phi) - \delta_i(\phi)| \leq |\delta^*(\phi) - \delta_i(\phi)|$. Hence the outcome set Φ is at least as close to δ_i as the outcome set Φ^*, as required. ■

PROOF OF THEOREM 7: Suppose the assumptions of Theorem 7 hold. Define $N_{accept\text{-}\phi}$ and $N_{reject\text{-}\phi}$ as in Section 7.3. Let $\delta: \{0,1\}^n \to \{0,1\}$ be supermajority voting with parameter $(m\text{-}1)/m$. Clearly, δ satisfies anonymity. Let $\{\delta_i\}_{i\in N}\in U$ be any profile. I show that there exists $i\in N$ such that $\{\phi\in X : \delta(\phi) = 1\} \subseteq \{\phi\in X : \delta_i(\phi) = 1\}$. As each δ_i is weakly consistent, there is no proposition $\phi\in X$ such that ϕ and $\neg\phi$ are both supported by a proportion of more than $(m\text{-}1)/m$ of the individuals. Hence $\{\phi\in X : \delta(\phi) = 1\}$ is weakly consistent. Partition $\{\phi\in X : \delta(\phi) = 1\}$ into equivalence classes of logically equivalent propositions. Let m^* be the number of such equivalence classes. Since $\{\phi\in X : \delta(\phi) = 1\}$ is a weakly consistent subset of X, we have $m^*\leq m\leq 2m$. Let the propositions $\phi_1, \phi_2, ..., \phi_{m^*} \in \{\phi\in X : \delta(\phi) = 1\}$ be representatives for these equivalence classes. Now $|N_{accept\text{-}\phi_1}|, |N_{accept\text{-}\phi_2}|, ..., |N_{accept\text{-}\phi_{m^*}}| > n(m\text{-}1)/m$. Then $N_{accept\text{-}\phi_1}$ and $N_{accept\text{-}\phi_2}$ have more than $n(m\text{-}1)/m\text{-}n/m = n(m\text{-}2)/m$ members in common. Further, $N_{accept\text{-}\phi_1}$, $N_{accept\text{-}\phi_2}$ and $N_{accept\text{-}\phi_3}$ have more than $n(m\text{-}2)/m - n/m = n(m\text{-}3)/m$ members in common. Continuing, $N_{accept\text{-}\phi_1}, N_{accept\text{-}\phi_2}, ..., N_{accept\text{-}\phi_{m^*}}$ have *more than* $n(m\text{-}m^*)/m$ members in common. Now, since $m^*\leq m$, $n(m\text{-}m^*)/m\geq 0$, and hence $N_{accept\text{-}\phi_1}, N_{accept\text{-}\phi_2}, ..., N_{accept\text{-}\phi_{m^*}}$ have more than zero members in common, i.e. at least one. Thus there exists $i\in N$ such that $i\in N_{accept\text{-}\phi_1}\cap N_{accept\text{-}\phi_2}\cap\ldots\cap N_{accept\text{-}\phi_{m^*}}$. But since every proposition in $\{\phi\in X : \delta(\phi) = 1\}$ is equivalent to one of $\phi_1, \phi_2, ..., \phi_{m^*}$ and δ_i is complete, weakly consistent and deductively closed, we also have $i \in \cap_{\phi \in \{\phi \in X : \delta(\phi) = 1\}} N_{accept\text{-}\phi}$. But then $\{\phi \in X : \delta(\phi) = 1\} \subseteq \{\phi \in X : \delta_i(\phi) = 1\}$, and the strong consistency of δ follows from the strong consistency of δ_i. ■

PROOF OF THEOREM 10:

(i) implies (ii): Suppose δ satisfies anonymity and, for any $\{\delta_i\}_{i\in N} \in \mathcal{R}$, $M(\delta, \Omega)$ is almost complete and invariant under changes of the decision-path. By Theorem 1, δ is strongly consistent (and thus also weakly consistent). By Theorem 3, δ is deductively closed. By Lemma 3, for every $\phi \in X$ (and any decision-path Ω), $\phi \in M(\delta, \Omega)$ if and only if $\delta(\phi)=1$. Since $M(\delta, \Omega)$ is almost complete, so is δ. Hence δ satisfies anonymity, and, for any $\{\delta_i\}_{i\in N} \in \mathcal{R}$, $\delta_{\{\delta_i\}_{i\in N}}$ is almost complete, weakly consistent and deductively closed. By Theorem 9, δ is majority voting.

(ii) implies (i): Suppose δ is majority voting. By Theorem 9, δ satisfies anonymity, and, for any $\{\delta_i\}_{i\in N} \in \mathcal{R}$, $\delta_{\{\delta_i\}_{i\in N}}$ is almost complete, weakly consistent and deductively closed, and hence strongly consistent. Choose any $\{\delta_i\}_{i\in N} \in \mathcal{R}$. By Theorem 3, $M(\delta, \Omega)$ is invariant under changes of the decision-path. By Lemma 3, $M(\delta, \Omega)=\{\phi \in X : \delta(\phi) = 1\}$. But since δ is almost complete, so is $M(\delta, \Omega)$. This completes the proof. ■

PROOF OF THEOREM 11: Suppose δ is majority voting. By Theorem 10, for any $\{\delta_i\}_{i\in N} \in \mathcal{R}$, $M(\delta, \Omega)$ is invariant under changes of the decision-path. By Lemma 3, $M(\delta, \Omega) = \{\phi \in X : \delta(\phi) = 1\}$. The result follows immediately from Theorem 8. ■

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