Understanding Preferences: “Demand Types”, and the Existence of Equilibrium with Indivisibilities

Elizabeth Baldwin* and Paul Klemperer†

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elizabeth-baldwin.me.uk/papers/demandtypes.pdf and
www.nuff.ox.ac.uk/users/klemperer/demandtypes.pdf

We propose new techniques for understanding agents’ valuations. Our classification into “demand types”, incorporates existing definitions (such as substitutes, complements, “strong substitutes”, etc.), and permits additional distinctions. We obtain an easy-to-check necessary and sufficient condition for the existence of a competitive equilibrium for indivisible goods. Our condition generalises many existing results, and provides new insights: contrary to much popular belief, there are more classes of purely-complements preferences than classes of purely-substitutes preferences for which competitive equilibrium always exists. Our techniques are also powerful when equilibrium cannot be guaranteed from the “demand type”. For a specific set of individual valuations, we often can check for equilibrium existence by simply counting the number of intersection points of the geometric objects we study! Our methods also have applications to matching, and to the Product-Mix Auction, introduced by the Bank of England in response to the financial crisis.

Keywords: consumer theory; equilibrium existence; general equilibrium; competitive equilibrium; duality; indivisible goods; geometry; tropical geometry; convex geometry; auction; product mix auction; product-mix auction; substitute; complement; demand type; matching

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*London School of Economics, UK; e.c.baldwin@lse.ac.uk

†Nuffield College, Oxford University, UK; paul.klemperer@economics.ox.ac.uk

‡This paper extends and supersedes much of the material in Baldwin and Klemperer (2014). However, large parts of Sections 2.5, 6.2, and 6.3.1 of that paper will be incorporated and developed in Baldwin and Klemperer (in preparation-a), and Sections 4.2, 4.3, and 5, form the main part Baldwin and Klemperer (in preparation-b). This work was supported by ESRC grant ES/L003058/1. Acknowledgements to be completed later.

1 Introduction

This paper introduces new ways of thinking about demand for indivisible goods, and obtains new results about the existence of competitive equilibrium.[1]

"Demand types"

Our first key idea is to classify economic agents' individual and aggregate valuations into "demand types". A "demand type" is defined by a list of vectors that give the possible ways in which the individual or aggregate demand can change in response to a small generic price change. These vectors are analogous to the rows of a Slutsky matrix; with indivisibilities the dimensionality is low enough that we can characterize a *class of* valuations *globally* in this way.

So the vectors defining a "demand type" specify the possible comparative statics of any demand of that "type", and thus much of what economists think important about valuations. For example, a purchaser of lenses and spectacle frames who is interested in having spare pairs might always buy in the ratio 2:1, so always increases or reduces her demand for lenses and frames in this ratio, whatever the individual prices of the goods. We will describe *any* such valuation as being of "demand type $\pm\{(2,1)\}$". As another example, a consumer who views apples and bananas as substitutes would have preferences of "demand type $\pm\{(1,0),(0,1),(-1,1)\}$" if, whenever prices change slightly, the bundle she chooses only ever changes in the direction of adding or subtracting an apple, or of adding or subtracting a banana, or of substituting one piece of one kind of fruit for one piece of the other kind.[2]

Our classification is parsimonious. For example, the "type" that comprises all possible substitutes preferences for indivisible goods is defined by the set of all vectors with at most one positive integer entry, at most one negative integer entry, and all other entries zero; the "type" that is all complements preferences for indivisible goods is defined by the set of all vectors in which all the non-zero entries (of which there may be any number) are integers of the same sign; the class of all "strong substitutes"[3] preferences for n goods is a "demand type" with just $n(n+1)$ vectors.

Our classification clarifies the relationships between different classes of preferences. For example, our "demand types" descriptions show clearly *why* the conditions for three or more indivisible goods to all be (ordinary) substitutes are far more restrictive than the conditions for them to all be complements–although they are, of course, symmetric in the two-good case.

Our classification is also easy to work with, and very general. It always permits multiple units of each good; the agents can include sellers, buyers, and traders who can both buy and sell; and we will see that an aggregate valuation retains the "type" of the

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[1] Baldwin and Klemperer (in preparation-c) shows our techniques also help analyse demand for divisible goods. We assume, as is most common in the indivisible-goods literature, that preferences are quasilinear, so there is no distinction between compensated and uncompensated demand.

[2] For example, in an auction in which goods' characteristics suggest natural rates of substitution, bidders can be asked to express preferences that come from the corresponding "demand type". Indeed the original version of the Bank of England's Product-Mix Auction had one-for-one substitution built into its design (see Klemperer, 2008, 2010).

[3] See Section 3.2: Milgrom and Strulovici's (2009) "strong substitutes" generalised many existing valuation structures.

individual valuations it is based on. The classification also applies to matching models; in this case, the demand type is just the set of possible coalitions.

Moreover, using "demand types" leads to powerful new results:

Equilibrium existence

We give a simple necessary and sufficient condition, which generalises existing results, about whether or not competitive equilibrium always exists, whatever is the market supply, if all agents' valuations are drawn from a given class of valuations (i.e., are of a specific "demand type").

Our condition can easily be checked using the determinants of sets of the vectors in the "demand type". So we can quickly see whether any demand structure guarantees equilibrium existence. Several well-known results are easy special cases. Moreover, our results demolish the popular perception that the existence of equilibrium with indivisible goods depends on substitutes valuations (or re-packagings of goods for which valuations are substitutes). Indeed *every* demand type for which equilibrium is guaranteed can be obtained as a basis change of a demand type involving only *complementary* relationships (and for which equilibrium is also guaranteed)–and the corresponding result is not true for substitute preferences.

Our geometric methods also give beautiful answers to whether or not competitive equilibrium exists for any market supply, for a *specific* set of agents' valuations, when they are not all drawn from a "demand type" for which existence is always guaranteed. These answers are also easy to apply.

In particular, we will show how the existence of equilibrium is related to the number of price vectors at which more than one agent is indifferent between more than one bundle. As an elementary illustration, suppose a hotel has two bedrooms. Paul's family would like both or neither. Elizabeth and her partner want at most one. We will see that in this particular case there are at most *two* price vectors at which both Paul and Elizabeth are indifferent between more than one bundle, and that competitive equilibrium exists for valuations such that there are exactly two such prices. For example, imagine Paul is indifferent between paying £100 for both rooms, and looking elsewhere. Elizabeth is prepared to pay up to £30 for the smaller room, or £60 for the larger room. There are thus exactly *two* pairs of prices, (£30, £70) and (£40, £60), for the smaller and larger rooms respectively, such that both Paul and Elizabeth are indifferent between more than one option (Paul between taking neither room and both rooms, and Elizabeth between taking neither room and the one room she considers good value). And as predicted, competitive equilibrium exists: there exist prices such that demand exactly equals supply. (Indeed any pair of prices that exceed £30 for the smaller and £60 for the larger, but add to less than £100, will clear the market). However, suppose Paul is prepared to pay only £70 for the rooms. Now there is just *one* pair of prices (£20,£50) such that both Paul and Elizabeth are indifferent between more than one option (Paul between both rooms and neither, and Elizabeth between the two rooms). And we can check there exists no competitive equilibrium for these valuations: at any prices at which Paul is prepared to take both rooms, Elizabeth will also demand a room.[4]

The geometric objects we study, as described below, contain precisely those points at which an agent is indifferent between multiple bundles. So a simple count of the

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[4] Section 5.3 explains this example in detail.

number of intersection points of these geometric objects often suffices to determine whether competitive equilibrium exists!

Detailed description of the paper

The reason our "demand types" are a mathematically convenient way to categorize valuations is that the vectors they comprise describe how price space is divided into the different regions in which an agent demands different bundles. This division creates *precisely* the geometric structures studied in "tropical" geometry.[5] So we can apply the tools of convex and tropical geometry. The duality between the geometric object representing a valuation in price space, and the geometric object corresponding to the same valuation in quantity space, is particularly fruitful.

So we begin, in Section 2, by translating some existing mathematics literature into economics. We describe the properties of a "tropical hypersurface", a geometric object containing price vectors at which the agent is indifferent between two or more bundles. The economic interest is that these are the prices at which the agent's demand changes. Moreover, we observe that any geometric structure of this kind corresponds to a valuation function, so we can develop our understanding of demand by working directly with these geometric objects; we believe this is the first paper to do this. We then explore duality for indivisible demand.

Section 3 defines a "demand type" using the set of vectors describing the ways in which the bundles demanded by the agent change with prices. These are associated in a simple geometric way with the tropical hypersurface. It is then elementary to check whether a demand type is, for example, substitutes, or complements, or "strong substitutes", or "gross substitutes and complements", etc. Importantly, the "demand type" of the aggregate valuation of multiple agents is simply the union of the vectors of the individual "demand types".

Section 4.1 therefore turns to aggregate valuations and competitive equilibrium for "demand types": whether or not equilibrium exists depends on the nature of the intersections of agents' tropical hypersurfaces, and in particular on their "multiplicities". So we prove that equilibrium always exists for *any* set of agents who all have concave valuations of a given "demand type" on n goods, *if and only if* every subset of n of the "type's" vectors has determinant 0 or ± 1 (plus an additional condition if the demand type's set of vectors is in fewer than n dimensions).

Our necessary and sufficient condition on demand types is easy to check. We simply find the demand type of each individual agent *separately*, and it is then obvious whether or not the demand type of their aggregate satisfies the criterion. By contrast, Bikhchandani and Mamer's (1997) and Ma's (1998) conditions for existence of equilibrium for a set of agents need to be checked against every possible combination of agents–this seems both less practical and to give less insight into why agents' valuations do or don't permit equilibrium.

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[5] Tropical geometry is a branch of algebraic geometry recently developed by, among others, Mikhalkin (2004, 2005). We believe it has not previously been applied to economics. Goeree and Kushnir (2012) have used convex geometry (see, e.g., Rockafellar, 1970), on which tropical geometry builds, in a different context, while Richter and Rubinstein (Forthcoming) have shown the economic applicability of a more general and abstract definition of convexity. However, Danilov and Koshevoy and their co-authors' methods of discrete convex analysis have closer connections to ours (see, in particular, Danilov et al., 2001, Danilov et al., 2003 and Danilov and Koshevoy, 2004), as we discuss later in the introduction, and in detail in Section 4.1.

Our *sufficient* condition yields a class of results, each stating that equilibrium always exists when every individual valuation has a certain property. An example of such a result is that equilibrium always exists when every agent's valuation is "strong substitutes". This specific result is *not* new (see Milgrom and Strulovici, 2009), but it follows immediately from our theorem, as do others, such as some in Kelso and Crawford (1982), Sun and Yang (2006), Hatfield et al. (2013), and Teytelboym (2014), and extensions of many of these.

New economic properties that guarantee equilibrium are also easy to generate. For example, we exhibit a "demand type" which involves only complementarities and, moreover, which is not related (via any basis change) to any set of preferences for substitutes.

Because our condition is also *necessary*, we can quickly check whether equilibrium will always exist if agents' valuations are of any particular demand type. It follows easily, for example, that with (multiple units of) three or fewer goods, equilibrium always exists if and only if goods are either "strong substitutes" or a basis change of strong substitutes.[6] (However, this is not true with four or more distinct goods.) Furthermore, our geometric approach immediately provides an example of failure of equilibrium whenever our condition fails.

This theorem is closely related to the work in a remarkable series of papers by Danilov and Koshevoy and their co-authors. In particular, Danilov et al. (2001) provide a sufficient condition for equilibrium, which is mathematically very similar to our sufficient condition. However, our concept of "demand types" shows how this condition can be applied–for example, none of the papers listed above, whose equilibrium existence results are obvious corollaries of our theorem, present their results as applications of Danilov et al., since the latter's relevance was not clear. Our "demand types" also illuminate the condition's economic meaning for individual agents and, moreover, they make clear the sense in which the *same* condition is also *necessary* for equilibrium, which is not proved in Danilov et al. (2001).[7]

Finally, Section 5 shows that our methods, unlike Danilov et al.'s, also yields additional existence results of a quite different kind, namely about when combinations of *specific* valuations always yield equilibria (as distinct from when any set of valuations from some class always yield equilibria):

Tropical hypersurfaces can be understood as transformations of "ordinary" geometric objects, and versions of "ordinary" intersection theory apply tropically. In particular, there is a tropical version of Bézout's classic theorem that the number of intersections of two curves, taking into account "multiplicities" such as tangencies, is equal to the product of the degrees of their defining polynomials.[8] Since, we will see, failures of

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[6] Observe that our necessity result contrasts with "necessity" results of the kind given in several of the works listed above, which show only that equilibrium always exists if all agents' valuation functions have a certain property, but may fail if just one valuation function does not.

[7] We especially thank Gleb Koshevoy for very helpful discussions. We discuss the relationships to, and other distinctions from, Danilov and Koshevoy and their co-authors' work in Section 4.1. Analysing and interpreting our concept of "demand types" in price space allows us to develop economic implications further than they do, since they almost exclusively study quantity space. Moreover, though our techniques are novel, they are more straightforward than theirs. However, their work deserves *far* more attention than it has thus far received.

[8] For example, in "ordinary" geometry, two lines intersect once (possibly at infinity). A quadratic and a line intersect twice including intersections with complex coordinates (and counting intersections at infinity, and double-counting tangencies). Two quadratics intersect four times (correctly counted),

competitive equilibrium correspond to multiplicities, we can therefore often determine the existence or failure of equilibrium by simply counting the intersections of the tropical hypersurfaces!

Furthermore, even when this count does not suffice, our methods yield a recipe for determining whether or not equilibrium always exists for a given set of individual agents' valuations (for any possible market supply). Moreover, our recipe requires checking the properties of the valuations at only a finite collection of prices, whose number we bound.

Section 6 presents applications: Sections 6.1–6.2 observe that our model encompasses classic models such as Kelso and Crawford (1982), Hatfield et al (2013), and cycles of complements. This clarifies the relationships between these papers, reveals how they can be extended, and shows that their equilibrium-existence results are immediate special cases of ours. Section 6.3 uses the fact that equilibrium properties are unaffected by basis changes to show yet more results are easy corollaries of our work. Section 6.4, by contrast, exhibits a new, purely-complements, demand type for which equilibrium is guaranteed, but which is not a simple basis change of any standard demand type.

Section 6.5–6.6 notes that our approach yields new results about when stable matches exist in matching models, not only in the well-studied bipartite case, but also for more general multi-agent matchings, and has additional applications to the theory of an individual agent's demand.

Finally, Section 6.7 observes that our geometric techniques can develop extensions to the Bank of England's "Product-Mix Auction".[9] Our methods show what kinds of bids are needed to represent different kinds of preferences, analyse the implications of different restrictions on bids, reveal how to efficiently solve for equilibrium (and when it exists), etc.

Section 7 concludes. Appendix A contains proofs of results in the text.

This paper has been written for economists. Our ideas (and the economic context) have been translated for a mathematical audience by Tran and Yu (2015), and mathematicians may find it easier to read that paper first.[10]

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etc.

[9] Bidders in these auctions make sets of either/or bids for alternative objects. The Bank of England represents these bids geometrically as sets of points in multi-dimensional price space.

The then-Governor of the Bank (Mervyn King) told *The Economist* that the Product-Mix Auction "is a marvellous application of theoretical economics to a practical problem of vital importance to financial markets"; an Executive Director of the Bank described it as "a world first in central banking", and "potentially a major step forward in practical policies to support financial stability"; and current-Governor Mark Carney announced plans for greater use of the auction, and introduced an updated version endogenising total quantity and permitting more dimensions (i.e., more goods)–see Bank of England, 2010, 2011, Milnes, 2010, Fisher, 2011, Fisher et al., 2011, and *The Economist*, 2012.

(In principle, of course, the loans that the Central Bank auctions are almost continuously divisible, but we can use some of our same indivisible-good techniques to analyse this auction.)

[10] In addition to providing a mathematical exposition of our work, they also give an additional proof (via integer programming) of theorem 4.2.

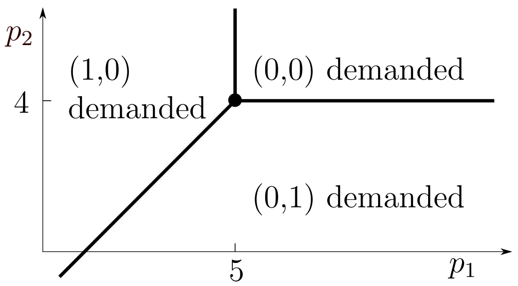

Figure 1: A simple tropical hypersurface (TH). The bundle demanded on each side of the TH is labelled.

2 Representing Indivisible Demand Geometrically

2.1 Assumptions, and Tropical Hypersurfaces (THs)

There are n goods, which come in indivisible units. Each agent has quasilinear utility, with valuation function $u : A \to \mathbb{R}$ on bundles in a finite *domain* $A \subsetneq \mathbb{Z}^n$.[11] We make no further restrictions on the domain A; it need not be discrete-convex,[12] and it may include negative bundles so agents can sell as well as buy. So at a price vector $\mathbf{p} \in \mathbb{R}^n$, the agent demands

$$D_u(\mathbf{p}) := \arg\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\}.$$

We will be particularly interested in the prices at which demand changes, that is, those prices at which the agent is indifferent among more than one bundle, namely the set of $\mathbf{p}$ for which $\#D_u(\mathbf{p}) > 1$. This set (with some additional structure–see Definition 2.1) is known as a "tropical hypersurface" (TH).[13] We will see that we have an essentially perfect correspondence between THs and concave valuation functions (Theorem 2.7), but believe ours is the first paper to use THs in economics.[14]

2.2 The Structure of Tropical Hypersurfaces

Fig. 1 shows a simple example of a TH. The agent's valuations are $u(0,0) = 0$, $u(1,0) = 5$ and $u(0,1) = 4$. So it demands a unique bundle in each of the three *unique demand regions* (UDRs), but switches between bundles along the three line segments, which together form the TH. In general, a TH is made up of $(n-1)$-dimensional linear components, which we call *facets*, and which separate the n-dimensional UDRs from each

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[11] That is, utility is linear in money (with no budget constraint): utility from bundle $\mathbf{x}$ at prices $\mathbf{p}$ is $u(\mathbf{x}) - \mathbf{p}.\mathbf{x}$. We assume different units of the same good always have the same price (which can be negative)–we can, of course, model different units of a homogeneous good which are priced independently by treating them as different goods.

We initially restrict to a single agent. We will later consider a finite set of agents with valuations u^j on domains A_j, $j = 1, \dots, m$, and will then write A for the "aggregate" domain, $\{\sum_j \mathbf{x}^j \mid \mathbf{x}^j \in A_j\}$.

[12] A is discrete-convex if all the integer points in its convex hull, Conv A, are in A, that is, $(\text{Conv } A) \cap \mathbb{Z}^n = A$.

[13] See Mikhalkin (2004) and others. In fact, our TH is "upside down" compared with Mikhalkin, who considers the non-smooth locus of $\mathbf{p} \mapsto \max_{\mathbf{x}\in A}\{\mathbf{x}.\mathbf{p} - u(\mathbf{x})\}$, but his convention is not universal, and our definition seems the most natural for economics.

[14] We developed these ideas first in our working paper, Baldwin and Klemperer (2012).

other.[15] A facet is defined to be closed, i.e., to contain its boundary; that boundary is itself made up of finitely many $(n-2)$-dimensional linear components, and this pattern continues on down the dimensions.[16] Any geometric object satisfying this description is called a *polyhedral complex.* It is also *rational* if, as will always be the case for us, each of its components can be defined by (linear) equations with integer coefficients. The k-dimensional components are called *k-cells.* So, for example, the TH of Fig. 1 contains three 1-cells, and one 0-cell (where the 1-cells meet). Observe that each 1-cell is the complete set of prices where two specific bundles are demanded, while the 0-cell is the unique price where all three possible bundles are demanded. More generally, any k-cell is the set of prices at which a particular set of bundles is demanded. Appendix A.1.1 gives a full, formal, taxonomy of THs and their economic interpretation.

Demand is constant in each UDR, since demand cannot switch from one unique bundle to another without passing through a price at which demand is non-unique. At a price in a facet, the agent is indifferent between the bundles $\mathbf{x}$ and $\mathbf{x}'$ demanded in the UDRs on either side of the facet. That is, $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$ for every $\mathbf{p}$ in any facet, F. So $\mathbf{p}.(\mathbf{x}' - \mathbf{x})$ is a constant for $\mathbf{p} \in F$, and the vector $\mathbf{x}' - \mathbf{x}$ is therefore normal to F. We call the greatest common divisor of the entries of $\mathbf{x}' - \mathbf{x}$ the *weight* of the facet, $w(F)$. So $\frac{1}{w(F)}(\mathbf{x}' - \mathbf{x})$ is a "primitive" integer vector (the greatest common divisor is 1), and it points from the UDR where $\mathbf{x}'$ is demanded to the UDR where $\mathbf{x}$ is. But since F is $(n-1)$ dimensional, there is a unique primitive integer vector normal to F and pointing from the UDR of $\mathbf{x}'$ to that of $\mathbf{x}$. So if we know F, $w(F)$ and $\mathbf{x}$, we can derive $\mathbf{x}'$. More generally, therefore, if we know the demand in any one UDR, we can work out the demand in every UDR, if we also know (1) all the facets (i.e., all the prices at which demand is non-unique), and (2) all the facet weights–and (1) and (2) are precisely the information that defines a TH:

Definition 2.1 (Mikhalkin, 2004, Example 2)**.** The *tropical hypersurface* $\mathcal{T}_u$ associated with any valuation u is the weighted rational polyhedral complex such that:

(1) its underlying set is $\{\mathbf{p} \in \mathbb{R}^n | \ \#D_u(\mathbf{p}) > 1\}$;

(2) the *weight* $w_u(F)$ of the facet F is the integer defined by $w_u(F)\mathbf{v}_F = \mathbf{x}' - \mathbf{x}$, in which $\mathbf{x}'$ is demanded in the UDR on one side of F; $\mathbf{x}$ is demanded in the UDR on the other side; and $\mathbf{v}_F$ is the primitive integer normal vector pointing from the former to the latter.

This definition is mathematically equivalent to Mikhalkin's, but the mathematical literature has not, of course, interpreted them in an economic context (that is, understood the $D_u(p)$ as demand sets).

2.3 The correspondence between *specific* valuations and THs

If we follow an agent along a price path that ends where it started, the demand at the end must be the same as that at the beginning. So the weights on the facets must satisfy the *balancing condition*:

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[15] Note that facets meet only at their boundaries. So in two dimensions, for example, the continuation of a straight line past an intersection with another facet, forms a new and distinct facet.

[16] So in three dimensions, for example, the facets are pieces of planes, whose boundaries are the line segments where they meet; the boundaries of the line segments are the points where line segments meet.

Definition 2.2 (Mikhalkin, 2004, Definition 3)**.** An $(n-1)$-dimensional weighted rational polyhedral complex $\Pi \subsetneq \mathbb{R}^n$ is *balanced* if for every for every $(n-2)$-dimensional cell $G \subsetneq \Pi$, the weights $w(F_j)$ on the facets $F_1, \ldots, F_l$ that are adjacent to G, and primitive integer normal vectors $\mathbf{v}_{F_j}$ for these facets that are defined by a fixed rotational direction about G, satisfy $\sum_{j=1}^{l} w(F_j)\mathbf{v}_{F_j} = 0$.[17]

This balancing condition is in fact the *only* condition that a weighted rational polyhedral complex has to satisfy to be the TH of some valuation function:[18]

Theorem 2.3 (Mikhalkin, 2004, Proposition 2.4; also Mikhalkin, 2005, Theorem 3.15)**.** *Suppose that* Π *is an* $(n-1)$*-dimensional balanced weighted rational polyhedral complex in* $\mathbb{R}^n$*. Then there exists a finite set* $A \subsetneq \mathbb{Z}^n$ *and a function* $u : A \to \mathbb{R}$ *such that* Π *is the TH,* $\mathcal{T}_u$*.*

It follows that a set in $\mathbb{R}^n$ is the TH of some quasilinear valuation *if and only if* it is a rational polyhedral complex and there exist weights for the facets such that it is balanced. It is often much easier to develop our ideas and intuitions by working with these geometric objects than by thinking of examples of valuations, and in the subsequent sections we will see how describing the geometry of the objects gives us insights into their economics.

We will be particularly interested in concavity of valuation functions in the standard discrete sense:

Definition 2.4. A function $u : A \to \mathbb{R}$ is *concave* if A is a discrete-convex set and u can be extended to a weakly-concave function on $\mathbb{R}^n$.

The significance of concavity is that it is a standard result that concave valuations are precisely those for which every possible bundle is demanded at some price, and for which the demand set at any price is discrete-convex, just as for divisible, weakly-concave valuations, and for essentially the same reasons:[19]

Lemma 2.5. $u : A \to \mathbb{R}$ *is concave*
iff A *is a discrete-convex set and for all* $\mathbf{x} \in A$ *there exists* $\mathbf{p}$ *such that* $\mathbf{x} \in D_u(\mathbf{p})$
iff $D_u(\mathbf{p})$ *is discrete-convex for all* $\mathbf{p}$.

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[17] This is just the n-dimensional generalisation of the requirement in 2 dimensions that, when moving in a sufficiently small circle around any point, the weights on any facets crossed be coherent. To choose a rotational direction around G, pick a 2-dimensional affine subspace H of $\mathbb{R}^n$ orthogonal to G, such that the intersection of each F_j with H is 1-dimensional. The intersection of H with the TH is then a collection of 1-cells meeting at the 0-cell which is $G \cap H$. An ordinary choice of rotational direction in this two-dimensional picture gives a rotational direction around G in $\mathbb{R}^n$.

[18] There do not necessarily exist weights to balance a general rational polyhedral complex. For example, in two dimensions, consider three points (0-cells), each with three adjacent facets, such that each pair of points has an adjacent facet in common. There are six weights, which must satisfy six equations (three balancing conditions in each of the two dimensions). But since the conditions are trivially satisfied by setting all weights equal to zero, the conditions can only be satisfied by positive integer weights if the conditions are not linearly independent–which is non-generic.

[19] For the divisible case see, e.g., Mas-Colell et al. (1995) pp. 135-8, especially Prop. 5.C.1(v), since a quasilinear valuation is equivalent to a standard profit function with a single-output technology. These results are also clear from considering the example in the next subsection (2.4).

Furthermore, it is easy to see that any non-concave valuation has the same TH as the minimal concave function that weakly exceeds it, since increasing any never-demanded bundle's value has no effect until the bundle is just marginally demanded, when the value function becomes locally affine. The marginally defined bundle is now added to the demand at some prices, but is never demanded uniquely, and all other bundles are demanded exactly as they were previously, so:

Lemma 2.6. *Let u' be the minimal concave function that weakly exceeds u.*[20] *Then* $\mathcal{T}_{u'} = \mathcal{T}_u$.

Clearly, adding a constant to $u(\mathbf{x})$ leaves the TH unchanged, as does increasing every available bundle by a fixed bundle and making a corresponding shift in the valuation.[21] So we have full equivalence between THs and concave valuation functions, up to shifts by a constant:

Theorem 2.7 (Mikhalkin, 2004, Remark 2.3)**.** *THs with an identified "demand 0" UDR are in bijective (1-1) correspondence with concave valuations u such that* $u(0) = 0$ *and such that demand is* 0 *at prices in the "demand 0" UDR.*

Importantly, therefore, *any* balanced weighted rational polyhedral complex also corresponds to some concave valuation, so we can develop our understanding of valuations by working directly with these geometric objects. However, we will *not* restrict attention to concave valuations.

2.4 Duality; and Subdivided Newton Polytopes (SNPs)

We constructed the TH in price space. We now construct a dual geometric object–the Subdivided Newton Polytope (SNP)–in quantity space. This presents much of the same information in a complementary way.[22]

Just as in the standard duality construction for a divisible, strictly-concave valuation, we will see that any price vector defines a hyperplane, tangent to the graph of the agent's valuation, which meets this graph at the agent's demand set for that price. But in our case, because the demand set sometimes contains more than one bundle, some tangent hyperplanes meet the graph at more than one point.

For example, Fig. 2a shows a valuation function, u, and Fig. 2b gives its graph, using bars to associate a bundle, $\mathbf{x}$, with its valuation, $u(\mathbf{x})$. We will always present the feasible bundles increasing to the *left*, and *down.* This will show the duality between the SNP and the TH most clearly.

The bundles, $\mathbf{x}$, demanded at a price, $\mathbf{p}$, are those that maximise $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = (-\mathbf{p}, 1).(\mathbf{x}, u(\mathbf{x}))$. So $\mathbf{x}$ is demanded at $\mathbf{p}$ if the point $(\mathbf{x}, u(\mathbf{x}))$ is "farthest out" from the origin in the "direction of that price" (i.e., the direction $(-\mathbf{p}, 1)$). Thus, the bundles which are demanded for *some* price are those bundles, $\mathbf{x}$, for which $(\mathbf{x}, u(\mathbf{x}))$ lies on the

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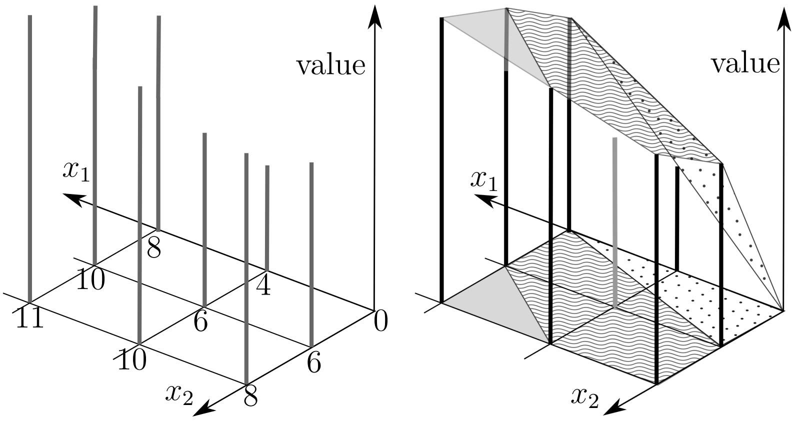
[20] If u's domain is not discrete-convex, then u''s domain must be the minimum discrete-convex set containing it.

[21] Of course, the bundle demanded in each UDR is then increased by the fixed bundle.

[22] The construction uses Legendre-Fenchel duality; e.g. see Murota (2003) but is not a precise duality: as we will see, information is lost, so that a single SNP corresponds to a set of THs. For more on these 'regular subdivisions' and on polytopes in general see Thomas (2006) and De Loera et al. 2010.

x_1				
2	*1*	*0*	$u(\mathbf{x})$	
8	4	0	*0*	
10	6	6	*1*	x_2
11	10	8	*2*	

(a) Tabular representation of a valuation, $u(\mathbf{x})$.



(b) A representation of the valuation using bars.

(c) The "roof" of the valuation.

Figure 2: A valuation and its roof.

minimal concave function which is everywhere weakly greater than u (in this example, they are every bundle except $(1, 1)$). We extend this latter function to the convex hull of the set of feasible bundles, and call its graph the *roof* of the valuation. Looking at Figs. 2b and 2c shows that the roof is simply the "top" of the convex hull of the points $(\mathbf{x}, u(\mathbf{x}))$ with respect to the final coordinate.

Because the agent is indifferent between several bundles at some prices, the roof is composed of linear pieces, that meet along lower-dimensional linear pieces; like the TH, it is a "polyhedral complex". (In our 2-good example, the roof includes pieces of planes, line segments, and points.) Each of the roof's vertices is at a bundle which is the unique demand for some prices. Conversely, if a top-dimensional cell of the roof is n-dimensional (a piece of hyperplane), this will correspond to a set of bundles among which the agent is indifferent at some unique price, $\mathbf{p}$, which is farthest out in a unique direction, $(-\mathbf{p}, 1)$, from the origin. More generally, *any cell of the roof is the intersection of some tangent hyperplane(s) with the roof, and so is the convex hull of the demand set for some price(s).*

By projecting downwards the top dimensional pieces of the roof, we can subdivide the convex hull of the set of feasible bundles–see Fig. 2c. Since the convex hull is called a "Newton Polytope" (in standard terminology), we call the resulting object a *Subdivided Newton Polytope* (SNP). We call the projection of a vertex of the roof a *vertex* of the SNP (which therefore corresponds to a bundle that is uniquely demanded at some price), and call the projection of a line segment of the roof an *edge* (which therefore corresponds to the line joining two uniquely-demanded bundles).

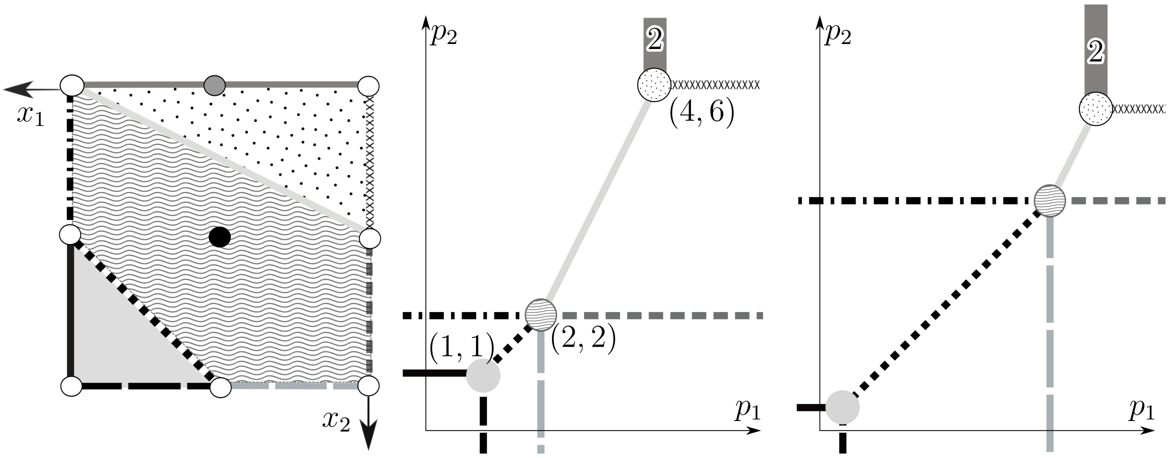
Note in particular, therefore, that an edge of the SNP with endpoints $\mathbf{x}$ and $\mathbf{x}'$ indicates the existence of prices, $\mathbf{p}$, for which the demand set contains both these bundles. Moreover, these prices form a facet of the TH: the family of tangent hyperplanes passing through a line segment of the roof is $(n-1)$-dimensional, and so the space of vectors $(-\mathbf{p}, 1)$ normal to these hyperplanes is similarly $(n-1)$-dimensional. As we noted in Section 2.2, $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$, that is, $\mathbf{p}.(\mathbf{x}' - \mathbf{x}) =$ *constant*, for all these price vectors, $\mathbf{p}$. So each (1-dimensional) edge of the SNP is normal to the ($(n-1)$-dimensional) facet that corresponds to it in the TH.

More generally, each k-cell of the SNP is the convex hull of the bundles which form the demand set for some price; and the set of prices, $\mathbf{p}$, for which these bundles are contained in the demand set $D_u(\mathbf{p})$ is an $(n-k)$-cell of the TH that is orthogonal to it:

Lemma 2.8. *The k-cells of the SNP and the $(n-k)$-cells of the TH are in bijective correspondence. Each TH cell is orthogonal to its corresponding SNP cell. That is,* $(\mathbf{p}'-\mathbf{p}).(\mathbf{x}'-\mathbf{x})=0$, *for all* $\mathbf{p},\mathbf{p}'$ *in the TH cell and* $\mathbf{x},\mathbf{x}'$ *in the SNP cell.*

In particular, *each edge of the SNP is normal to its corresponding facet in the TH.*

The SNP and TH of the valuation of Fig. 2a are pictured in Figs. 3a and 3b respectively,[23] depicting each cell in the same style as its dual cell in the other figure.



(a) The SNP.

(b) The TH of the valuation given in Fig. 2a, and so corresponding to the SNP in (a).

(c) Another TH corresponding to the SNP in (a), but with a different valuation.

Figure 3: (a) The SNP and (b) TH of the valuation shown in Fig. 2a, with dual geometric objects drawn with the same style and shading; (c) another TH of the same combinatorial type.

Thus the 0-cells (vertices) of the TH at the prices (4,6), (2,2), and (1,1) correspond to the dotted-, wavy-, and light-grey-, shaded 2-cells (areas) of the SNP, respectively, and also to the three correspondingly-shaded pieces of planes of the roof in Fig. 2c; the nine separately-distinguished 1-cells (line-segments) of the TH correspond to the nine correspondingly-styled 1-cells of the SNP; and each of the seven UDRs of the TH corresponds to one of the seven bundles that are the 0-cells of the SNP, and are shown as white circles in the SNP–we discuss the dark grey and black SNP bundles below.[24]

Notice that the dark grey horizontal edge at the top of the SNP passes through the grey bundle. This edge consists of two copies of the primitive integer vector, $(1,0)$, in its direction, so we say that this edge has *length* 2.[25] It is dual to the dark grey vertical facet of the TH which correspondingly has *weight* 2, and is so labelled. Recall

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[23] We typically draw a SNP without axes, since replacing A with $A+\mathbf{c}$ for some $\mathbf{c}\in\mathbb{Z}^n$ and re-defining u to correspond gives us the same SNP and TH.

[24] Clockwise from the top right of the TH, the bundles demanded in the UDRs are (0,0), (0,1), (0,2), (1,2), (2,2), (2,1), and (2,0).

[25] All other edges of this SNP have length 1 in this sense, which is of course *not* the Euclidean length.

from Section 2.2 that a facet's "weight" times its primitive integer normal vector is the change in demand between the UDRs it separates.

Neither the grey bundle, nor the black bundle, is a 0-cell of the SNP, and, therefore, no UDR corresponds to them: neither is *uniquely* demanded for any price. Furthermore, *neither* the TH *nor* the SNP tells us whether a non-vertex bundle such as one of these is demanded at all, for any price. If the valuation is affine in the relevant range, then $(\mathbf{x}, u(\mathbf{x}))$ is in the roof for such a bundle, $\mathbf{x}$, so it is demanded. But if the valuation is non-concave at the non-vertex bundle, this bundle's value lies strictly below the roof. In such a case the bundle is "jumped over" as we cross between UDRs.

The grey bundle is an example of the former case. Its valuation, 4, is precisely the average of the valuations, 0 and 8, of the bundles $(0, 0)$ and $(2, 0)$; see Figs. 2a and 2c. It is therefore demanded for some prices (here, $(4, p_2)$ for $p_2 \geq 6$–see the TH) but is not the unique demand at any price.

However, the black bundle's value is strictly below the level of the "roof" (see Fig. 2c), so it is never demanded at any price.[26]

Note that because the central wavy-shaded (five-sided) SNP cell is the only SNP cell that the black bundle lies in, the corresponding wavy-shaded 0-cell (at price (2,2)) of the TH in which it is currently "hidden", was the only price at which this bundle *might* have been demanded: when a bundle *is* demanded, its position in the SNP dictates at which price(s). That is:

Lemma 2.9. *For $\mathbf{x} \in A$, exactly one of the following holds:*

(1) $\mathbf{x}$ is not demanded for any price;

(2) $\mathbf{x}$ is in an SNP cell iff, for every $\mathbf{p}$ in the corresponding TH cell, $\mathbf{x} \in D_u(\mathbf{p})$.

The results of this section are laid out formally in Appendix A.1.3

2.5 The correspondence between *sets of* valuations and SNPs

It is easy to see that multiple valuation functions, $u(\mathbf{x})$, yield the same SNP. That is, there are many ways of changing the values of the bundles without affecting which sets of bundles can jointly form the demand set. (For example, consider how we can change the valuation of Fig. 2a without changing the connections in Fig. 2c.) Such changes cannot affect the correspondences and orthogonality relationships discussed in the previous subsection between the cells of the SNP and of the TH.

So a single SNP corresponds to a *set* of THs, all of which have cells with the same dimensions and slopes, connecting to one another in the same way. We say that such a set of THs (and the set of corresponding valuation functions) are all of the same *combinatorial type*; they all correspond to agents who make the same trade-offs between additional units of goods, even if not always at the same prices.

In sum,

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[26] But if its valuation were greater, so the corresponding bar in Figs. 2b and 2c just touched the roof, then it would still not be a vertex, but it would be demanded at the price corresponding to the wavy-shaded 0-cell (that is, (8,8)) that is a vertex of the TH. And if it had an (even) higher valuation (so "poked through" the current roof), then the corresponding SNP point would become a vertex, and the corresponding TH 0-cell would "open up" to form a new UDR corresponding to the range of prices at which the bundle (1,1) would then be demanded.

Theorem 2.10 (Mikhalkin, 2004, Proposition 2.1)**.** *There is a bijective correspondence between SNPs of THs and combinatorial types of THs.*

Fig. 3c gives a TH of another valuation that has the same combinatorial type as the valuation of Fig. 2a–its SNP is therefore also that shown in Fig. 3a.

It should be clear from our example that, starting from any SNP, it is easy to find the combinatorial type of the TH; the exact location of the TH for any specific valuation function can then easily be worked out from the values of the different bundles.[27] Conversely, given any TH, it is easy to determine which bundle is demanded in each UDR starting from the demand in any one UDR (see discussion above Definition 2.1), and then also easy to find the corresponding SNP.[28]

Furthermore, if the set of feasible bundles is not too large, it is easy to list all the possible SNPs, and so also all the possible combinatorial types of THs, that is, every possible distinct structure of trade-offs that an agent might make between the goods. Figs. 9 and 10 in Appendix A.1.4 give examples.

2.6 Representation in Price Space *vs.* Representation in Quantity Space

Although the TH and SNP are "dual", the price and quantity representations have different properties and are useful in different contexts.

We will see (in Section 3) that the relationship between the economic properties of a valuation and the geometric properties of its TH in price space allows us to classify valuations into "demand types", such as substitutes, complements, etc. It would be almost equivalent to categorise valuations using the SNP in quantity space. However, an important distinction is that any geometric object satisfying the simple "balancing condition" of Definition 2.2 is the TH of some valuation (see Theorem 2.3), but *not* every subdivision of every Newton polytope arises from some valuation. Moreover, there seems to be no straightforward check of whether or not a given subdivision corresponds to any valuation function.[29] This distinction is very important for developing (counter)examples and deepening our understanding.

So it seems easier to specify all the geometric objects in price space that represent a particular economic property, than to do this in quantity space where we have to take care to restrict attention to cases that can actually arise. And Theorem 2.3 guarantees that if we develop examples to, e.g., test conjectures, working in price space, then the

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[27] For example, for the valuation of Fig. 2a, it is clear from the valuations of bundles (1,0) and (0,1) that the dotted-shaded 0-cell of the TH is at $\mathbf{p} = (4,6)$, since 4 and 6 are the prices below which the agent will first buy any of goods 1 and 2, respectively, when the other good's price is very high. And the coordinates of the wavy-shaded 0-cell must be (2,2) since 8-6=2 is the incremental value of a second unit of good 2, when the agent has no unit of good 1, and 10-8=2 would be the incremental value from then adding a unit of good 1, etc.

[28] In two dimensions, we know each UDR (area) in the TH corresponds to a vertex (point) in the SNP. A facet (line-segment) in the TH corresponds to an edge in the SNP in the orthogonal direction, joining the vertices corresponding to the UDRs on either side; its length is given by the weight of the facet. So we can immediately draw all the vertices and edges.

[29] Maclagan and Sturmfels (2015, Fig. 2.3.9) show a subdivision that corresponds to *no* valuation function.

corresponding valuations will exist; we have found in practice that this is considerably easier than developing valuation functions directly.

Working in price space also makes it much easier to aggregate agents' valuations (see Section 3.3).

Furthermore, while an SNP shows only the collections of bundles among which the agent is indifferent for some price vectors, THs show clearly which bundles are demanded in which regions of prices. Thus THs are often easier to interpret.

So we will mostly develop our ideas in price space.

However, the different perspective offered by the geometric objects in quantity space is also valuable. In particular, some essential information that is only implicit in the TH becomes obvious in the SNP. For example, we will see in Sections 4 and 5 that a 0-dimensional, or low-dimensional, cell of the TH sometimes "hides" important detail that is much more easily seen and interpreted in the higher-dimensional dual object in the SNP.

Another virtue of the SNP is that the easiest way to compute the THs of specific valuations is often via first computing the SNPs, so even when the TH is an easier-to-understand representation, the SNP helps us construct it more quickly.[30]

The fact that the different representations are useful in different contexts makes the ability to move easily between them, using duality, especially valuable.

3 "Demand Types"

3.1 Defining "demand types"

The previous section suggests classifying valuations according to the vectors that are normal to their THs' facets:[31]

Definition 3.1. A valuation is of *demand type* $\mathcal{D}$ if all the primitive integer normals to the facets of its associated TH lie in a set, $\mathcal{D}$, of primitive integer vectors in $\mathbb{Z}^n$, such that if $\mathbf{v} \in \mathcal{D}$ then $-\mathbf{v} \in \mathcal{D}$.[32]

For example, the valuation of Fig. 1 is of demand type $\pm\{(1,0),(0,1),(-1,1)\}$ as, of course, are many other valuations, for example, all those shown in Figs. 8a-c. Note that a valuation is of *any* demand type which contains the facet normals of its TH; we do not restrict to the minimal such set.[33]

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[30] For example, in constructing the TH of the valuation of Fig. 2a, going via the SNP (see Section 2.4 and note 27) both separates the question "in what directions are there line segments?" from the question "where in space are they?", and also clarifies which bundles have to be compared with which.

[31] See Manzini et al. (2015) for recent work offering a different approach in classifying valuations.

[32] We will write "demand type $\mathcal{D}$" for the set of *valuations* defined by the set $\mathcal{D}$ of *vectors*; given a demand type, we will refer to the defining vectors as the "demand type's vectors". These will all be non-zero. Note our definition does not consider the weights on facets; see Baldwin and Klemperer (2012, note 25, and 2014, note 42).

[33] For example, the valuations of Figs. 1 and 8a-c are also of demand type $\pm\{(1,0),(0,1),(-1,1),(-2,1)\}$ which is the minimal demand type of the valuations of Figs. 2–3.

Recalling Lemma 2.8, we can equivalently classify valuations according to the vectors in the directions of their SNPs' edges.[34] Recall also, however, that all THs correspond to valuations of the demand type that their facet normals' vectors define, but *not* all subdivisions of Newton polytopes whose edges are among the vectors of a demand type correspond to (any) valuations.

3.2 Comparative Statics, and Substitutes, Complements, etc.

Since the vectors defining a demand type are the set of all the possible directions of the TH's facet normals, and since these in turn specify the possible directions of demand changes as we cross the facets between UDRs, combinations of these vectors specify all the possible changes in demand between prices in UDRs. Since the UDRs are dense in price space, these are the possible changes in demands that can generically result from a small change in prices.[35]

It follows straightforwardly that demand types provide simple characterisations of concepts such as substitutes and complements. (This characterisation is *not* symmetric, as we will explain.)

For substitutes, an increase in a good's price, between prices at which demand is unique, might decrease but cannot increase the demand for that good, and cannot result in the agent decreasing its demand for other goods. (See, for example, Figs. 1 and 3b-3c, for this property holding, and Fig. 4 for it failing.) So the vectors that are normal to a facet may have two non-zero entries of opposite signs, but cannot have two non-zero entries of the same sign (see Appendix A.2.1 for details).

Definition 3.2. A valuation u is *ordinary substitutes*[36] if, for any prices in UDRs such that $\mathbf{p}' \geq \mathbf{p}$, if $D_u(\mathbf{p}) = \{\mathbf{x}\}$ and $D_u(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \geq x_k$ for all k such that $p_k = p'_k$.[37]

Proposition 3.3. *A valuation is of a demand type whose vectors each have at most one positive and at most one negative coordinate entry* iff *it is an ordinary substitutes valuation.*

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[34] Danilov, Koshevoy and their co-authors examine these vectors in quantity space in the course of their impressive body of work that, we will see in Section 4.1, has close connections to ours (see Danilov et al., 2001, and Danilov et al., 2003, 2008, 2013). However, they do not use them to create a taxonomy of demand–we, by contrast, develop a general framework to understand them in economic terms (see also Baldwin and Klemperer, 2012, 2014 and in preparation-b). In particular, as Danilov et al. work almost exclusively in quantity space, they do not see these vectors as giving changes in demand as we move around in price space.

[35] See Baldwin and Klemperer (2014, in preparation-b) for full discussion of possible demand changes to and from prices at which demand is non-unique.

[36] We call "ordinary substitutes" what most others (e.g., Ausubel and Milgrom, 2002) simply call "substitutes". We do this for clarity, since some have defined "substitutes" in other ways. In particular, although Kelso and Crawford's (1982) definition is equivalent in their model, it is not generally equivalent if it is extended to multiple units of three or more goods (see Danilov et al., 2003, Ex. 6 and Thm 1). Our definition (3.2) seems the most natural one in the general case. It is also equivalent to several properties that seem to naturally characterise "substitutes", and to the indirect utility function ($\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\}$) being submodular–see Baldwin, Klemperer and Milgrom (in preparation). See also Baldwin and Klemperer (2014). Hatfield et al. (2013, see Section 6.1) and Danilov et al. (2003) use definitions equivalent to 3.2, and the latter authors make a similar observation to our next proposition (3.3) when they say "each cell of a valuation's parquet is a polymatroid".

[37] Here we write, as is standard, $\mathbf{p}' \geq \mathbf{p}$ when the inequality holds component-wise.

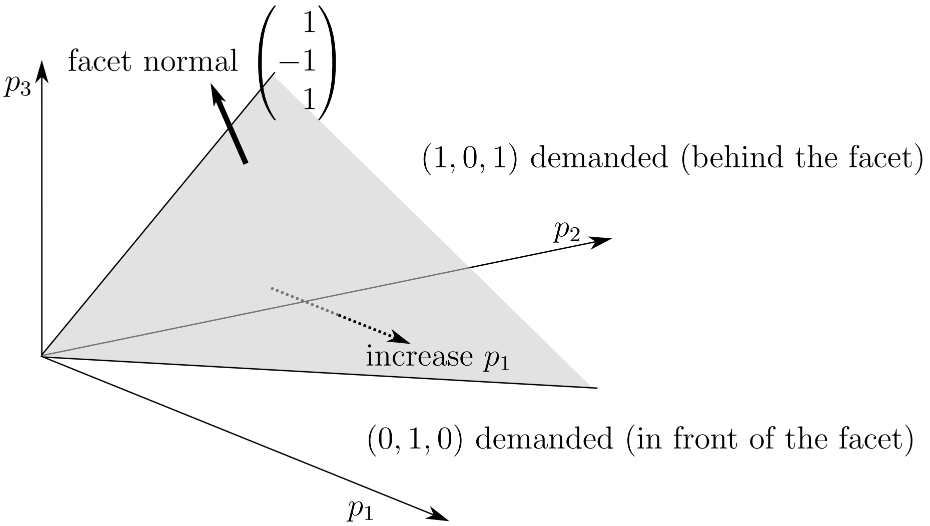

Figure 4: A facet (shaded) with its normal (the arrow shown in bold). Increasing either p_1 (as shown with a dotted arrow), or p_3, demonstrates complementarities between goods 1 and 3, as the bundle demanded switches from (1,0,1) to (0,1,0).

We can thus refer to the set of all primitive integer vectors satisfying this description as the "ordinary substitutes vectors".

Similarly, for complements, if any good's price increases, then the agent may reduce, but cannot increase, her demand for other goods, so there is no facet whose normal vector has two non-zero entries of different signs:

Definition 3.4. A valuation u is *ordinary complements* if, for any prices in UDRs such that $\mathbf{p}' \geq \mathbf{p}$, if $D_u(\mathbf{p}) = \{\mathbf{x}\}$ and $D_u(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \leq x_k$ for all k such that $p_k = p'_k$.

Proposition 3.5. *A valuation is of a demand type whose vectors' non-zero coordinate entries are all of the same sign* iff *it is an ordinary complements valuation.*

So we can similarly refer to the set of all primitive integer vectors whose entries are all of the same sign as the "ordinary complements vectors".

Note that, by contrast with standard definitions such as 3.2 and 3.4, our way of classifying valuations using "demand types" clearly reveals both the lack of symmetry between substitutes and complements, and the reason for it: the substitutes demand type includes only vectors for which each pair of non-zero entries are of opposite signs, while the complements demand type includes only vectors for which each pair of non-zero entries have the same signs–but this implies that in more than two dimensions complements permits vectors with *any* number of non-zero entries, whereas substitutes permits at most two non-zero entries.

The reason for the asymmetry is that, if any one good can trade-off against both of two other goods simultaneously across a single facet, the two other goods *must* be complementary. Consider, for example, Fig. 4, which illustrates a facet with normal (1,-1,1), defined by $\{\mathbf{p} \in \mathbb{R}^3 \mid p_1 + p_3 = p_2;\ p_{1,}p_2, p_3 \geq 0\}$. Moving from the UDR with $p_1 + p_3 < p_2$ ("behind" the facet) to the UDR with $p_1 + p_3 > p_2$ ("in front of" the facet), by increasing the price of either good 1 or good 3, changes the bundle demanded from (1,0,1) to (0,1,0) and so reduces demand for both goods 1 and 3. So even when all goods are mutual substitutes there can never be trade-offs between more than two of them across the whole of a facet.[38]

It is easy to find the demand types that correspond to other standard classes of valuations. For example (see Baldwin and Klemperer, 2014, Corollary 5.20, and in preparation-b):

Proposition 3.6. *A valuation is strong substitutes*[39] *iff it is concave and is of a demand type whose vectors each have at most one +1 entry, at most one -1 entry, and no other non-zero entries.*

We can thus refer to this set of vectors as the "strong substitutes vectors". Note that with n goods, there are just $n(n+1)/2$ strong substitutes vectors (and their negations).

We discuss some other demand types of interest in Sections 6.3-6.4.

3.3 Aggregate Demand, and the "Demand Type" of an Aggregate Valuation of Multiple Agents

An important feature of our "demand types" classification–that, in particular, greatly facilitates the study of equilibrium–is that the demand type of the aggregate valuation of multiple agents is just the union of the sets of vectors that form the individual agents' demand types.

We now consider a finite set of agents, $j = 1, \dots, m$: agent j has valuation u^j for integer bundles in a finite set, A_j. It is obvious that the agents' *aggregate demand*, $D_U(\mathbf{p})$, at any price $\mathbf{p}$ is simply the Minkowski sum of their individual demands at that price, that is, $D_U(\mathbf{p}) = \left\{ \sum_j \mathbf{x}^j \mid \mathbf{x}^j \in D_{u^j}(\mathbf{p}) \right\}$. So it is also clear that the superimposition of the individual agents' THs is the TH of an "aggregate valuation" that corresponds to the aggregate demand.[40] More precisely, let $\mathcal{T}_U$ be the union of the individual THs, $\mathcal{T}_{u^j}$, with facet weights given by adding the weights of facets that coincide. $\mathcal{T}_U$ is clearly balanced since the individual THs are, and so we can apply Theorem 2.3 to see that there exist

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[38] One mutual substitute might trade-off against two others at prices where more than one facet meets, if at least one of those facets has weight greater than 1. For example, an agent might switch 2 units of A for 1 each of two other goods, B and C, which it treats as indistinguishable, in the intersection of all three weight-2 facets where the agent switches between two of the three goods.

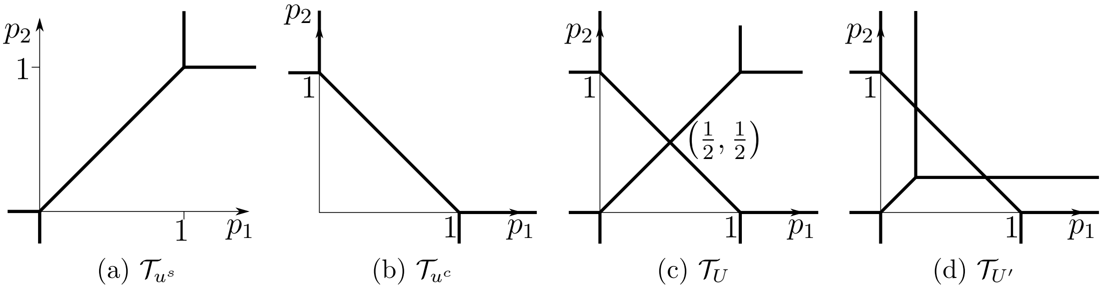
To illustrate why the conditions for indivisible goods to be substitutes are so restrictive, consider a consumer who regularly makes three kinds of trips: journey A can be made only by bus or train; journey B can be made only by car or train; journey C can be made only by car or bus. Thought of as divisible goods, the three modes of transport are all mutual substitutes. But if the price of *either* bus tickets or train tickets is slightly raised, a consumer might buy a car and reduce her use of *both* forms of public transport. which are therefore locally complements–that is, the car takes the role of good 2 in the situation pictured in Fig. 4.

[39] Milgrom and Strulovici (2009) define valuations to be "strong substitutes" if every unit of every good is a substitute for every other unit of every good, in the sense of Kelso and Crawford (1982). For other equivalent definitions see Milgrom and Strulovici (2009), Baldwin and Klemperer (2014) and Baldwin, Klemperer and Milgrom (in preparation). In particular, Danilov et al. (2003, Proposition 7) show valuations are their "step-wise gross substitutes" if they are both concave and (in our language) the edges of all their SNP faces are strong substitutes vectors. (We use Milgrom and Strulovici's later terminology because it seems to have become more standard). Figs. 1, 5a, and 8a-c show examples of THs of strong substitutes valuations.

[40] Mathematically, finding aggregate demand corresponds to tropically multiplying tropical polynomials.

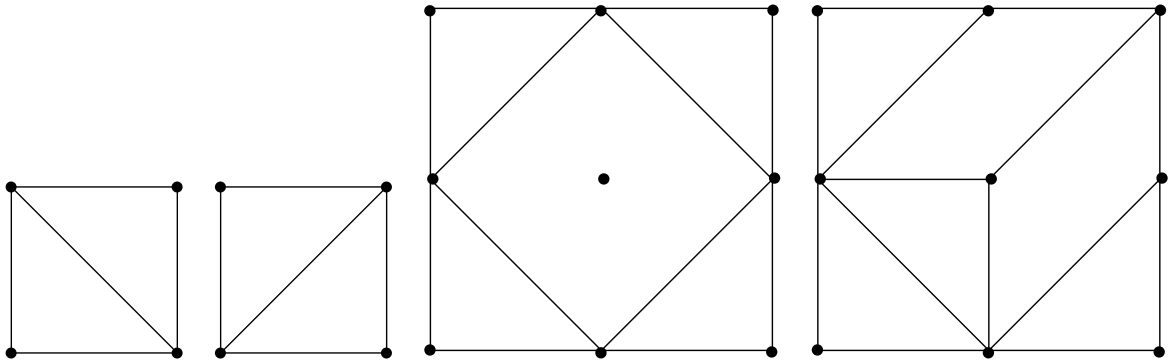
quasilinear valuations corresponding to $\mathcal{T}_U$. And we can confirm our interpretation by observing that $\mathcal{T}_U$ implies aggregate demand at a price is unique iff all the individual demands are, and the change in aggregate demand between any prices is just the sum of the changes in the individual demands as we cross facets of the individual THs. (We cannot in general uniquely identify the aggregate valuation U from $\mathcal{T}_U$ as U need not be concave, even when individual valuations are. We give the precise form of U below, but it is cumbersome to work with and we seldom do so).

Fig. 5 illustrates this by showing the THs of two simple valuations, with domain $\{0,1\}^2$: a substitutes valuation, $u^s(x_1, x_2) = 1$ if $x_1 \geq 1$ *or* $x_2 \geq 1$, $u^s(x_1, x_2) = 0$ otherwise (Fig. 5a); a complements valuation, $u^c(x_1, x_2) = 1$ if $x_1 \geq 1$ *and* $x_2 \geq 1$, $u^c(x_1, x_2) = 0$ otherwise (Fig. 5b); and of the aggregate of these two valuations (Fig. 5c).



(a) $\mathcal{T}_{u^s}$ (b) $\mathcal{T}_{u^c}$ (c) $\mathcal{T}_U$ (d) $\mathcal{T}_{U'}$

Figure 5: The THs of (a) a simple substitutes valuation; (b) a simple complements valuation; (c) the aggregate of the simple substitutes and simple complements valuations shown; (d) the aggregate of the simple complements valuation shown and a simple substitutes valuation with lower values for each unit.



(a) SNP of $\mathcal{T}_{u^s}$ (b) SNP of $\mathcal{T}_{u^c}$ (c) The SNP of $\mathcal{T}_U$ (d) The SNP of $\mathcal{T}_{U'}$

Figure 6: SNPs corresponding to the THs shown in Fig. 5.

Note that when cell interiors from different agents intersect, the cells are split up into new, smaller cells in the aggregate TH, with a new, lower-dimensional, cell at their intersection. For example, in Fig. 5c, the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a 0-cell, on the boundary of four distinct 1-cells.

It is now immediate that demand "type" is preserved under aggregation:

Proposition 3.7. *Valuations u^j are of demand type $\mathcal{D}$ for $j = 1, \ldots, m$ iff the aggregate TH, $\mathcal{T}_U$, is of demand type $\mathcal{D}$.*

The ability to "add" THs straightforwardly is a merit of working in price space for these purposes.

Working in quantity space would be possible, using the standard result–see Appendix A.2.2–that, since agents' preferences are quasilinear, $D_U(\mathbf{p})$ is just what would be demanded by a single agent whose valuation, $U(\mathbf{y})$, is the greatest sum of the valuations, u^j, that can be attained by dividing the bundle, $\mathbf{y}$, between the agents, that is, $U(\mathbf{y}) = \max\left\{\sum_j u^j(\mathbf{x}^j) \mid \mathbf{x}^j \in A_j, \sum_{j=1}^m \mathbf{x}^j = \mathbf{y}\right\}$. But, because finding any value of $U(\mathbf{y})$ requires considering all possible partitions of $\mathbf{y}$ among the agents, which is both time-consuming and unintuitive, doing this in quantity space is harder.

We also cannot find the SNP of an aggregate valuation from the individual SNPs, in quantity space, since neither the aggregate valuation, nor its combinatorial type, is uniquely defined by the *combinatorial* types of the individual valuations (that is, by the individual SNPs). However, working in price space, starting with *specific* THs, we can easily find the aggregate TH, and hence the aggregate SNP (and other information about aggregate demand) for the specific case.

For example, it is easy to see that if we alter the substitutes valuation of Fig. 5a above, to $u^{s*}(x_1, x_2) = 1/4$ if $x_1 \geq 1$ *or* $x_2 \geq 1$, $u^s(x_1, x_2) = 0$ otherwise, its TH is of the same combinatorial type as before. However, the 0-cell that was at $(1, 1)$ has moved to $\left(\frac{1}{4}, \frac{1}{4}\right)$, so the TH of the aggregate, U', of it with the complements valuation, u^c, of Fig. 5b is that shown in Fig. 5d. And it is also straightforward that the SNP of *both* u^s and u^{s*} is that of Fig. 6a, and that the SNP of u^c is that of Fig. 6b. However, the SNP of the aggregate valuation U of u^s and u^c is that of Fig. 6c, while the SNP of the aggregate valuation U' of u^{s*} and u^c is that of Fig. 6d. Clearly, there is no unique aggregate SNP corresponding to the SNPs of Fig. 6a and Fig. 6b.

4 The Existence of Competitive Equilibrium for a Demand Type

A beautiful aspect of our "demand types" classification is that it leads us naturally to a very broad characterisation of the individual trade-offs for which equilibrium is and is not guaranteed (under our assumption of quasilinear preferences).

This theorem requires much weaker assumptions about agents' preferences than used in the existing leading economics literature, so our condition for equilibrium is correspondingly much more general. It immediately generalises, for example, the equilibrium results of Kelso and Crawford (1982), Hatfield and Milgrom (2005), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014). In particular it is *not* necessary for all agents to have "strong substitutes" valuations (or some basis change thereof) for equilibrium to always exist; complements valuations guaranteeing equilibrium are easy to find. Instead, concavity and a "unimodularity" condition explained below are all that are required.

Concavity The crucial role of concavity is that, since concave functions are precisely those for which every possible bundle is demanded at some price (Lemma 2.5), there exists a competitive equilibrium price vector for every possible market supply iff the agents' aggregate valuation is everywhere concave. And, although each individual agent's valuation being concave is *not* sufficient for their aggregate valuation to be concave, our geometric approach shows us a simple condition that *is* sufficient.

Unimodularity

Definition 4.1. A set of vectors in $\mathbb{Z}^n$ is *unimodular* if every linearly independent subset can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1.

By "the *determinant*" of n vectors we mean the determinant of the $n \times n$ matrix which has them as its columns.[41] Alternative equivalent conditions for unimodularity are given in Remark A.15.

To understand the geometric importance of unimodularity, note that a set of n linearly independent integer vectors are the edges of an n-dimensional parallelepiped. This shape contains no integer point (either in its boundary or in its interior) aside from its vertices, iff its volume is 1. But this volume is just the absolute value of the determinant of the vectors along its edges.

Moreover, if this volume *is* 1 then it follows that any lower-dimensional parallelepiped spanned by a subset of these vectors *also* contains no integer point other than its vertices. So if unimodularity is satisfied, this critical property must hold. Unimodularity is also necessary for this property (see Remark A.15). And this property implies that we can move between *any* integer bundles in the linear span of this parallelepiped by taking *integer* combinations of these vectors: the "integer lattice" in this linear span is made up of repeated copies of the parallelepiped. That is, the vectors are an "integer basis" for this subspace.

If the set of vectors spans $\mathbb{R}^n$, then there exist sets of n of them that are linearly independent; it is therefore, of course, sufficient to check that all n-element sets have determinant ± 1 or 0.

Importantly, therefore, *unimodularity of a set $\mathcal{D}$ defining a demand type* is not too hard to check, and we refer to "unimodular demand types" in the obvious way.

4.1 Necessary and Sufficient Condition for Equilibrium to *always* Exist for a Demand Type

Since, of course, an individual agent with a non-concave valuation function fails to always have a competitive equilibrium, we now only consider concave individual valuations. Similarly, we only consider bundles in the domain of the aggregate valuation, since a bundle clearly cannot be demanded if it is outside the set considered by the agents. For brevity, we will refer to the domain of aggregate valuation as "the domain":

Theorem 4.2. *A competitive equilibrium exists for* every *set of agents with concave valuations of demand type $\mathcal{D}$ and any supply bundle in the domain* iff *$\mathcal{D}$ is unimodular.*[42]

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[41] We ignore the order of the vectors since we are only ever interested in the absolute values of determinants.

[42] Tran and Yu (2015) call this result the Unimodularity Theorem in their recent exposition of our work. They also provide an additional proof, via integer programming.

In particular, it is immediate from the discussion in the previous subsection that

Corollary 4.3. *With n goods, if the vectors of $\mathcal{D}$ span $\mathbb{R}^n$, then a competitive equilibrium exists for* every *set of agents with concave valuations of demand type $\mathcal{D}$ and any supply bundle in the domain* iff *every subset of n vectors from $\mathcal{D}$ has determinant* 0 *or* ± 1.

We sketch the proof and intuition for these results in the next subsection (4.2). Full details are in the Appendix.

A remarkable series of papers by Danilov, Koshevoy and their co-authors, has developed results that are very closely related to ours. In particular, Theorems 1, 3 and 4 of Danilov et al. (2001) together provide a *sufficient* condition for equilibrium, which is analogous to our condition on demand types.[43] However, the interpretation or usefulness of their result is not made clear; by contrast, our theorem both demonstrates the applicability of the result, and clarifies the connections to existing economic results.[44]

Danilov et al. also prove no *necessity* result. Because they have not developed their definition as a taxonomy of demand, in the way we do with demand types, they do not show the necessity of unimodularity for the existence of competitive equilibrium. Once our concept of demand types is introduced, however, a necessity result can easily be developed.[45]

Danilov et al. moreover state their results under different assumptions from ours. They assume the domain, A, of every agent's valuation is $\mathbb{Z}^n_{\geq 0}$, which precludes, for example, the application to agents who both buy *and* sell which our more general assumption permits.[46]

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[43] Their sufficient condition for a class of valuations to have equilibrium for any supply is that the valuations be "$\mathscr{D}$-concave" for some "class of discrete convexity" $\mathscr{D}$. Here, "$\mathscr{D}$-concave" valuations are concave valuations such that every demand set $D_u(\mathbf{p})$ belongs to the set "$\mathscr{D}$" of subsets of $\mathbb{Z}^n$. A "class of discrete convexity" is a collection of sets such that every set is discrete convex, and every Minkowski sum and every Minkowski difference of the sets is discrete convex. They also show that $\mathscr{D}$ *is* a class of discrete convexity if the edges of the convex hulls of the sets in $\mathscr{D}$ form a unimodular set of vectors. The proof of this, their Theorem 4, is given by Danilov and Koshevoy (2004, Thm. 2). Note that Danilov et al.'s use of the notation $\mathscr{D}$ is *not* connected with our use of $\mathcal{D}$ to represent demand types.

[44] We will see that Theorem 4.2 generalises the results on equilibrium in work subsequent to Danilov et al.'s, including in Hatfield and Milgrom (2005), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014). The absence in Danilov et. al's work of our notion of demand types or of any economic interpretation of their concept of "$\mathscr{D}$-concavity", and the presentation of their work in relatively unfamiliar terms (namely the relationships between sets of primitive integer vectors which are parallel to edges of specific collections of integral pointed polyhedra and the "classes of discrete convexity" that they define) seems to have resulted in leading economists being unaware of their work or of its implications. (We were also unaware of their work until after we had developed our own results.)

[45] The sufficiency part of our theorem follows from combining Theorems 1, 3 and 4 of Danilov et al. (2001). To understand the relationship between these theorems and our Theorem 4.2, observe that in their Theorem 4 certain sets of "primitive integer vectors, which are parallel to edges of" a certain "collection of integral pointed polyhedra" are analogous to our demand types; furthermore, the "classes of discrete convexity" they define are analogous to a set of demand sets $D_u(\mathbf{p})$ which are all discrete-convex and such that this property is preserved under aggregation. It is not hard to also show, using our Lemma 2.9, the necessity of a demand type giving rise to a "class of discrete convexity" for competitive equilibrium to always exist, and in this way we can also derive our *necessity* result from their work.

[46] For example, our model, unlike theirs, applies to (and extends) Hatfield et al. (2013)–see Section 6.1. In fact Danilov et al.'s assumption seems unnecessary for them, so we could develop our full theorem by extending their work. See our Note 45, above. See also our discussion about the distinction

Finally, although the techniques we use to prove our results are novel, they seem simpler and more accessible to economists than Danilov et al.'s *very* advanced mathematical techniques. So we will prove the theorem using our alternative method, which understands the result as an application of "intersection multiplicities" in tropical geometry.[47]

4.2 Intuition and Sketch of Proof for Theorem 4.2

4.2.1 The Role of Intersections

The first insight is that we can determine whether equilibrium exists by focusing on the intersection of individual THs: we know equilibrium always exists, that is, every bundle is demanded at some price, *iff* the aggregate valuation is concave *iff* the aggregate demand set is discrete-convex at every price (Lemma 2.5). But if all but one of the agents have unique demand at some price, the aggregate demand set is simply the shift of the remaining agent's demand set by the other agents' (unique) demands. And this set must be discrete-convex, since we assumed that every individual valuation is concave. So we only need to check prices at which two or more agents have non-unique demand. That is:

Lemma 4.4. *Equilibrium exists for every supply bundle in the domain* iff *the aggregate demand set is discrete-convex at the* intersection *of agents' THs.*

4.2.2 Necessity

Consider, therefore, a price at which two or more agents' TH facets of weight 1 intersect (and other agents have unique demand).[48] The corresponding cell in the aggregate SNP is then a parallelepiped whose edges are the normals to those intersecting facets. So these edges are vectors of the demand type of the agents' valuations.

If the demand type is *not* unimodular then, as discussed earlier in Section 4, we can find a set of its vectors for which such a parallelepiped contains integer point(s) that are not its vertices.[49] And we saw in Section 2.4 that bundles that are not SNP vertices are "hidden" inside the corresponding cells of the corresponding TH, and may not be demanded at the corresponding prices. Indeed, in this case, each of the relevant agents has just two bundles in its demand set, so with s agents there are only 2^s different possible aggregate demands, and these must correspond to the parallelepiped's 2^s different integer vertices. The non-vertex bundle(s) therefore *cannot* be demanded at the corresponding price.[50] So the aggregate demand set is *not* discrete-convex at this price, and competitive

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between their approach and ours in Note 34. Their work also covers some of the examples in Sections 4.3.3, 4.3.4 and 6.3, as we note in those sections.

[47] It was this theory that inspired our (independent) development of our results. Full details of our proof are in Appendix A.3.2.

[48] We assume that *only* facets (and no lower-dimensional cells in the individual THs) intersect at this price. This scenario is generic when $n = 2$ but not for $n \geq 3$.

[49] Unimodularity is equivalent to the *tropical intersection multiplicity* being equal to one in such a case (see Section 5 and Appendix A.4).

[50] Because we specified the facets all had weight 1, there are necessarily just 2 bundles in each demand set. If any facet had a greater weight, integer points that are not vertices will be "weakly-demanded" (i.e., demanded, but never uniquely demanded), like the "grey" bundle, (1,0) of Fig. 3, that we discussed

equilibrium therefore fails if the supply is such a non-vertex bundle (see Lemma 2.9).

This logic shows necessity (see Proposition A.16 in Appendix A.3.2). It also demonstrates how to easily construct examples of failure of equilibrium for any non-unimodular demand type.

4.2.3 Sufficiency for Simple Cases

If THs intersect only in the simple form just discussed, a parallel argument to the one above demonstrates sufficiency: if the demand type *is* unimodular, then no parallelepiped corresponding to an intersection price contains a non-vertex integer point, so no bundles are "hidden" in the intersection and the aggregate demand set *is* discrete-convex at every price, implying our sufficiency result.

The importance of unimodularity will carry over to general intersections. The broader intuition is that unimodularity of the set of facet normals means we can reach all bundles by taking integer combinations of this set of vectors. That is, all the bundles are connected by edges in quantity space and, correspondingly, all aggregate demands can be achieved by crossing appropriate facets in price space.

4.2.4 Sufficiency when the TH Intersection is "transverse"

To develop a general proof for sufficiency, we begin by focusing on "transverse" TH intersections:

Definition 4.5. THs $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ intersect *transversally* at $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, in which C^i is the minimal cell of $\mathcal{T}_{u^i}$ containing $\mathbf{p}$, for $i = 1, 2$, and $C^1 + C^2$ is the set-wise (Minkowski) sum of these cells.

The intersection of THs $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ is *transverse* if they intersect transversally at every point of their intersection.

THs $\mathcal{T}_{u^1}, \ldots, \mathcal{T}_{u^k}$ intersect *transversally* at $\mathbf{p}$ if $\mathcal{T}_{u^{j+1}}$ intersects the TH of the aggregate of the valuations $u^1, \ldots, u^j$ transversally at $\mathbf{p}$, for all $j = 1, \ldots, k-1$.[51]

Thus transverse intersections are generic intersections, as we make precise below (see Proposition 4.6). For example, in two dimensions, two lines crossing at a single point are intersecting transversally, but two coincident lines are not, and nor are three lines crossing at a single point. (To understand the last case, observe that if there are just two intersecting THs, the one that contains two of the lines has a 0-cell at this point; and if we are considering three THs, the aggregate of the first two has a 0-cell at this point. Either way, aggregating with the remaining TH then involves aggregating a line with the 0-cell.) In three dimensions, a line meeting a plane in a single point is a transverse intersection (point), as is two planes meeting in a line, or three planes meeting in a single point.

The important point about prices at transverse intersections is that the changes in bundles considered by different agents at any such price are linearly independent–that is, the spaces of the possible changes in the individual agents' bundles have zero intersection

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at the end of Section 2.4; the SNP cell will consist of consecutive copies of the parallelepiped described here.

[51] It is straightforward that this definition is independent of the order in which the THs are taken.

(Lemma A.25 makes this point precise). This means that there is only one possible way to apportion a change in the aggregate supply bundle among the different agents.

To show that a supply bundle, which would be demanded at a transverse intersection price if it were demanded anywhere, is in fact demanded, we start from any supply bundle that is demanded in a UDR adjacent to the intersection. We then consider two different ways of thinking about dividing up the change between the two supply bundles among the agents; the fact that these two divisions must be the same will show that the supply bundle in question is also demanded.

The first way we think about dividing up the change in aggregate supply observes that the SNP cell in the aggregate SNP is the Minkowski sum of the individual SNP cells. So any change in total supply within the aggregate SNP cell must be decomposable as a sum of individual changes, each of which is within an individual SNP cell and therefore within the convex hull of an individual's demand set. So we have assigned each agent a bundle in the convex hull of its demand set at this price, although we have not yet demonstrated that these new bundles are integer bundles.

For example, when two agent's THs intersect transversally in two dimensions, the corresponding cells in the aggregate SNP are just parallelograms of the kind discussed above, so in each case the aggregate change can be broken down into parts along the edges of this parallelogram, and each agent can be allocated the additional supply corresponding to "its" edge of the parallelogram. (In more than two dimensions, the geometry is a little more complicated, because the cells in the aggregate SNP that correspond to transverse intersection prices need not be parallelepipeds.)

The second way we think about dividing up the change in aggregate supply uses the unimodularity of $\mathcal{D}$: we can fix a basis for each individual's change in demand at the intersection price, made up of edges of that individual's SNP cell (equivalently, its facet normals at the price). Transversality means that taking all these bases together creates a basis for the space of aggregate changes in demand. And, because the set of all edge vectors is unimodular, this is an "integer basis". So any integer change in aggregate supply can be presented as an integer combination of these basis vectors, thus assigning an *integer* change in bundle to each agent, although we have not yet demonstrated that each agent's new bundle is in the convex hull of its demand set.

However, since we assumed the intersection was transverse at this price, the allocation of bundles to agents is unique. So the two allocations are the same. Thus both assign each individual agent an integer bundle in the convex hull of its demand, which the agent therefore demands, since its individual valuation is concave.

4.2.5 Sufficiency for the General Case

The full proof of sufficiency can be completed using the standard convex-geometric methods used thus far (see the Appendix of Baldwin and Klemperer, 2014). However, it is quickest to appeal to the tropical-geometric result that generically all THs intersect transversally:

Proposition 4.6 (Maclagan and Sturmfels, 2015, Proposition 3.6.12)**.** *For any THs* $\mathcal{T}_{u^1}$ *and* $\mathcal{T}_{u^2}$*, and generic* $\mathbf{v} \in \mathbb{R}^n$*, the intersection of* $\mathcal{T}_{u^1}$ *and* $\epsilon\mathbf{v} + \mathcal{T}_{u^2}$ *is transverse, for all sufficiently small* $\epsilon > 0$*.*

So there always exist small perturbations of agents' valuations that make their THs' intersection transverse, so for which, by the previous argument, every bundle is demanded on aggregate at some price. But the aggregate value of any bundle that was not demanded before the perturbation must be a finite amount beneath the value that would be required for it to be demanded, which contradicts the bundle being demanded after an arbitrarily small perturbation of valuations.

Full details of the proof are in Appendix A.3.2.

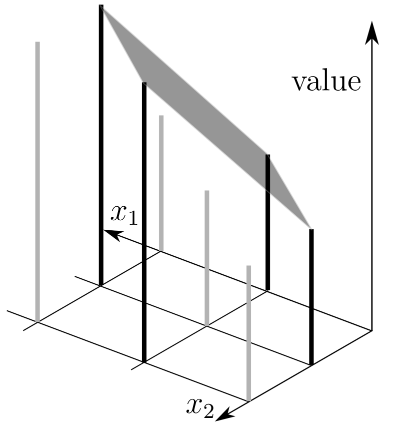
4.3 Examples

4.3.1 Simple Illustration of Necessity of Theorem 4.2

We can illustrate our result on necessity of unimodularity for equilibrium by considering individual agents with the simple two-goods substitutes and complements valuations shown in Figs. 5a and 5b, respectively, and their aggregate valuation, shown in Fig. 5c. Note that both the individual valuations are concave. However, their aggregate valuation, given in Fig. 7a, is *not*, as can easily be seen by observing that $(U(1,0)+U(0,1)+U(2,1)+U(1,2))/4 > U(1,1)$. The failure of aggregate concavity is also clear in Fig. 7b, which shows the aggregate valuation together with the cell of its roof that corresponds to the price vector $\left(\frac{1}{2},\frac{1}{2}\right)$.

x_1				
2	*1*	*0*	$U(\mathbf{x})$	
1	1	0	*0*	
2	1	1	*1*	x_2
2	2	1	*2*	

(a) The aggregate valuation corresponding to Fig. 5c.



(b) The aggregate valuation corresponding to Fig 5c, showing the cell of its roof that corresponds to the price vector $\left(\frac{1}{2},\frac{1}{2}\right)$.

Figure 7: The aggregate valuation of Fig. 5c.

It is apparent that all the bundles $(1,0)$, $(0,1)$, $(2,1)$, and $(1,2)$ are demanded at this price, but the bundle $(1,1)$ is not, and so also is never demanded at any price. So there is no equilibrium if the supply is $(1,1)$.

Theorem 4.2 (and Corollary 4.3) warned of this possibility, since they imply equilibrium may fail for some supplies when the demand type is not unimodular. The reason is that the minimal demand type containing both individual valuations contains both $(1,-1)$ (from the substitutes individual valuation) and $(1,1)$ (required for the complements individual valuation), and no set of vectors containing both $(1,-1)$ and $(1,1)$ can be unimodular: the matrix formed by $(1,-1)$ and $(1,1)$ has determinant 2. Moreover, the discussion in Section 4.2.2 told us that there would be an example of equilibrium failure of exactly this kind.

Working in quantity space, we can understand the failure of equilibrium by looking at the SNP of the aggregate valuation, in particular at the cell corresponding to price $\left(\frac{1}{2},\frac{1}{2}\right)$. This diamond, with edges in the directions parallel to $(1,-1)$ and $(1,1)$, has area 2 (the determinant of its edges)–see Fig. 6c. So there is a bundle inside the diamond, namely the quantity $(1,1)$, which is *not* a vertex of the aggregate SNP; it is correspondingly "hidden" at the intersection of the diagonals at the price $\left(\frac{1}{2},\frac{1}{2}\right)$ in the aggregate TH (in Fig. 5c), and it is indeed not demanded in this case.

The equivalent price-space perspective is to observe that we cannot start from, for example, the UDR in which in the quantity $(0,1)$ is demanded, and then move across TH facets normal to $(1,-1)$ and $(1,1)$, to arrive any price at which the bundle $(1,1)$ is demanded, because aggregate demand cannot change by $(1,0)$ $(=(1,1)-(0,1))$. The reason, as above, is that the demand type is not unimodular and so, in particular, it is impossible to write $(1,0)$ as a sum of *integer* multiples of $(1,-1)$ and $(1,1)$.[52]

4.3.2 Basis Changes

A benefit of our method of categorising valuations into "demand types" is that it is straightforward (see Appendix A.3.3) that:

Proposition 4.7. *"Having equilibrium for* every *set of agents with concave valuations and any supply bundle in the domain" is a property of a demand type that is preserved under unimodular basis changes.*[53]

Making such a basis change is equivalent to re-packaging the goods so that any integer bundle can still be obtained by buying and selling an integer selection of the new packages (and, conversely, any integer selection of the new packages can be obtained as an integer combination of the original goods).

To illustrate, consider the previous subsection's example. Create a new good, 3, from two units of good 1 plus one unit of good 2, and consider the economy in which the goods traded are 1 and 3. Note that we can recreate one unit of good 2 by buying one unit of good 3 and selling two units of good 1, and we can convert any bundle expressed in terms of goods 1 and 2 (as a column vector) to a bundle of goods 1 and 3 by pre-multiplying by $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.[54] So the fact that there are no prices at which the bundle $(1,1)$ is aggregate demand when (only) goods 1 and 2 are traded implies that

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[52] When the set of vectors is *not* unimodular, the number of non-vertex bundles in such a parallelepiped is equal to one less then the determinant (see Fact A.45). Thus, using the determinants, we should be able to derive bounds on the extent to which we need to relax supply constraints in order to achieve competitive equilibrium. For explorations of this idea in the related field of matching *without* transfers, see Nguyen and Vohra (2014) and Nguyen et al. (2015).

[53] A unimodular matrix G is an integer matrix with integer inverse, i.e., an integer matrix of determinant 1. Premultiplying bundles of goods by G is equivalent to premultiplying the prices at which bundles are demanded by G^T. This transforms the facet normals of the TH, and hence the vectors of any corresponding demand type, by G^{-1}. Details are in Appendix A.3.3.

[54] This matrix plays the role of "G^{-1}" in Note 53.

there are also no prices at which the bundle $(-1,1)$, is demanded in an economy in which (only) goods 1 and 3 are traded.

Observe that the "substitutes" agent of the original economy (who bought either $(1,0)$ or $(0,1)$ at price $\left(\frac{1}{2},\frac{1}{2}\right)$) corresponds to an agent in the new economy who would "buy" either $(1,0)$ or $(-2,1)$. We can interpret this as an agent with an endowment of -2 units of good 1 (a contract to sell), and who buys either three units of good 1 or one unit of good 3. Thus this agent treats goods 1 and 3 as 3:1 substitutes. Similarly, the "complements" agent of the original economy (who bought neither or both of goods 1 and 2) corresponds to an agent in the new economy with an endowment of -1 unit of good 1, who buys one unit of either good 1 or good 3 (so is indifferent between bundles $(0,0)$ and $(-1,1)$) that is, an agent who treats goods 1 and 3 as 1:1 substitutes. So this is a pure substitutes economy in which equilibrium fails. Since the demand type of the original economy was defined by the columns of $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, which have determinant 2, the demand type in the new economy is defined by the columns of $\begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$, that is, by $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, which also have determinant 2.

We illustrate below (Section 6.3) the usefulness of unimodular basis changes in finding new demand types for which equilibrium *is* guaranteed.

More generally a unimodular basis change simply distorts the TH by a linear transformation which leaves its important structure unaffected (see Proposition A.21 in the Appendix).[55] So *other* important properties of demand are also unaffected–see Baldwin and Klemperer (2014, and in preparation-b).

4.3.3 Equilibrium with Complements

Mathematical results from Grishukhin et al. (2010) imply that *every* unimodular demand type is a unimodular basis change of a demand type that contains only vectors in $\pm\{0,1\}^n$ (and so contains only complements valuations). So from Proposition 4.7:

Proposition 4.8. *Every demand type for which equilibrium is guaranteed (i.e., exists for every set of agents with concave valuations and any supply bundle in the domain) is a unimodular basis change of a demand type which contains* only *complements valuations and for which equilibrium is guaranteed.*

Furthermore, the corresponding statement *cannot* be made about substitutes: see Proposition 4.11.[56] This is in stark contrast to much conventional wisdom about the "necessity" of substitutes for competitive equilibrium.[57]

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[55] We lay out the general behaviour in Appendix A.3.3. Analogous results about "basis changes" of valuations for divisible goods were developed by Gorman, 1976, pp. 219–220. Related results for specific cases of indivisible goods are in, e.g., Sun and Yang 2006, Sun and Yang 2008, and Hatfield et al., 2013.

[56] For two goods (but not more–see Section 3.2), substitutes are a unimodular basis change of complements via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see Section 4.3.2) so for two goods (but only this case), competitive equilibrium fails "as often" for substitutes as for complements .

[57] See, for example, Kelso and Crawford (1982). Also Gul and Stacchetti's (1999, p. 96) state "in a

4.3.4 The Relationship between Equilibrium and Strong Substitutes

It has been known for over 100 years (Poincaré, 1900) that what we call the "strong substitutes vectors" (see Proposition 3.6) form a unimodular set. So Theorem 4.2 immediately implies the results of Danilov et al. (2001, 2003) and Milgrom and Strulovici (2009) that

Proposition 4.9. *Equilibrium exists for* every *set of agents with strong substitutes valuations and any supply bundle in the domain.*[58]

Furthermore, in any dimension, the set of (all) strong substitutes vectors form a *maximal* unimodular set. That is, adding any other vector, $\mathbf{w}$, that is not a strong substitutes vector, contradicts unimodularity: the determinant of $\mathbf{w} = \mathbf{e}^i + \mathbf{e}^j$ together with $\mathbf{e}^i - \mathbf{e}^j$, and all $\mathbf{e}^k$ such that $k \neq i, j$, has absolute value 2; and the determinant of any $\mathbf{w}$ with $|w_j| > 1$ for some j with all $\mathbf{e}^i$ such that $i \neq j$ is w_j. So we also have Gul and Stacchetti's (1999, Thm. 2), Hatfield and Milgrom's (2005, Thm. 2), and Milgrom and Strulovici's (2009, Thm. 16), result that:

Proposition 4.10. *Given any one agent who does not have a strong substitutes valuation, there exist strong substitutes valuations for other agents such that equilibrium fails to exist for some supply in the domain.*[59]

Danilov and Grishukhin (1999) provided a characterisation of all maximal unimodular sets of vectors, including a list giving, up to unimodular basis change, all such sets up to dimension 6. This shows that for $n \leq 3$ equilibrium is guaranteed *iff* the demand type is a unimodular basis change of strong substitutes, or a subset thereof.

However, we now demonstrate that with more goods there are demand types for which equilibrium always exists, and which are not basis changes, even of *ordinary* substitutes:

Proposition 4.11. *With $n > 3$, there exist demand types which are* not *a unimodular basis change of ordinary substitutes, or a subset thereof, for which equilibrium is guaranteed.*

To see this, consider the demand type whose vectors are the columns of:

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

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sense, the GS [gross substitutes] condition is necessary to ensure existence of a Walrasian equilibrium", and the specificity of their model in which such claims are valid often seems to be forgotten. (Azevedo et al's (2013, p.286) remark that "adding a continuum of consumers ... eliminates the existence problems created by complementarities" can also be misinterpreted.) And applying our results to matching (see Section 6.5, and Baldwin and Klemperer, 2014) shows that stable allocations arise for a broader class of preferences than many people assume from Hatfield and Milgrom's (2005, p.915) statement "preferences that do not satisfy the substitutes condition cannot be guaranteed always to select a stable allocation", though the Proposition (p.921) that their introductory remark loosely summarises is, of course, correct in its context.

[58] We show in Sections 6.1 and 6.3 that equilibrium existence results of Kelso and Crawford (1982), Hatfield and Milgrom (2005), Sun and Yang (2006, 2009), and Hatfield et al. (2013), are easy corollaries.

[59] It also follows from Section 6.1 that Hatfield et al. (2013, Thm. 7) is an easy corollary.

It is routine to check that D is unimodular. A mathematical argument that it is not a basis change of *strong* substitutes is given in Baldwin and Klemperer (2014); that it is not a basis change of *ordinary* substitutes can be demonstrated by a machine proof.[60]

Now $n > 4$ is straightforward: extend the vectors of D by zeros and append the coordinate vectors $\mathbf{e}^i$ for $i \geq 5$. It is clear that any basis change taking such a demand type to a purely substitutes demand type would restrict to a basis change taking D to ordinary substitutes, contradicting the above.

5 Existence of Equilibrium for Specific Valuations

Theorem 4.2 shows which demand types *always* have a competitive equilibrium, and for which demand types equilibrium fails for some valuations. For example, we saw equilibrium does not always exist for the demand type of Section 4.3.1, and indeed it did not exist for the valuations given there. But we will see below that if, for example, the "substitutes" agent of that example has a low enough valuation (e.g., the valuation u^{s*} discussed in Section 3.3), then the demand type would be unchanged, but equilibrium would exist.

So we now show that tropical intersection theory also provides results about *for which valuations* equilibrium exists, for demand types for which equilibrium does not always exist. As in our development of Theorem 4.2 (see Section 4.2), the key is that we can show that only certain isolated TH intersection points need be analysed. Moreover, tropical intersection theory provides bounds on the number of such points and, remarkably, tells us that a simple count of them may suffice to demonstrate the existence or failure of equilibrium.

5.1 The Tropical Bézout-Kouchnirenko-Bernshtein theorem

The crucial point is that the celebrated Bézout theorem extends to tropical geometry:

Bézout's (1779) theorem tells us that in two dimensions the number of intersection points of two (ordinary) geometric curves equals the product of their degrees, except in degenerate cases when the curves have a component in common. In counting "intersection points", we include those with complex coordinates and those at infinity (where, e.g., parallel lines "meet") and we assign an appropriate "multiplicity" to each intersection point. For example, a tangency which is a "double" root (as, e.g., between a line and a parabola) has multiplicity 2 and so counts twice. So, for example, a quadratic (degree 2) and a line (degree 1) always intersect twice; a quadratic and a cubic (degree 3) intersect $3 \times 2 = 6$ times. More recent (Bernshtein, 1975 and Kouchnirenko, 1976, see e.g. Gelfand et al. 1994) versions of the theorem have extended it, including to geometric objects in higher dimensions.[61]

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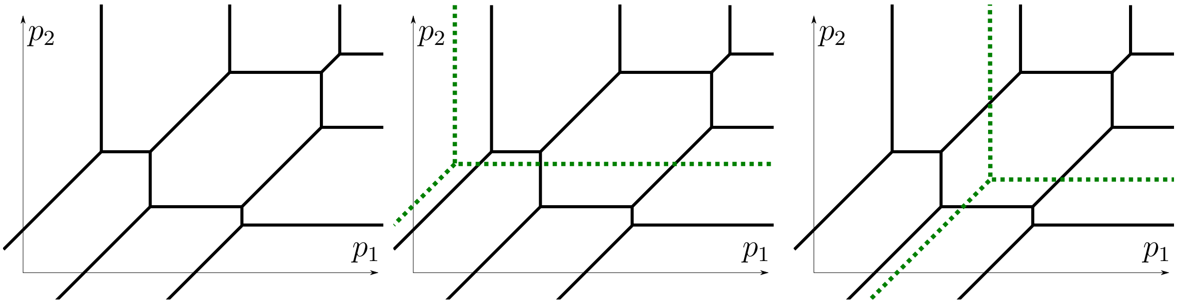
[60] We are very grateful to Tim O'Connor for helping us with this. The Matlab code, with notes, is available at elizabeth-baldwin.me.uk/papers/BasesCompilation.m
and www.nuff.ox.ac.uk/users/klemperer/BasesCompilation.m .

[61] The classical forms of these theorems also have applications in economics: McKelvey and McLennan (1997), McLennan (2002, 2005) and McLennan and Berg (2005) use them, among other techniques, to bound various characterisations of Nash equilibria.

A deep insight of tropical geometry is that THs can be obtained as particular transformations of "ordinary" geometric objects, and intersection properties are preserved under these transformations.[62] Thus similar intersection theorems hold: once we have defined "multiplicity" correctly, the TH provides exactly the right intersection counts. Furthermore, a TH is in real (not complex) space so we can "see them".

Our contribution is to observe that a "too-high" multiplicity at a transverse TH intersection price corresponds to a failure of discrete-convexity of the demand set there, that is, a "hidden" bundle, and so a failure of equilibrium when the supply is that bundle. So, if the intersection is transverse, a sufficient total number of intersection 0-cells guarantees equilibrium, and too few means that equilibrium may fail.

So, for example, Fig. 8 shows the TH of a "strong substitutes" valuation for up to 3 units in total, of 2 goods. All its facets have weight 1. Figs. 8b and 8c show the intersection of this TH with the THs of two different "strong substitutes" valuations for up to just 1 unit in total of the 2 goods. Notice that although the intersection prices are all in the same cell of the "1 unit" TH in Fig. 8b, and all in different cells of that TH in Fig. 8c, both TH intersections contain three points: it should be clear from these figures that *any* "1-unit strong substitutes" TH will intersect the first TH *exactly* three times if the intersection is transverse. (Recall from Propn. 3.6 that all "strong substitutes" facet normals for two goods are in ±{(1,0), (0,1), (-1,1)}.) And this is *precisely* because the TH of any valuation for up to 3 units in total is the tropical transformation of an "ordinary" cubic, and the TH of any valuation for up to 1 unit in total is the tropical transformation of an "ordinary" line, and–as we already know from Proposition 4.9–"strong substitutes" valuations always give rise to equilibrium, so these THs must intersect exactly $3 \times 1 = 3$ times if their intersection is transverse.



(a) TH of a strong substitute valuation for up to 3 units.

(b) The TH from (a), and a TH of a valuation for 1 unit

(c) The TH from (a), and a TH of another valuation for 1 unit

Figure 8: The THs of two generic valuations, one for up to 3 units, and one for a single unit, intersect exactly $3 \times 1 = 3$ times if equilibrium exists for any supply (as it does for strong substitutes valuations, illustrated).

By contrast, recall Figs. 5c and 5d giving the intersection between a single "complements" agent with one of two possible "substitutes" agents. In Fig. 5d, there are two intersections; in Fig. 5c there is only one. This corresponds *precisely* to the facts that competitive equilibrium exists for every supply in the former case, but–as we saw in Section 4.3.1–fails for some supply in the latter case.

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[62] THs are particular limits of logarithmic transformations of hypersurfaces (in complex projective space) in algebraic geometry. See, e.g., Maclagan and Sturmfels (2015) for a full discussion.

5.2 Condition for Equilibrium to Exist for Every Supply, For a Given Set of Valuations

Bézout's theorem was extended by Kouchnirenko and Bernshtein by use of "mixed volumes". Taking the sum of sets to be their Minkowski sum, define:

Definition 5.1. The n-dimensional *mixed volume* of n convex sets $X_1, \dots, X_n \subset \mathbb{R}^n$ is $MV_n(X_1, \dots, X_n) = \sum_{k=1}^{n} (-1)^{n-k} \left[\sum_{I \subset \{1,\dots,n\} | |I|=k} \mathrm{Vol}_n \left(\sum_{i \in I} X_i\right)\right]$.

Write $MV_n(X, Y, (k, n-k))$ for the mixed volume of k copies of X and $n-k$ copies of Y, for any $0 \leq k \leq n$.

Here, $\mathrm{Vol}_n(X_i)$ is the n-dimensional volume of X_i (so $\mathrm{Vol}_n(X_i) = 0$ if $\dim(X_i) < n$). So, in two dimensions, $MV_2(X, Y) = \mathrm{Vol}_2(X+Y) - \mathrm{Vol}_2(X) - \mathrm{Vol}_2(Y)$ (in which $\mathrm{Vol}_2(.)$ is, of course, just the two-dimensional area), and in three dimensions, $MV_3(X, Y, Z) = \mathrm{Vol}_3(X+Y+Z) - \mathrm{Vol}_3(X+Y) - \mathrm{Vol}_3(Y+Z) - \mathrm{Vol}_3(Z+X) + \mathrm{Vol}_3(X) + \mathrm{Vol}_3(Y) + \mathrm{Vol}_3(Z)$, etc. Thus the "mixed volume" is a linear combination of ordinary volumes.

An important special case is that it can be shown that $MV_n(X, \dots, X) = n!\mathrm{Vol}_n(X)$. See Appendix A.4.3 for further discussion.

We also generalise the concept of "weight", that we previously defined for TH facets, so that it applies to any TH cell. We do this by letting the *weight* of a $(n-k)$-cell of a TH be the (k-dimensional) volume of the corresponding SNP cell divided by the (k-dimensional) volume of its fundamental simplex.[63] We will give examples of use of mixed volumes, and weights, below.

Finally, we define the "naïve weighting"of any 0-cell at which two THs, $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$, intersect transversally, as $w_1 w_2$, in which w_i is the weight of the minimal cell of $\mathcal{T}_{u^i}$ that contains the 0-cell.[64]

As prefigured in the previous subsection (5.1) the tropical Bézout-Kouchnirenko-Bernshtein theorem, as presented by by Bertrand and Bihan (2007, 2013), now leads to a straightforward way to determine whether equilibrium exists for valuations whose THs' intersection is transverse. We first give the important theorem, and then explain it in more detail in Section 5.3 (full details are in Appendix A.4).

The key geometric result requires (only) that for some *fixed* k, the intersection of two THs is transverse at all 0-cells which are an intersection of a k-cell of the first TH, and an $(n-k)$-cell of the second:

Lemma 5.2. *Let u_1 and u_2 be any concave valuations on domains whose convex hulls are $\tilde{A}_1$ and $\tilde{A}_2$, and such that the domain of the aggregate valuation has dimension n, and such that for some $k \in \{1, \dots, n-1\}$, any intersection between a k-cell of $\mathcal{T}_{u^1}$ and a $(n-k)$-cell of $\mathcal{T}_{u^2}$ is transverse.*

The naïvely-weighted count of 0-cells at such cell intersections is bounded above by $MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k))$. If this count equals this upper bound, then equilibrium exists for u_1 and u_2 and any supply in the convex hull of demand at any price at such intersections.

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[63] A "simplex" on r vectors is the convex hull of those vectors together with $\mathbf{0}$. For the fundamental simplex, the relevant vectors are any integer basis of the minimal linear space parallel to the SNP cell. Its volume is $1/(k!)$ of that of the fundamental parallelepiped, namely the parallelepiped whose edges are this basis. See Definition A.31.

[64] Note this weight is *not* the "multiplicity" of the tropical Bézout-Kouchnirenko-Bernshtein theorem; indeed it is the distinction between them that yields our results.

If the bound is not met with equality and $n \leq 3$, then equilibrium fails for some supply in the convex hull of demand at some price at such an intersection.

It follows immediately from summing over k (and since a TH has no n-cells) that:

Theorem 5.3. *Let u_1 and u_2 be any concave valuations on domains whose convex hulls are $\tilde{A}_1$ and $\tilde{A}_2$, and such that the domain of aggregate demand has dimension n, and whose TH intersection is transverse.*

The naïvely-weighted count of 0-cells in their TH intersection is bounded above by $\sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k))$. If this count equals this upper bound, then equilibrium exists for u_1 and u_2 and any supply in the domain. If the bound is not met with equality and $n \leq 3$, then equilibrium fails for some supply in the domain.

To test whether equilibrium is guaranteed to exist for $m > 2$ valuations, we can sequentially check whether it always exists for the $(l+1)^{\text{th}}$ valuation and the aggregate of the first l valuations, for $l = 1, \ldots, m-1$. Of course, if the aggregate domain is of dimension less than n, we can just make a basis change to reduce the dimension of the goods-space so that the aggregate domain is of full dimension, and Theorem 5.3 (and Lemma 5.2) apply.

For example, consider $n = 2$, so that the cell weights are just the facet weights. Suppose $\tilde{A}_i$ contains all non-negative bundles of up to a total of d_i units, $i = 1, 2$. Then it is straightforward that $\text{Vol}_2(\tilde{A}_i) = (d_i)^2/2$. Also, $\tilde{A}_1 + \tilde{A}_2$ is the convex hull of all bundles of up to a total of $d_1 + d_2$ units so $\text{Vol}_2(\tilde{A}_1 + \tilde{A}_2) = (d_1 + d_2)^2/2$. Thus, $\sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k)) = MV_2(\tilde{A}_1, \tilde{A}_2) = d_1 d_2$.[65] So consider two valuations on these goods for up to a total of d_1 units, and d_2 units, respectively. Assume their THs intersect transversally (as is generic). Then the number of points in their intersection, each weighted by the product of the weights of the facets that intersect at that point, equals $d_1 d_2$ if equilibrium exists for these valuations for every supply, and is lower than $d_1 d_2$ otherwise. The simple example illustrated in Figs. 8b-c in the previous subsection (5.1) is a special case.

5.3 Explanation and Illustration of Theorem 5.3

Theorem 5.3 is an application of the tropical Bézout-Kouchnirenko-Bernshtein theorem, which tells us that if the intersection of two THs is transverse, then the "*multiplicity*"-weighted count of 0-cells in the intersection equals the relevant mixed volume. Our contribution is demonstrating that the multiplicity-weighting is equal to the "naïve" weighting precisely when equilibrium is guaranteed (or, when $n \leq 3$, precisely when it exists).

It is easiest to illustrate it in the two good case. Then a transverse intersection of two THs consist only of 0-cells, the "multiplicity" of such a cell is just the area of the corresponding aggregate SNP cell, and these SNP cells are all parallelograms with integer area. Furthermore, the minimal (indeed only) cells of the individual THs that contain these intersection points are the corresponding facets, so the relevant cell weights are just the facet weights.

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[65] More generally, if $\tilde{A}_i$ contains all bundles of $n > 2$ goods up to a total of d_i units, $i = 1, 2$, then $\sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k)) = \sum_{k=1}^{n-1} d_1^{n-k} d_2^k$.

Begin with the case in which all the facet weights are 1, so the *naïvely*-weighted count of Theorem 5.3 is just the unweighted count. It follows from the tropical Bézout-Kouchnirenko-Bernshtein theorem that if count of Theorem 5.3 equals the upper bound, the areas of the SNP cells corresponding to the intersection prices must all be 1 (since they must all be positive integers). So, as argued in Section 4.2 above, the aggregate demands are discrete-convex at all these points so, by Lemma 4.4, equilibrium exists for every supply.

But if the equality of Theorem 5.3 fails, then some multiplicity, and hence the area of some SNP cell at an intersection price, must exceed 1. So there is a "hidden bundle" at this price, and equilibrium *must* fail for this supply bundle. The reason is exactly as explained at Section 4.2.2: both agents have just two bundles in their demand set at this price, these can create only the $2 \times 2 = 4$ possible aggregate demands that correspond to the four vertices of the aggregate SNP cell, and so the "hidden bundle" cannot be an aggregate demand.

We described such an example in the introduction (hotel rooms). Here we give more details, using the simple two-good substitutes and complements valuations, u^s and u^c, whose individual THs and aggregate TH are shown in Figs. 5a-5c, and whose individual SNPs and aggregate SNP are shown in Figs. 6a-6c. The domain of each individual valuation is $\{0,1\}^2$, so the domain of the aggregate is $\{0,2\}^2$, and the relevant mixed volume is therefore $4-1-1=2$. The *individual* THs intersect only at the price $\left(\frac{1}{2},\frac{1}{2}\right)$, and this is transverse (so Theorem 5.3 applies). So, since the two facets containing it both have weight 1, the naïvely-weighted count is just 1, and equilibrium therefore fails for some supply. Indeed, since the relevant mixed volume is 2, we know that the area of the (aggregate) SNP cell at the single intersection point must be 2, as we can see in Fig. 5c; the relevant cell is the central diamond, and contains the "hidden" bundle $(1,1)$ at which equilibrium fails, as we saw in Section 4.3.1.

On the other hand, u^{s*} (see Section 3.3) is a valuation of the same demand type (indeed same combinatorial type), and on the same domain as u^s, so the mixed volume relevant to testing for equilibrium of u^{s*} and u^c is still 2, but the individual THs now intersect at two prices, $\left(\frac{1}{4},\frac{3}{4}\right)$ and $\left(\frac{3}{4},\frac{1}{4}\right)$–see Fig. 5d. Both these intersection points are transverse, so Theorem 5.3 still applies, and the total naïvely-weighted count is now 2. Equilibrium therefore exists for every supply, as can be seen in Fig. 6d: the two SNP cells corresponding to the intersection points are the two parallelograms which both have area 1, and every supply bundle is a vertex of the SNP, and so is demanded for some price.

It is not hard to generalise to the case in which the intersecting facets' weights, w_1 and w_2, may exceed 1. Because we assumed that the individual agents have concave valuations, each agent, j, is indifferent, at prices on its facet, among the (w_j+1) bundles on the corresponding edge in its individual SNP.[66] Agent, j's SNP edge is, of course, of "length" w_j (that is, it is w_j times its primitive integer vector), and the SNP cell of the aggregate valuation is now a "large" parallelogram, which can be divided by a grid into $w_1 w_2$ copies of a small parallelogram whose edges are the minimal (i.e., primitive) integer vectors in the same directions.

It is clear that all the bundles on the vertices of this grid are demanded at the

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[66] The reason is the same as for the "grey" bundles, not "black" ones, in our discussion in Section 2.4: j's valuation is linear along the SNP edge.

intersection price; they correspond to the different ways in which we can give either agent one of the bundles between which it is indifferent. (Again, these bundles correspond to "grey" bundles, not "black" ones, in our discussion in Section 2.4.) Furthermore, there are no other bundles in the "large" parallelogram if and only if each of the (identical) small parallelograms has area 1. So there is no problem bundle at the intersection price if and only if the area of the large parallelogram (that is, its multiplicity) equals $w_1 w_2$ (that is, the naïve weighting of the corresponding 0-cell). It follows again, therefore, that equilibrium holds for every supply in the domain of the aggregate valuation iff the naïvely-weighted count of all the intersection 0-cells equals the multiplicity-weighted count, and the tropical Bézout-Kouchnirenko-Bernshtein theorem tells us the latter equals the relevant mixed volume.

For more than two goods, the logic is similar in spirit, but more complicated in detail:

First, the definition of cell weight is more intricate, and so is that of "multiplicity".

Next, the intersection does *not* consist only of 0-cells, but (in the transverse case) will be a "rational polyhedral complex" (that is, have a similar structure to a lower-dimensional TH) of dimension $(n-2)$. So we first simplify the situation. To do this, we recall (Lemma 4.4) that we are focussing on intersection prices because, if equilibrium fails for any supply, there must be a failure of discrete-convexity of the aggregate demand at some price in the intersection. Any such price lies in some k-cell of the aggregate TH. It follows easily from the duality construction of Section 2.4 (see Corollary A.11) that because we assume the domain of the aggregate valuation is n-dimensional, this k-cell has some 0-cell(s) in its boundary. So these 0-cell(s) also lie in the individual THs' intersection. And it also follows that the bundle demonstrating failure of discrete-convexity at the original price *also* demonstrates failure of discrete-convexity at the 0-cell price. So we can restrict attention to just the 0-cells of the intersection, that is, a finite number of points.

Finally, a transverse intersection 0-cell in more than two dimensions need not correspond to a parallelepiped in the SNP. However, we can draw a parallelepiped which contains the right information. Start with a vertex at some bundle which is demanded, on aggregate, at this 0-cell price. Identify what each individual agent will receive when this bundle is the aggregate demand, and then, for each agent, choose a basis for possible changes in demand at this price. We use the collection of these basis vectors as the edges of our parallelepiped. Some *subset* of this parallelepiped is contained in the aggregate SNP cell corresponding to the 0-cell of the TH at the intersection.

The parallelepiped contains additional bundle(s) to those that are either on its vertices or on the vertices of a grid of "small" parallelepipeds (in the case that any of its edges' weights exceed 1) iff its volume exceeds the product of its edges' weights. One can show that, if the parallelepiped does *not* contain such a potentially-problematic bundle, then there are no problems in the SNP cell itself. However, if the parallelepiped *does* contain such a bundle, these bundle(s) *may or may not* lie in the SNP cell itself.

In three dimensions this creates no ambiguity: the TH cell at the intersection is at least half a parallelepiped (and its vertices are vertices of the parallelepiped). By symmetry, if there is any additional bundle in a parallelepiped then there is one in both halves. But, in four or more dimensions, the equality of the Theorem 5.3's condition is sufficient, but no longer necessary, for equilibrium, as we illustrate in Example A.29 in the Appendix. However, we can then use Lemma 5.2 for each individual k separately, to narrow down where, if anywhere, equilibrium might fail; we make use of this in the next subsection, where we create a general recipe for checking the existence of equilibrium.

5.4 Checking Equilibrium in the General Case

It might be conjectured that we could apply Theorem 5.3 when intersections are not transverse. Recall from our analysis in Section 4.2.5 that, if equilibrium fails at an intersection price, then it also fails after a small perturbation in valuations to make the intersection transverse. However, it is *not* true that if equilibrium exists, it also exists after a small generic perturbation: there are "fragile" equilibria which only arise at a non-transverse intersection (see Appendix Example A.36). To determine whether equilibrium exists for such cases, we first need a little more tropical-intersection theory:[67]

Definition 5.4. The *stable intersection,* $\mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2}$, of THs $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ consists of all cell intersections $C^1 \cap C^2$ where C^i is a cell of $\mathcal{T}_{u^i}$ and $\dim(C^1 + C^2) = n$.

Recall that two THs intersect transversally at $\mathbf{p}$ if C^j is the *minimal* cell of $\mathcal{T}_{u^j}$ containing price $\mathbf{p}$ and $\dim(C^1 + C^2) = n$. So the stable intersection of two THs contains all cells at which the THs intersect transversally, and may contain some or all of any additional cells where they intersect.

For example, in Fig. 5c, if the (downward-sloping) "complements" TH were translated by a parallel shift to pass through (1,1)–the vertex where three line-segments of the "substitutes" TH meet–then the THs would not intersect transversally anywhere (the intersection at (1,1) is non-transverse since the substitutes TH has a 0-cell there). However, the point (1,1) would be the stable intersection, since it is the intersection of the "complements" facet with any one of the three facets of the "substitutes" TH. Two identical substitutes THs, both of the kind shown in Fig. 5a, also have no transverse intersection, while the 0-cells (0,0) and (1,1) would then form the stable intersection. Note that the coincident line segments of the intersection are *not* in the stable intersection. More generally, a stable intersection of two THs in n dimensions has the same "cell structure" as a TH (that is, it is a rational polyhedral complex) and is of dimension $n-2$.[68]

An alternative way to define the "stable intersection" both gives additional insight into the relationship with transverse intersections, and yields an important result. The "stable intersection" is the limit-set of any series of "perturbed" TH intersections that

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[67] We conjecture that there are other "counting" methods of determining whether equilibrium exists when the intersection is transverse. For example, we could perturb each agent's original valuation by a small amount to make it "strictly concave", that is, so that all the cells of the individual TH where "hidden" bundles just touch the roof are "opened up" (so, for example, all the facet weights of the individual THs are then 1, and all bundles that correspond to "grey" ones in our earlier discussion of Section 2.4 turn "white"). And we could make the perturbation small enough that no new facet that has been created (or moved) in either individual TH is far enough from the location of the original facet from which it was derived to otherwise disturb the structure of the aggregate TH. We could then check whether the number of UDRs in the new aggregate TH equals the number of bundles in the aggregate domain. However, this would both be cumbersome, and also require detailed knowledge of the aggregate TH to ensure the perturbation was small enough. Moreover, there is no obvious way to extend such an approach to analyse the general case which we now discuss.

[68] See Maclagan and Sturmfels, 2015, Thm. 3.6.10. If $n = 1$ then the stable intersection is empty.

can be created by fixing one of the THs, and imposing on the other a series of arbitrarily small (and decreasing to zero) translations such that in every case the intersection between the THs is transverse. It is a standard result of tropical geometry that this limit is well-defined.[69]

Now recall that we showed (in our proof of Theorem 4.2) that equilibrium can fail at an intersection only if it also fails for arbitrarily close valuations that intersect transversally. So if equilibrium fails at an intersection, we can take a series of infinitesimally small and decreasing perturbations that yield transverse intersections at which it also fails. It follows that equilibrium must *also* fail in the limit of these transverse intersection prices, that is, at the stable intersection. So any failure of equilibrium must show up at prices in the stable intersection (since, from Lemma 4.4, we only need to check intersections of THs). That is:

Theorem 5.5. *If the domain of the aggregate valuation of two concave valuations has dimension n, then equilibrium exists for every supply bundle in the domain* iff *the aggregate demand set is discrete-convex at every* 0*-cell of the stable intersection of agents' THs.*

In particular, if equilibrium fails for a supply $\mathbf{x}$, then this supply must exhibit failure of discrete-convexity (that is, $\mathbf{x} \in \operatorname{Conv} D_U(\mathbf{p})$, but $\mathbf{x} \notin D_U(\mathbf{p})$) at some price $\mathbf{p}$ in such a 0-cell. As usual, if the aggregate domain is of dimension less than n, we can just make a basis change to reduce the dimension of the goods-space so that the aggregate domain is of full dimension, and Theorem 5.5 then applies.

So even if the intersection of two THs contains a continuum of points, we only need to check a finite number of points to find out whether equilibrium always exists.[70] And, as before, to test whether competitive equilibrium always exists for $m > 2$ valuations, we can sequentially check, for $l = 1, \ldots, m-1$, whether equilibrium always exists for the aggregate of the $(l+1)^{\text{th}}$ valuation and the aggregate valuation of the first l valuations.

Furthermore, the Tropical Bézout theorem now gives us a bound on the number of the points that we have to check:

Theorem 5.6 (cf. Bertrand and Bihan, 2007, 2013)**.** *The number of* 0*-cells in the stable intersection of THs $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ of valuations on domains whose convex hulls are $\tilde{A}_1$ and $\tilde{A}_2$, respectively, which are inside a k-cell of $\mathcal{T}_{u^1}$ and a $(n-k)$-cell of $\mathcal{T}_{u^2}$, is bounded above by $MV_n(\tilde{A}_1, \tilde{A}_2, (n-k,k))$. The total number of* 0*-cells in the stable intersection of the THs is bounded above by $\sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k,k))$.*

The second part of this result is obvious from the fact that any 0-cell in the stable intersection of two THs is contained in a k-cell of one TH and an $(n-k)$-cell of the other TH for some $1 \leq k \leq n-1$ (see Definition 5.4; a TH has no n-cells, of course), although k need not be uniquely defined.

Observe that the two-dimensional case ($n = 2$) is particularly straightforward; the bound on the total number of 0-cells is then just $MV_2(\tilde{A}_1, \tilde{A}_2)$.

In sum, therefore:

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[69] Maclagan and Sturmfels' 2015 Prop. 3.6.12, stated as our Appendix Proposition A.37. Note this definition gives us another easy way to see, in the examples just above, that the 0-cell at price (1,1) is in the stable intersection, but is not transverse.

[70] But if equilibrium does fail at such a point, it might then also fail at a continuum of prices.

Corollary 5.7. *For any concave valuations, we can check whether or not equilibrium exists at every supply in the domain by checking only a finite set of prices, the number of which we have bounded.*

Moreover, combining Theorems 5.3, 5.5 and 5.6 and Lemma 5.2 yields a recipe to test whether equilibrium exists for every supply in the domain, for the aggregate of any two concave valuations:[71]

Recipe 5.8.

(1) If the domain of the aggregate valuation is in less than full dimension, make a basis change so that it is in full dimension.
(2) If the intersection is not transverse, go to (5).
(3) If the naïvely-weighted count of 0*-cells in the intersection equals* $\sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k))$*, equilibrium exists for all supplies.*
(4) If not, and $n \leq 3$*, equilibrium fails for some supply.*
(5) Equilibrium exists for all supplies iff for each k*, one of the following holds:*
  (i) every 0*-cell in both a* k*-cell of* $\mathcal{T}_{u^1}$ *and a* $(n-k)$*-cell of* $\mathcal{T}_{u^2}$ *is a transverse intersection point, and the naïvely-weighted count of these* 0*-cells equals* $MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k))$
  or (ii) for every 0*-cell in both a* k*-cell of* $\mathcal{T}_{u^1}$ *and a* $(n-k)$*-cell of* $\mathcal{T}_{u^2}$*, the aggregate demand set (found from the two agents' individual demand sets) can be seen directly to be discrete-convex.*

6 Applications

6.1 Interpreting Classic Models in a Unified Framework

Our model encompasses some classic studies as special cases, so clarifies connections between them. It also facilitates our understanding of these papers. In particular, it makes many of their equilibrium-existence results straightforward. And Baldwin and Klemperer (2014, in preparation-b) use our framework to study the implications of their assumptions on preferences.

Kelso and Crawford's (1982) seminal analysis of n_1 firms, each of which is interested in hiring some of n_2 workers, can be understood as a model with $n_1 n_2$ distinct "goods", each of which is the "transfer of labour" by a specified worker (a "seller") to a specified firm (a "buyer"); the "price" of a good is the salary to be paid. So the full set of bundles we consider is $\{-1, 0, 1\}^n$, in which $n = n_1 n_2$, but each agent's valuation is defined only on the subset of this domain that is relevant to it.

Specifically, each worker has preferences only over a subset of the domain of the form $\{-1, 0\}^{n_1}$ (that is, it has preferences only over the n_1 goods that correspond to its own labour), and only over the subset of these vectors that have at most one non-zero entry (it can work for at most one firm). Obviously, their only possible SNP edges are the strong substitute vectors (non-zero vectors with at most one $+1$ entry, at most

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[71] Of course, if 5(i) holds, then so does 5(ii). Also, if $n \leq 3$, then if 5(i) fails for any k (for which every 0-cell in a k-cell of $\mathcal{T}_{u^1}$ and a $(n-k)$-cell of $\mathcal{T}_{u^2}$ is a transverse intersection point), then so will 5(ii), so there is no need to proceed further.

one -1 entry, and no other non-zero entries, see Proposition 3.6). Furthermore, each firm only has preferences over a subset of the aggregate domain of the form $\{0,1\}^{n_2}$ (that is, it has preferences only over the n_2 goods that correspond to workers it can employ). Kelso and Crawford assume firms have ordinary substitutes preferences over workers, but with only one unit of each good, the only substitute changes of demand are the strong substitutes vectors, so all valuations are of this type.[72] Individual concavity always holds when individual domains have this form.

It is perhaps less obvious that Hatfield et al.'s (2013) model of networks of trading agents, each of whom can both buy and sell, both fits into our framework, and is also closely related to Kelso and Crawford's model. To show this is so, we (again) treat each transfer of a product from a specified seller to a specified buyer as a separate good, so each agent again has preferences over a subset of $\{-1,0,1\}^n$, where n is now the number of "separate goods".

Since Hatfield et al. restrict each agent to be either a seller or a buyer (or neither) on any one good, an agent i which is the specified seller in n_1^i potential trades and is the specified buyer in n_2^i potential trades simply has preferences over a subset of the domain which, after an appropriate re-ordering of the goods for that agent, is of the form $\{-1,0\}^{n_1^i} \times \{0,1\}^{n_2^i}$. (As in Hatfield et al., we can restrict an agent's domain of preferences further so that, e.g., it cannot sell good 1 unless it also buys one of goods 2 or 3.) Furthermore, although Hatfield et al. describe goods to be sold as complements of goods to be bought, this is because they measure both buying and selling as non-negative quantities. So, since in our framework selling is just "negative buying", the "complementarities" disappear and it is clear that the condition they impose is exactly ordinary substitutes.[73] Just as for Kelso and Crawford's model, the only SNP edges of such a domain that are vectors of the ordinary substitutes demand type are also vectors of the strong substitutes demand type.

Trivially, any valuation over any subset of $\{-1,0\}^{n_1}$ or $\{0,1\}^{n_2}$ or $\{-1,0\}^{n_1^i} \times \{0,1\}^{n_2^i}$ is concave so, in both Kelso and Crawford's and Hatfield et al.'s models, the existence of equilibrium follows immediately from Proposition 4.9.

Reformulating models in our framework also shows clearly how we can generalise them. It is immediate, for example, that as long as we retain concavity and the strong substitutes demand type, we can remove Hatfield et al.'s restriction that an agent cannot be both a buyer and a seller on any one good (by simply extending their domain to be *any* subset of $\{-1,0,1\}^n$) and can also permit their agents to trade multiple units of the same products (by enlarging the domain to any subset of $\mathbb{Z}^n$).

Other models that fit into our framework are Bikhchandani and Mamer (1997) (this is just the restriction of our model to $A = \{0,1\}^n$), and Hatfield and Milgrom's (2005) famous model of "contracts" (since this can be embedded in Kelso and Crawford's model–see Echenique, 2012).[74]

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[72] Kelso and Crawford actually make a more restrictive assumption than this for their substitute preferences, but in fact the characterisation follows from the weaker assumption mentioned here. See Danilov et al. (2003), Baldwin and Klemperer (2014), and Baldwin, Klemperer and Milgrom (in preparation).

[73] Their "choice language" definition differs superficially from Definition 3.2, but Hatfield et al., 2015, Thm B.1 confirms the equivalence.

[74] Hatfield and Kojima (2010) does not fit into our framework, since it is inconsistent with quasi-linear preferences.

6.2 Analysing when Equilibrium Exists

As an example, consider "complements" consumers, each of whom is only interested in a single, specific, pair of goods, and that these pairs form a cycle. Thus there are n kinds of consumers and n goods, and we can number both goods and consumers $1, \ldots, n$, such that every consumer of kind $i < n$ demands goods i and $i + 1$, which it sees as perfect complements, while consumers of kind n demand goods n and 1. It is easy to check that:

$$\det \left(\begin{array}{ccccccc} 1 & 0 & 0 & & & 0 & 1 \\ 1 & 1 & 0 & & & 0 & 0 \\ 0 & 1 & 1 & & & 0 & 0 \\ \cdot & 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & 1 & 0 \\ 0 & 0 & 0 & & & 1 & 1 \end{array} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

So if n is odd, Corollary 4.3 tells us equilibrium does *not* always exist. Furthermore, our proof of Theorem 4.2 shows that we can find an example of equilibrium failure by simply selecting a single consumer of each type, each of which values its desired pair at v, so that they are all indifferent between purchase and no purchase (and hence their facets all intersect) if every good's price is $v/2$.[75] But the tropical-Bézout methods of Section 5 can determine, for any *given* set of agents' valuations, whether equilibrium exists for every supply in the domain.

If n is even, the columns of this matrix are not linearly independent. However, if we exclude the i^{th} column, for any i, the remaining $n-1$ columns are then linearly independent, and can trivially be extended to n linearly independent vectors with determinant 1 by adding the column $\mathbf{e}^i$. So using Theorem 4.2, equilibrium *always* exists if n is even, since the valuations are, trivially, concave.[76,77]

6.3 Basis Changes to find new Demand Types that always have Equilibrium

We can use the fact (Section 4.3.2) that equilibrium existence, and other properties,[78] are unaffected by basis changes, together with knowledge of these properties for any one

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[75] So equilibrium fails if aggregate supply is exactly 1 unit of each good (the "middle of the parallelepiped") since the minimum and maximum aggregate demands are zero, and 2 units of each good, respectively, at this price. (It is easy to check failure of equilibrium for $x_i = 1$, for all i, by contradiction. At least one good, w.l.o.g. good 1, would not be part of a pair being allocated together. So good 1 has value 0 to whoever receives it, hence $p_1 \leq 0$. Therefore $p_2 \geq v$, since otherwise consumer 1 would demand both goods 1 and 2. Therefore $p_3 \leq 0$, since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore $p_4 \geq v$, etc., so $p_j \leq 0$ if j is odd. But consumer n then wishes to buy goods n and 1, which is a contradiction.)

[76] For example, the aggregate demand of 1 unit of each good is supported by price $v/2$ for every good, when there is exactly one consumer of each kind, each of which values its preferred pair at v.

[77] Sun and Yang (2011) and Teytelboym (2014) have independently used alternative methods to show these results for a version of this model; the even n case is also a special case of the "generalised gross substitutes and complements" demand type that we discuss in Section 6.3. Once one understands the relationship with matching – see Section 6.5 – the $n = 3$ case is given by Pycia (2008).

[78] See Baldwin and Klemperer (2014, especially Section 5) and Baldwin and Klemperer (in prepn-c).

demand type, to obtain useful results about other demand types.[79]

For example, a number of transformations of strong substitutes are of interest:

Interval Package Valuations[80] Premultiplying the strong substitutes vectors $\mathbf{e}^i$ and $(\mathbf{e}^i - \mathbf{e}^j)$, by the upper triangular matrix of 1s (of the appropriate dimension) yields the vectors $\sum_{k=1}^{i} \mathbf{e}^k$ and $\sum_{k=j+1}^{i} \mathbf{e}^k$ for $i > j$, respectively (and their negations). This is the "interval package valuations" demand type for goods which have a natural fixed order, and for which any contiguous collection of goods may be considered as complements by any agent. For example, valuations for bands of radio spectrum, or for "lots" of sea bed to be developed for offshore wind (see Ausubel and Cramton, 2011) may be of this form.

So this is an important purely-complements demand type for which equilibrium always exists when valuations are concave.

Generalised Gross Substitutes and Complements Premultiplying the strong substitutes vectors by a matrix formed of $\{\mathbf{e}^i | \, i \leq k\} \cup \{-\mathbf{e}^i | \, i > k\}$, for some k, yields the demand type whose valuations are those satisfying Sun and Yang's (2006, see also 2009) definition of "gross substitutes and complements"extended to permit multiple units of goods, and sellers as well as buyers.[81] These are valuations such that goods can be separated into two groups, with goods within the same group being strong substitutes, and each good also may exhibit 1:1 complementarities with any good in the other group. As above, it is immediate that equilibrium always exists for concave "generalised gross substitutes and complements" valuations.[82]

6.4 Other new Demand Types

Our proof of Proposition 4.11 exhibited a matrix D defining a unimodular demand type containing only complements demands that has not previously been studied. This might, for example, model a firm's demand for "bundles of" four kinds of workers–three sorts of specialist (the first three goods) and a supervisor (the fourth good). The first three columns of D show that any one of the three kinds of specialist has value on his own; a supervisor on her own is worthless[83]; but the middle three columns of D show that a supervisor increases the value of any specialist (that is, there are pairwise complementarities between any one of the first three "goods" together with the fourth); and the last three columns of D show that there are also complementarities between any pair of different specialists if (but only if) a supervisor is also present.[84]

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[79] We implicitly gave an example above, when we observed that Hatfield et al's (2013) model of complements could be understood as being restricted to strong substitutes by relabelling the good "purchase of a unit" as -1 units of a good "sale of a unit".

[80] These valuations were introduced by Danilov et al. (2008, 2013).

[81] The demand type's vectors are $\{\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^i - \mathbf{e}^{i'}, \mathbf{e}^i + \mathbf{e}^j, \mathbf{e}^j - \mathbf{e}^{j'} \mid i, i' \in \{1, \ldots, k\}, j, j' \in \{k + 1, \ldots, n\}\}$.

[82] Shioura and Yang (2013) have independently made the same extension of gross substitutes and complements, and shown that equilibrium always exists for it.

[83] The reason is that $\mathbf{e}^4 \notin \mathcal{D}$; perhaps each firm's owner is a supervisor herself, and an additional supervisor without any workers would merely "spoil the broth".

[84] There is an infinite family of related unimodular demand types in higher than four dimensions. There are, of course, many basis changes of D (and of any unimodular demand type) that include all

6.5 Matching

The example in the previous subsection can be interpreted as a multi-player matching problem in which the columns of D are the coalitions of workers that create value: Baldwin and Klemperer (2014, in preparation-a) show that, assuming perfectly transferable utility, a stable matching in which no subset of workers can gain from re-matching (that is, an allocation in the core of the game among the workers) corresponds exactly to a competitive-equilibrium allocation of workers in our model (in which every worker receives its competitive wage, and no further gains from trade are possible). So, since the demand type is unimodular, it describes a class of multi-player matching problems for which a stable match always exists.

More generally, Baldwin and Klemperer (2014, in preparation-a) show any matching problem with perfectly transferable utility corresponds to a demand type containing only vectors in $\pm\{0,1\}^n$. If, as in the "workers" example above, the demand type is unimodular, a stable match always exists. If it is not unimodular, the tropical-Bézout methods of Section 5 tell us for what coalitions' valuations there are stable matchings.[85]

6.6 Understanding Individual Demand

Our techniques are powerful tools for understanding individual demand. In particular, Mikhalkin's important observation (our Theorem 2.3) tells us that *any* balanced rational polyhedral complex is the TH of some quasilinear valuation *and conversely*. This allows us to explore properties of valuations by drawing and analysing appropriate geometric diagrams *without* needing to undertake the typically much more challenging task of constructing valuations that generate these diagrams.

Baldwin and Klemperer (2014, in preparation-b) further explores the comparative statics of individual demand, in order to better understand demand changes at non-UDR prices. Unimodularity turns out to have important implications for the structure of *individual* demand, as well as (as we saw in our discussion of the existence of equilibrium) for aggregate demand. This work also leads to a generalisation of Gul and Stacchetti's (1999) "Single Improvement Property".

Related work (joint with Paul Milgrom) uses our framework to help understand implications of different notions of substitutability for indivisible goods that have been suggested in the literature.[86]

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the coordinate vectors.

Another intriguing example of a unimodular demand type that is *not* a member of this family, but which is also not a unimodular basis change of strong substitutes, contains all vectors in $\mathbb{Z}^6$ with three +1 entries and three -1 entries, thus incorporating (all) valuations for six distinct goods in which all changes in an agent's demand involve swapping three of the goods for the other three goods as prices move between UDRs. Danilov and Grishukhin's (1999) characterisation of maximal unimodular sets of vectors provides many more examples (including a basis change of D, but not a pure-complements one with this interpretation).

[85] Baldwin and Klemperer (in preparation-a) develops the application to matching in detail; preliminary work is in Baldwin and Klemperer (2014). Since our framework allows us to consider multiple players of each kind, it easily yields results along the lines of Chiappori, Galichon, and Salanié (2014, see also Balinski, 1970).

[86] Baldwin, Klemperer, and Milgrom (in preparation). This paper also develops the relationship between the existence of equilibrium for substitutes and properties of the Vickrey auction and the core.

6.7 Auction Design

Practical auctions need to restrict the kinds of bids that can be made, thus restricting the preferences that bidders can express. Restricting to a demand type is often natural, since the economic context often suggests appropriate trade-offs between goods. For example, the Bank of England expected bidders to have £1:£1 trade-offs between any pair of the several different "kinds" of money it loaned in the financial crisis.[87] Since such trade-offs can be represented by strong substitutes preferences, if we permit rationing, the Bank chose auction rules that made it easy for bidders to communicate such preferences, and the Bank was also unconcerned about ruling out the expression of other preferences.[88]

Knowing that the bids in an auction must all express preferences of a demand type also clarifies the meaning, and the implications, of the restrictions that have been imposed on the bidders.[89] In particular, the motivation of the Product-Mix Auction is to find competitive equilibrium, given bidders' and the bid-taker's reported preferences. Since the Bank of England's implementation of the Product-Mix auction allows rationing (which makes "goods" divisible) ensuring the existence of equilibrium is not too hard.[90] But in many contexts rationing is less sensible. For example, a too-small piece of radio spectrum may not be useful. Similarly, a government may be interested in offers to build gas-fired plants, nuclear-power stations, wind farms, etc., and these may be indivisible. So results about equilibrium with indivisibilities tell us when Product-Mix Auctions can easily be used.[91]

Our techniques also facilitate the analysis of Product-Mix Auctions. Individuals' bids in these auctions are aggregated in exactly the same simple way that THs are combined to find aggregate demand. This also makes the auctions more "user-friendly", and is critical for getting them implemented in practice. Moreover, geometric analysis can develop methods for finding equilibrium in new versions of the Product-Mix Auction; this may help resolve problems currently facing regulators such as the U.S. Federal Communications Commission, the U.K.'s Ofcom and the U.K. Department for Energy and Climate Change.[92]

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[87] The different "goods" were long-term loans (repos) against different qualities of collateral.

[88] *Any* strong substitutes preference could be expressed if the Bank's "Product-Mix" Auctions (described in Klemperer, 2008, 2010, and Baldwin and Klemperer, in preparation-c) were augmented by permitting "negative" bids (see Klemperer, 2010, and Baldwin and Klemperer, in preparation-c).

[89] Restricting to a demand type also permits relatively complex "bids" while still checking that they satisfy the restrictions, since there are easy software solutions to calculate the normal vectors of the TH for any valuation and so reveal the demand "type".

[90] So the updated (2014) implementation of the Bank's auction also permitted some complements preferences while maintaining the existence of equilibrium.

[91] Baldwin and Klemperer (in preparation-c) shows when equilibrium existence is guaranteed in new forms of the Product-Mix Auction.

[92] Product-mix auctions are "one-shot" auctions for allocating heterogeneous goods. Their equilibrium allocations and prices are similar to those of clock or Simultaneous Multiple-Round Auctions in private value contexts, but they permit the bid-taker to express richer preferences; they are more robust against collusive and/or predatory behaviour; and they are, of course, much faster. (They can also resolve clock auctions' problem of failing to find the exact equilibrium when it is unique, or the correct equilibrium when it is not–see Harbord et al., 2011.)

For other work in auction development, using these and other geometric techniques, see Candogan et al. (2015) and Lee (2015).

7 Conclusion

An agent's demand is completely described by its choices at all possible price vectors. So it can also be described by the *divisions* between the regions of price space in which the agent demands different bundles, and hence by the vectors that define these divisions. This suggests a natural way of classifying valuations into "demand types".

Using this classification, together with the duality between the geometric representations of valuations in price space and in quantity space, yields significant new insights into when competitive equilibrium exists.

A demand type's vectors also encode the possible comparative statics of demand, and we expect many other results can be understood more readily, and developed more efficiently, using our geometric perspective.

Companion papers[93] use our framework and tools to obtain new results about the existence of stable matchings in multiple-agent matching models; about individual demand; and further develop the Product-Mix Auction implemented by the Bank of England in response to the 2007 Northern Rock bank run and the subsequent financial crisis.

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[93] See Baldwin and Klemperer (in preparation-a,b and c). Preliminary work is in Baldwin and Klemperer (2014).

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A Additional formal definitions, and proofs of results in the text

A.1 More details for Section 2

A.1.1 The mathematics and economics of tropical hypersurfaces

Recall we defined the underlying set of a tropical hypersurface associated to valuation u to be

$$\mathcal{T}_u := \{\mathbf{p} \in \mathbb{R}^n \mid \#D_u(\mathbf{p}) > 1\}.$$

As stated in the text, a tropical hypersurface has the structure of a weighted rational polyhedral complex. Here we build up that structure by understanding the economic interpretation of its components. Throughout we work with the Euclidean topology.

Definition A.1.

(1) The *cell interior* of the TH $\mathcal{T}_u$ at a price $\mathbf{p}$ consists of points $\mathbf{p}'$ such that $D_u(\mathbf{p}) = D_u(\mathbf{p}')$.[94] A subset of $\mathcal{T}_u$ is a cell interior if it is the cell interior at some point in $\mathcal{T}_u$.

(2) A subset of $\mathcal{T}_u$ is a *cell* if it the closure of a cell interior of $\mathcal{T}_u$.

(3) The *affine span* of a cell of $\mathcal{T}_u$ is the smallest affine space containing the cell.[95]

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[94] Note that cells are subsets of the TH $\mathcal{T}_u$, and not, as one might intuitively guess from looking at Fig. 1, the open areas around the sides of the TH; those are the "unique demand regions".

[95] Recall that an affine space in $\mathbb{R}^n$ is a parallel shift of a linear subspace, that is, a set $\{\mathbf{v}+\mathbf{c} \mid \mathbf{v} \in U\}$ for some linear subspace $U \leq \mathbb{R}^n$ and some fixed vector $\mathbf{c}$.

(4) The *dimension* of a cell is the dimension of its affine span. A cell of dimension k is referred to as a k-cell. An $(n-1)$-cell is referred to as a *facet* (where n is the number of goods, and so the dimension of Euclidean space in which the TH lives).

(5) The *boundary* of a cell of $\mathcal{T}_u$ consists of those points in the cell that are not in its cell interior.

(6) A *unique demand region (UDR)* is a connected component of the complement of the TH.

Note that the cell interior is the largest set that is both contained in the cell and open in the affine span of the cell.[96] Naturally, "unique demand regions" are so-called because the demand set contains only one element for such prices.

A TH has the structure of an abstract "polyhedral complex":

Definition A.2.

(1) A set $\Pi \subsetneq \mathbb{R}^n$ is a *polyhedral complex* if:
   - (i) Π is the union of finitely many cells.
   - (ii) Each cell is a closed convex polyhedral set in $\mathbb{R}^n$. That is, each cell may be represented as an intersection of half-spaces $\{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p}.\mathbf{w} \geq \alpha\}$ for some vector $\mathbf{w}$ and scalar α.
   - (iii) The interiors of the cells do not intersect.
   - (iv) The boundary of a k-cell is the union of a finite number of $(k-1)$-cells.

(2) Π is a *rational* polyhedral complex if the slope of the affine span of each cell is rational. That is, in 1(ii), the vectors $\mathbf{w}$ may be taken to have integer coefficients.[97]

(3) Π is *k-dimensional* if it is contained in the union of its k-cells.

It is easy to see that any TH is an $(n-1)$-dimensional rational polyhedral complex. Properties 1(i) and 1(iii) follow by definition of the cells. Cell interiors are defined by a collection of equalities $\mathbf{p}.(\mathbf{x}-\mathbf{x}') = u(\mathbf{x})-u(\mathbf{x}')$ and inequalities $\mathbf{p}.(\mathbf{x}-\mathbf{x}') > u(\mathbf{x})-u(\mathbf{x}')$, and that a cell is defined by weakening these strict inequalities to weak inequalities. So properties 1(ii) and (2) follow. The boundary of a cell is where at least one of the weak inequalities holds with equality; when this is the case the price must lie in a lower dimensional cell, so property 1(iv) is satisfied. Finally, we note that it is generic for only one bundle to be demanded, and hence the UDRs are n-dimensional; moreover, they are also polyhedral sets. As the TH is the complement in $\mathbb{R}^n$ of the UDRs, it follows that the TH is $(n-1)$-dimensional.

It is useful to note the economic meaning of cells versus cell interiors:

Lemma A.3. *Let C be a cell, let C° be its (relative) interior, and fix $\mathbf{p}^\circ \in C^\circ$. Then $D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})$ iff $\mathbf{p} \in C$, with equality holding iff $\mathbf{p} \in C^\circ$.*

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[96] See the equations for the three objects, given below. One might strictly refer to the "cell interior" as the *relative interior* of the cell.

[97] We follow the convention of Mikhalkin (2004) in *not* restricting α in 1(ii) to be rational for rationality of the complex. This is because our u takes values in $\mathbb{R}$. One can alternatively specify $u : A \to \mathbb{Q}$ and $\alpha \in \mathbb{Q}$ to obtain an analogous version of Theorem 2.3–see Maclagan and Sturmfels (2015, Definition 2.3.2 and Proposition 3.1.1), in whose terms our polyhedral complexes are $\mathbb{R}$-rational.

In particular, if C is a 0-cell then $C^\circ = C$.

Proof of Lemma A.3. Identify the set of bundles $D_u(\mathbf{p}^\circ)$. As seen above, the cell interior prices are identified by a collection of strict inequalities, stating that each of the bundles which are in $D_u(\mathbf{p}^\circ)$ is strictly preferred to each of those that are not, together with a set of equalities, stating that the agent is indifferent between the bundles in $D_u(\mathbf{p}^\circ)$. The cell itself, being the closure of the "cell interior", is defined algebraically by replacing all the strict inequalities by their weak counterpart. But this now identifies the set of prices at which the agent is indifferent between the bundles in $D_u(\mathbf{p}^\circ)$, and *weakly* prefers them to all others. That is, for a price $\mathbf{p} \in C$, the demand set contains these bundles but may also contain others.

The equality when $\mathbf{p} \in C^\circ$ follows immediately from Definition A.1(1). $\square$

For completeness we re-iterate here Definition 2.1 from the text (see there for further discussion of the interpretation of weights):

Definition A.4 (Mikhalkin, 2004, Example 2)**.** The *tropical hypersurface* $\mathcal{T}_u$ associated with any valuation u is the weighted rational polyhedral complex such that:

1. its underlying set is $\{\mathbf{p} \in \mathbb{R}^n | \ \#D_u(\mathbf{p}) > 1\}$;
2. the *weight* $w_u(F)$ of the facet F is the integer defined by $w_u(F)\mathbf{v}_F = \mathbf{x}' - \mathbf{x}$, in which $\mathbf{x}'$ is demanded in the UDR on one side of F; $\mathbf{x}$ is demanded in the UDR on the other side; and $\mathbf{v}_F$ is the primitive integer normal vector pointing from the former to the latter.

A.1.2 Concavity of valuation functions: Proofs for Section 2.3

Proof of Lemma 2.5. Suppose that A is discrete-convex. It is a standard application of the supporting hyperplane theorem that a function $u : A \to \mathbb{R}$ is concave iff, for all $\mathbf{x} \in A$, there exists $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} \in D_u(\mathbf{p})$. As we have defined concavity to require discrete-convexity of domain, this provides the first equivalence. Moreover, since the intersection between a supporting hyperplane and a convex set is always itself convex, discrete-convexity of $D_u(\mathbf{p})$ also follows.

Conversely, suppose $D_u(\mathbf{p})$ is discrete-convex for all $\mathbf{p}$. Let $u' : A' \to \mathbb{R}$ be the minimal weakly-concave function everywhere weakly greater than u, where A' is the minimal discrete-convex set containing A. Consider any $\mathbf{x} \in A'$. By the previous equivalence, there exists $\mathbf{p}$ such that $\mathbf{x} \in D_{u'}(\mathbf{p})$. As the minimal weakly-concave function on $\mathbb{R}^n$ extending u and u' on $\mathbb{R}^n$ must coincide, it follows that $\operatorname{Conv} D_u(\mathbf{p}) = \operatorname{Conv}_{\mathbb{R}} D_{u'}(\mathbf{p})$. But by assumption it follows that $\mathbf{x} \in D_u(\mathbf{p})$. So the second property holds (and in particular A is discrete-convex). $\square$

Proof of Lemma 2.6. First see that, for any bundle $\mathbf{x}$, we have $D_u(\mathbf{p}) = \{\mathbf{x}\}$ iff $D_{u'}(\mathbf{p}) = \{\mathbf{x}\}$, by minimality of u'. So $D_u(\mathbf{p})$ is single valued iff $D_{u'}(\mathbf{p})$ is, and at such points the demand sets coincide. Hence both the underlying sets and the weights of the THs coincide. $\square$

A.1.3 Duality: proofs for Section 2.4

Here we build up the key results for duality in more detail.

Given $u : A \to \mathbb{R}$ with finite $A \subsetneq \mathbb{Z}^n$, we let $f_u : \operatorname{Conv}_{\mathbb{R}} A \to \mathbb{R}$ be the minimal weakly-concave function on $\operatorname{Conv} A$ which is everywhere weakly greater than u. Recall that we defined the "roof" of u to be the graph of f_u.

Lemma A.5. *The "roof" is a polyhedral complex.*

Proof. It is clear that the roof is the upper (with respect to the final coordinate) boundary of the convex hull of the points $(\mathbf{x}, u(\mathbf{x}))$. It is standard (see e.g. Grünbaum and Shephard, 1969) that this has the structure of a polyhedral complex. $\square$

We also now formally define the SNP:

Definition A.6. The *subdivided Newton Polytope (SNP)* associated to a valuation $u : A \to \mathbb{R}$ is the set $\operatorname{Conv} A$ with the structure of a rational polyhedral complex whose cells are the projections to the first n coordinates of the cells of the roof.

Again, the *dimension* of an SNP cell is the dimension of its affine span, and we refer to k-cells of the SNP in the same way as the TH. However, a TH only has cells in dimensions 0 to $n-1$, whereas an SNP may have cells in dimensions 0 to n.

To understand the SNP further we first show:

Lemma A.7. *For every* $\mathbf{p} \in \mathbb{R}^n$*, we have* $\operatorname{Conv} D_u(\mathbf{p}) = D_{f_u}(\mathbf{p})$*. Moreover,* $D_u(\mathbf{p})$ *is the intersection of* $D_{f_u}(\mathbf{p})$ *with those* $\mathbf{x} \in A$ *such that* $u(\mathbf{x}) = f_u(\mathbf{x})$*.*

Proof. We assume f_u is weakly concave, so the set bounded above by its graph is convex. By the supporting hyperplane theorem, for every $\mathbf{x} \in \operatorname{Conv} A$ there exists a supporting hyperplane to this graph at $\mathbf{x}$. That $\operatorname{Conv} D_u(\mathbf{p}) = D_{f_u}(\mathbf{p})$ now follows by minimality of f_u. Clearly, this supporting hyperplane also passes through $(\mathbf{x}, u(\mathbf{x}))$ if and only if $u(\mathbf{x}) = f_u(\mathbf{x})$. $\square$

We can now see clearly the economic meaning of the SNP, giving an alternative route to defining it:

Corollary A.8. *A subset* $\sigma \subseteq \operatorname{Conv} A$ *is a cell of the SNP iff it has the form* $\operatorname{Conv} D_u(\mathbf{p})$ *for some* $\mathbf{p} \in \mathbb{R}^n$*.*

Proof. Follows from Lemma A.7. $\square$

As in Corollary A.8, we will always use Greek letters to refer to the cells of an SNP, to distinguish them from the cells of a TH.

Now we show, if a bundle is demanded for *any* price, then whenever it is in the convex hull of a demand set, it is actually demanded at that price:

Lemma A.9. *If* $\mathbf{x} \in \operatorname{Conv} D_u(\mathbf{p})$ *and* $\mathbf{x} \in D_u(\mathbf{p}')$ *for some* $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n$*, then* $\mathbf{x} \in D_u(\mathbf{p})$*.*

Proof. Since $\mathbf{x} \in D_u(\mathbf{p}')$, by Lemma A.7, we know $u(\mathbf{x}) = f_u(\mathbf{x})$. But since also $\mathbf{x} \in \operatorname{Conv} D_u(\mathbf{p}) = D_{f_u}(\mathbf{p})$, we conclude from A.7 again that $\mathbf{x} \in D_u(\mathbf{p})$. $\square$

The duality between the SNP and the TH is stated in full as follows:

Proposition A.10 (this extends Lemma 2.8 from the body text)**.** *There is a bijective correspondence between cells σ of the SNP and the set encompassing both all cells and all unique demand regions of the TH, C_σ, such that:*

(1) $\sigma = \operatorname{Conv} D_u(\mathbf{p})$ *for all* $\mathbf{p} \in C_\sigma^\circ$*;*
(2) $C_\sigma = \{\mathbf{p} \in \mathbb{R}^n \mid \sigma \subseteq \operatorname{Conv} D_u(\mathbf{p})\}$*;*
(3) inclusions reverse: $\sigma \subsetneq \sigma' \Leftrightarrow C_{\sigma'} \subsetneq C_\sigma$*;*
(4) dimensions are dual: $\dim \sigma + \dim C_\sigma = n$*;*
(5) cells are orthogonal: $(\mathbf{p}' - \mathbf{p}).(\mathbf{x}' - \mathbf{x}) = 0$ *for all* $\mathbf{p}, \mathbf{p}' \in C_\sigma$*,* $\mathbf{x}, \mathbf{x}' \in \sigma$*.*

Proof. By Definition A.1(1) the demand set is constant in a cell interior, and by Corollary A.8 every SNP cell σ can be associated to some price $\mathbf{p}$ such that $\sigma = \operatorname{Conv} D_u(\mathbf{p})$. So (1) gives a well-defined correspondence between an SNP cell σ on the one hand, and a set C_σ° which is either a cell interior or a unique demand region. Next, recall from Lemma A.3 that a price $\mathbf{p}$ is in the cell C_σ iff $D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})$, where $\mathbf{p}^\circ$ is some representative element of C_σ°. Moreover, the latter holds iff $\sigma = \operatorname{Conv} D_u(\mathbf{p}^\circ) \subseteq \operatorname{Conv} D_u(\mathbf{p})$. Necessity is obvious, and sufficiency follows from Lemma A.9: if we assume $\operatorname{Conv} D_u(\mathbf{p}^\circ) \subseteq \operatorname{Conv} D_u(\mathbf{p})$, then any $\mathbf{x} \in D_u(\mathbf{p}^\circ) \subseteq \operatorname{Conv} D_u(\mathbf{p})$ must satisfy $\mathbf{x} \in D_u(\mathbf{p})$. So $\mathbf{p} \in C_\sigma$ iff $\sigma \subset \operatorname{Conv} D_u(\mathbf{p})$, i.e. (2) holds. Now (3) follows from the combination of (1) and (2).

For (4) note that the affine span of C_σ is given by the set of prices $\mathbf{p}'$ such that $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, i.e. all prices such that $\mathbf{p}'.(\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all such $\mathbf{x}, \mathbf{x}'$. If $\sigma = \operatorname{Conv} D_u(\mathbf{p})$ is k-dimensional, these equations impose k linearly independent constraints on such $\mathbf{p}'$, so $\dim C_\sigma = n - k$.

Moreover, it follows now that $(\mathbf{p}'' - \mathbf{p}').(\mathbf{x}' - \mathbf{x}) = 0$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$ and all $\mathbf{p}', \mathbf{p}'' \in C_\sigma$. Thus this equality holds for any $\mathbf{x}, \mathbf{x}' \in \operatorname{Conv} D_u(\mathbf{p}) = \sigma$, proving (5). ☐

Note that $\operatorname{Conv} D_u(\mathbf{p}) \neq \sigma$ for $\mathbf{p}$ which are in the boundary of C_σ but not in its interior. Lemma 2.9 now follows:

Proof of Lemma 2.9. Let σ be an SNP cell, and let the corresponding TH cell be C_σ. From Proposition A.10(1) and (2), we know $\mathbf{x} \in \sigma$ holds iff $\mathbf{x} \in \operatorname{Conv} D_u(\mathbf{p})$ for all $\mathbf{p} \in C_\sigma$. Now, by Lemma A.9, either $\mathbf{x}$ is demanded for *no* price, or the latter holds iff $\mathbf{x} \in D_u(\mathbf{p})$ for all $\mathbf{p} \in C_\sigma$. ☐

In Section 5 we will be particularly interested in 0-cells of the TH. We will show there that we can check for certain properties by only studying those isolated points. We will need to know, as is now simple:

Corollary A.11. *Given* $u : A \to \mathbb{R}$*, if* $\operatorname{Conv} A$ *is* n*-dimensional, then every* k*-cell* C_σ *of* $\mathcal{T}_u$ *has some* 0*-cell* C_τ *in its boundary, with* $\sigma \subseteq \tau$*. Moreover if* $\mathbf{x} \in \sigma$ *but* $\mathbf{x} \notin D_u(\mathbf{p}_\sigma^\circ)$ *for* $\mathbf{p}_\sigma^\circ \in C_\sigma^\circ$*, then also* $\mathbf{x} \notin D_u(\mathbf{p}_\tau^\circ)$ *for* $\mathbf{p}_\tau^\circ \in C_\tau^\circ$*.*

Proof. This is easy to see using the SNP: σ is an $(n-k)$-cell, and by assumption the SNP itself is n-dimensional, which means that σ is contained in an n-cell τ of the SNP. So there exists a 0-cell C_τ of the TH with $\sigma \subseteq \tau$ by construction and with $C_\tau \subseteq C_\sigma$ by Proposition A.10(3). That such $\mathbf{x} \notin D_u(\mathbf{p}_\tau^\circ)$ follows from Lemma A.9. ☐

A.1.4 Examples for Section 2.5

Example A.12. For a fixed A, it is easy to draw every possible SNP and so obtain every possible combinatorial type of TH, thus enumerating all possible "essentially-different" structures of demand. We do this for $A = \{0, 1\}^2$ in Fig. 9.

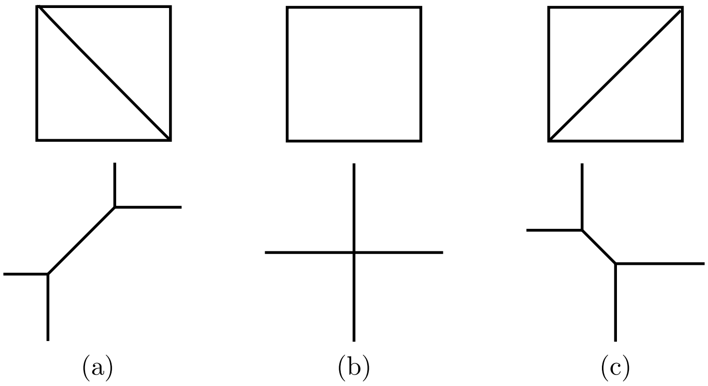

Figure 9: All the possible SNPs, and examples of their corresponding combinatorial types of TH when $A = \{0, 1\}^2$.

It is not hard to see that Fig. 9a applies when $u(0, 0) + u(1, 1) < u(1, 0) + u(0, 1)$, so represents substitutes; Fig. 9b applies when $u(0, 0) + u(1, 1) = u(1, 0) + u(0, 1)$, so is additively separable demand; and Fig. 9c applies when $u(0, 0)+u(1, 1) > u(1, 0)+u(0, 1)$, so is complements. (See Section 3.2 for these distinctions). Importantly, it is clear that these are the *only* possibilities.

Observe that Fig. 9b can be seen as a limit of Fig. 9a (or, equivalently, Fig. 9c). In the TH, the two 0-cells become arbitrarily close and then coincide in the limit; in quantity space, the faces of the "roof" tilt until they are coplanar, meaning that the SNP edge distinguishing them disappears.

Likewise, any SNP in which the subdivision is not maximal (that is, additional valid $(n-1)$-faces could be added) can be recovered by deleting $(n-1)$-faces from some SNP whose subdivision *is* maximal; the corresponding TH is a limit (or "degeneration"). Even for larger domains than $A = \{0, 1\}^2$, we can go on to enumerate all those combinatorial types of demand for which the SNP subdivision is maximal, knowing we can recover the remainder as their limits, as in the following example.

Example A.13. For $A = \{0, 1, 2\} \times \{0, 1\}$, we list the maximal subdivisions which correspond to THs in Fig. 10.

A.2 Proofs for Section 3: demand types

A.2.1 Proofs for Section 3.2: comparative statics

Proof of Proposition 3.3. If a valuation u is not of such a demand type, it must have a facet F with normal $\mathbf{v}$ where $v_i, v_j < 0$ for some $i \neq j$. Then $\mathbf{e}^i.\mathbf{v} \neq 0$, i.e., this coordinate vector is not parallel to the facet. So we may choose UDR prices $\mathbf{p}, \mathbf{p}' = \mathbf{p}+\epsilon\mathbf{e}^i$ which lie on either side of the facet. We know demand change from $\mathbf{p}$ to $\mathbf{p}'$ is an integer multiple

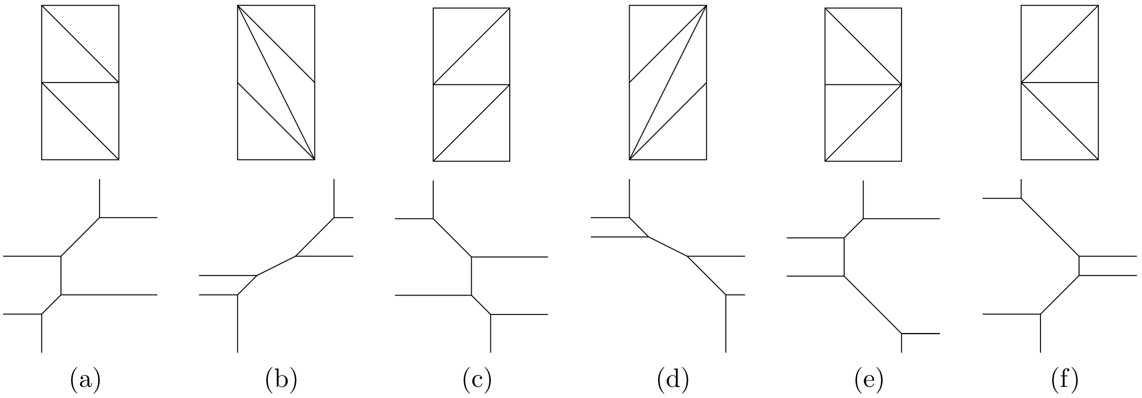

Figure 10: All the possible SNPs with maximal subdivision, and examples of their corresponding combinatorial types of TH, when $A = \{0,1\} \times \{0,1,2\}$.

w of $\mathbf{v}$. The price for good i has gone up, so by the law of demand, demand for good i must have gone down: $w > 0$. Hence demand for good j also decreases: goods i and j are not substitutes.

Conversely, suppose the valuation *is* of such a demand type. Choose prices $\mathbf{p}' \geq \mathbf{p}$ which both lie in unique demand regions. The straight line $[\mathbf{p}, \mathbf{p}']$ from the first price to the second need not cross only facets, but because the UDRs are open we can choose a small translation vector $\mathbf{w}$ such that $\mathbf{p} + \mathbf{w}$, and $\mathbf{p}' + \mathbf{w}$ are both respectively in the same unique demand regions as $\mathbf{p}, \mathbf{p}'$ and such that $[\mathbf{p} + \mathbf{w}, \mathbf{p}' + \mathbf{w}]$ *does* only cross facets. Let $\mathbf{x}^0, \ldots, \mathbf{x}^l$ be demanded in each UDR that this line meets (so in particular $\{\mathbf{x}^0\} = D_u(\mathbf{p})$ and $\{\mathbf{x}^l\} = D_u(\mathbf{p}'))$. Now, in every case, $\mathbf{x}^i - \mathbf{x}^{i-1}$ is an integer multiple of one of our allowed facet normals, and so has at most one positive and at most one negative coordinate entry. By the law of demand, demand must weakly decrease at each step for any good whose price is increasing, and hence demand must weakly increase at each step for any other good. As the overall change in demand is just the composition of such changes, this holds for the change in demand from $\mathbf{x}^0$ to $\mathbf{x}^l$. □

Proof of Proposition 3.5 This proof is completely analogous to that of Proposition 3.3. In the first step, we need only suppose that there exists a facet whose normal has one positive and one negative coordinate entry, and find that these goods are not complements. In the second step, we see that because demand weakly decreases at each step for any good whose price is increasing, it must *also* weakly decrease for all other goods (since facet normals have the same sign). □

The pattern of proof for these two propositions signals that there is potential for generalisation here. In Baldwin and Klemperer (2014 and in preparation-b) we develop these ideas much more broadly, defining "$\mathcal{D}$-steps" and showing how they greatly facilitate study of the comparative statics of demand.

A.2.2 Proofs for Section 3.3: aggregate demand

Suppose we have agents $j = 1, \ldots, m$ with valuations $u^j : A_j \to \mathbb{R}^n$. Define their *aggregate domain* to be $A := \sum_{j=1}^m A_j$, that is, $\mathbf{y} \in A$ iff $\mathbf{y} = \sum_{j=1}^m \mathbf{x}^j$ where $\mathbf{x}^j \in A_j$. Define their *aggregate valuation* to be $U(\mathbf{y}) := \max\left\{\sum_{j=1}^m u^j(\mathbf{x}^j) \mid \mathbf{x}^j \in A_j, \sum_{j=1}^m \mathbf{x}^j = \mathbf{y}\right\}$.

Proposition A.14. *With agents and aggregate demand as above,*

(1) $D_U(\mathbf{p}) = \sum_{j=1}^m D_{u^j}(\mathbf{p})$ *for all* $\mathbf{p} \in \mathbb{R}^n$.
(2) $\mathcal{T}_U$ *has underlying set equal to* $\bigcup_{j=1}^m \mathcal{T}_{u^j}$, *and the weight on any facet of* $\mathcal{T}_U$ *is equal to the sum of weights of individual TH facets that contain it.*

Proof. (1) For any $\mathbf{p} \in \mathbb{R}^n$, note that

$$\sum_{j=1}^m D_{u^j}(\mathbf{p}) = \sum_{j=1}^m \max_{\mathbf{x}^j \in A}\{u^j(\mathbf{x}^j) - \mathbf{p}.\mathbf{x}^j\} = \max\left\{\sum_{j=1}^m u^j(\mathbf{x}^j) - \mathbf{p}.\left(\sum_{j=1}^m \mathbf{x}^j\right) \mid \mathbf{x}^j \in A_j\right\},$$

and on the other hand, by definition of A, that

$$\begin{aligned}
&\max_{\mathbf{y}\in A}\{U(\mathbf{y}) - \mathbf{p}.\mathbf{y}\} \\
&= \max\left\{\max\left\{\sum_{j=1}^m u^j(\mathbf{x}^j) \mid \mathbf{x}^j \in A_j, \sum_{j=1}^m \mathbf{x}^j = \mathbf{y}\right\} - \mathbf{p}.\mathbf{y} \mid \mathbf{y} = \sum_{j=1}^m \mathbf{x}^j, \mathbf{x}^j \in A_j\right\} \\
&= \max\left\{\sum_{j=1}^m u^j(\mathbf{x}^j) - \mathbf{p}.\left(\sum_{j=1}^m \mathbf{x}^j\right) \mid \mathbf{x}^j \in A_j\right\},
\end{aligned}$$

and that the same arguments $\mathbf{x}^j \in A$, with $\mathbf{y} = \sum_{j=1}^m \mathbf{x}^j$, are maximising in either case.

(2) By (1), $D_U(\mathbf{p})$ is single-valued iff $\sum_{j=1}^m D_{u^j}(\mathbf{p})$ is single-valued, and hence iff $D_{u^j}(\mathbf{p})$ is single-valued for all j. Thus the underlying sets given coincide. Suppose F is a facet of $\mathcal{T}_U$ with adjacent UDRs R and R'; let $\mathbf{v}_F$ be a primitive integer vector pointing from R to R'. For $j = 1, \ldots, m$, write the demand of agent j in R as $\mathbf{x}^j$ and in R' as $\mathbf{x}^{j\prime}$ (for some agent these will be distinct, but not necessarily for all). Then $w_{u^j}(F)\mathbf{v}_F = \mathbf{x}^{j\prime} - \mathbf{x}^j$ for all j, and so

$$\sum_j w_{u^j}(F)\mathbf{v}_F = \sum_j \mathbf{x}^{j\prime} - \sum_j \mathbf{x}^j = w_U(F)\mathbf{v}_F,$$

as required. ☐

A.3 Proofs for Section 4: Equilibrium

A.3.1 Equivalent characterisations of unimodularity

Remark A.15. The following are equivalent, for a set of s linearly independent vectors in $\mathbb{Z}^n$:

(1) they are an integer basis for the subspace they span;[98]
(2) an s-dimensional parallelepiped in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$ and these vectors as edges contains no point in $\mathbb{Z}^n$ except its vertices;

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[98] This is made completely precise in Fact A.40 below.

(3) they can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ±1;
(4) among the determinants of all the $s \times s$ matrices consisting of s rows of the $n \times s$ matrix whose columns are these s vectors, the greatest common factor is 1.[99]

Proofs of these facts may be found in Cassels (1971).[100] We refer to a set of vectors in $\mathbb{Z}^n$ as unimodular if every linearly independent subset has these properties.

A.3.2 Proof of Theorem 4.2

As described in Section 4.2, we first prove necessity and sufficiency for transverse intersections, and then show that the general case follows. We start with necessity:

Proposition A.16. *Consider $s \leq n$ agents each of whose demand set includes precisely 2 bundles at price $\mathbf{p}$, i.e., $\#D_{u^i}(\mathbf{p}) = 2$, for $i = 1, \ldots, s$. Write $\mathbf{v}^i$ for the difference between the two bundles demanded by agent i (so $\mathbf{v}^i$ is normal to i's facet of demand at $\mathbf{p}$). Suppose the s vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s$ are linearly independent. Write U for the aggregate valuation. There exists an integer bundle in* Conv $D_U(\mathbf{p})$ *which is not demanded at any price iff vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s$ do not form a unimodular set.*

Proof. By Lemma 2.9, an integer bundle in Conv $D_U(\mathbf{p})$ is not demanded at any price iff it is not in $D_U(\mathbf{p})$. Now, each individual agent i's demand at $\mathbf{p}$ has the form $D_{u^i}(\mathbf{p}) = \{\mathbf{y}^i + \delta_i \mathbf{v}^i \mid \delta_i \in \{0, 1\}\}$, where $\mathbf{y}^i$ is the bundle demanded on the appropriate side of the TH facet. So the set of bundles demanded on aggregate at $\mathbf{p}$ is

$$D_U(\mathbf{p}) = \left\{\mathbf{y} + \delta_1 \mathbf{v}^1 + \cdots + \delta_s \mathbf{v}^s \mid \delta_i \in \{0, 1\}; i = 1, \ldots, s\right\},$$

where $\mathbf{y} = \sum_i \mathbf{y}^i$. These points are precisely the vertices of an s-dimensional parallelepiped in $\mathbb{Z}^n$ (since its edges, the $\mathbf{v}^i$, are linearly independent). There exists an integer bundle in Conv $D_U(\mathbf{p})$ which is not in $D_U(\mathbf{p})$ iff this parallelepiped contains an integer bundle which is not a vertex, and, by Remark A.15(2) and (3), this holds iff the set $\{\mathbf{v}^1, \ldots, \mathbf{v}^s\}$ is not unimodular. ☐

Next, sufficiency:

Proposition A.17. *Suppose price $\mathbf{p}$ is in the interior of an $(n - k_i)$-cell C^i of the TH $\mathcal{T}_{u^i}$ of each of s agents $i = 1, \ldots, s$, who have concave valuations u^i, and together have aggregate valuation U. Then every integer bundle in* Conv $D_U(\mathbf{p})$ *is demanded at $\mathbf{p}$ if each C^i is a subset of the intersection of a set of facets $F^i_1, \ldots, F^i_{k_i}$ of $\mathcal{T}_{u^i}$ (not necessarily comprising all facets of $\mathcal{T}_{u^i}$ that pass through C^i) with primitive integer normal vectors $\mathbf{v}^i_1, \ldots, \mathbf{v}^i_{k_i}$ and $\{\mathbf{v}^i_j \mid i = 1, \ldots, s; j = 1, \ldots, k_i\}$ are unimodular.*

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[99] This fact is especially helpful when developing examples.

[100] (1)⇔ (3) follows from Cassels (1971) Lemma I.1 and Corollary I.3. (1)⇔(4) is Cassels (1971) Lemma I.2. For (1)⇔(2) consider a parallelepiped P whose vertices are $\mathbf{y} + \sum_{i=1}^{s} a_i \mathbf{w}^i$ for $a_i \in \{0, 1\}$. If $\mathbf{z}$ is a non-vertex integer point in P, then $\mathbf{z} - \mathbf{y}$ exhibits the failure of (1). Conversely, if failure of (1) is exhibited by an integer $\sum_{i=1}^{s} b_i \mathbf{w}^i$ where b_i are not all integers, then $\mathbf{y} + \sum_{i=1}^{s} a_i \mathbf{w}^i$ exhibits failure of (2), where a_i is the non-integer part of b_i in each case.

Proof. All bundles demanded by agent i at $\mathbf{p}$ are demanded throughout the $(n-k_i)$-cell C^i, which corresponds to a k_i-dimensional polytope σ_i in the SNP of agent i. Moreover, σ_i possesses an edge in direction $\mathbf{v}^i_j$ for $j = 1, \dots, k_i$; each corresponds to the facet F^i_j. Thus, if $\mathbf{y}^i$ is some integer bundle in $D_{u^i}(\mathbf{p})$, then (by a dimension count) the affine span of σ_i is precisely $\left\{\mathbf{y}^i + \sum_{j=1}^{k_i} \beta^i_j \mathbf{v}^i_j \mid \beta^i_j \in \mathbb{R} \text{ for } j = 1, \dots, k_i\right\}$, and in particular, $D_{u^i}(\mathbf{p})$ is contained in this set.

Thus, since aggregate demand is the Minkowski sum of individual demand, we may express aggregate demand among these agents as $D_U(\mathbf{p}) = \left\{\mathbf{y} + \sum_{i=1}^{s}\sum_{j=1}^{k_i} a^i_j \mathbf{v}^i_j \mid \mathbf{y}^i + \sum_{j=1}^{k_i} a^i_j \mathbf{v}^i_j \in D_{u^i}(\mathbf{p}) \text{ for } i = 1, \dots, s\right\}$, where $\mathbf{y} := \sum_{i=1}^{s} \mathbf{y}^i$.

Now, suppose $\mathbf{x}$ is an integer bundle in $\operatorname{Conv} D_U(\mathbf{p})$. Then $\mathbf{x} - \mathbf{y}$ is in the span of the $\mathbf{v}^i_j$. But since they are an integer basis for their span, we can write $\mathbf{x} - \mathbf{y} = \sum_{i=1}^{s}\sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j$, for some $b^i_j \in \mathbb{Z}$. So we can define $\mathbf{x}^i := \mathbf{y}^i + \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j$, and know that $\mathbf{x}^i \in \mathbb{Z}^n$.

But we also know $\mathbf{x}^i \in \operatorname{Conv} D_{u^i}(\mathbf{p})$. To see this, observe that since $\mathbf{x} \in \operatorname{Conv} D_U(\mathbf{p})$, we can write $\mathbf{x} - \mathbf{y} = \sum_\beta \sum_{i=1}^{s}\sum_{j=1}^{k_i} \lambda_\beta a^i_{j,\beta} \mathbf{v}^i_j$ for some finite set of weights $\lambda_\beta \in [0,1]$ such that $\sum_\beta \lambda_\beta = 1$ and such that $\mathbf{y}^i + \sum_{j=1}^{k_i} a^i_{j,\beta} \mathbf{v}^i_j \in D_{u^i}(\mathbf{p})$ for each agent i and for each β. But since the $\mathbf{v}^i_j$ are linearly independent, there is an unique way to write $\mathbf{x} - \mathbf{y}$ as a weighted sum of the $\mathbf{v}^i_j$, so $b^i_j = \sum_\beta \lambda_\beta a^i_{j,\beta}$, and so $\mathbf{x}^i = \mathbf{y}^i + \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j = \mathbf{y}^i + \sum_{j=1}^{k_i} \sum_\beta \lambda_\beta a^i_{j,\beta} \mathbf{v}^i_j \in \operatorname{Conv} D_{u^i}(\mathbf{p})$.

So $\mathbf{x}^i$ is an integer vector in $\operatorname{Conv} D_{u^i}(\mathbf{p})$. By concavity of u^i there exists *some* price at which $\mathbf{x}^i$ is demanded by agent i (Lemma 2.5), and so by Lemma 2.9 we know $\mathbf{x}^i \in D_{u^i}(\mathbf{p})$. Thus $\mathbf{x} = \sum_{i=1}^{s} \mathbf{x}^i \in D_U(\mathbf{p})$. That is, $\mathbf{x}$ is demanded at $\mathbf{p}$, as required. ☐

Finally we deal with the non-transverse case. Start by recalling Proposition 4.6: given valuations u^1 and u^2, for generic $\mathbf{v} \in \mathbb{R}^n$ and small enough ϵ, the intersection $(\mathcal{T}_{u^1}) \cap (\epsilon\mathbf{v} + \mathcal{T}_{u^2})$ is transverse.

Strictly speaking, we should have noted that $\epsilon\mathbf{v} + \mathcal{T}_{u^2}$ is also a TH to state this result, although it is obviously a balanced weighted rational polyhedral complex of dimension $n-1$, so we can apply Theorem 2.3. However, in the following we will need to know explicitly the corresponding valuation:

Lemma A.18. *Let $u : A \to \mathbb{R}$ be a valuation and let $\epsilon > 0$ and $\mathbf{v} \in \mathbb{R}^n$. Define $u_\epsilon : A \to \mathbb{R}$ by $u_\epsilon(\mathbf{x}) = u(\mathbf{x}) + \epsilon\mathbf{v}.\mathbf{x}$. Then:*

(1) $D_{u_\epsilon}(\mathbf{p}) = D_u(\mathbf{p} - \epsilon\mathbf{v})$ for all $\mathbf{p} \in \mathbb{R}^n$;
(2) $\mathcal{T}_{u_\epsilon} = \epsilon\mathbf{v} + \mathcal{T}_u$;
(3) $|u_\epsilon(\mathbf{x}) - u(\mathbf{x})| \le R\epsilon\|\mathbf{v}\|$, where R satisfies $R > \|\mathbf{x}\|$ for all $\mathbf{x} \in A$.

Proof. First see

$$D_{u_\epsilon}(\mathbf{p}) = \arg\max_{\mathbf{x} \in A}\{u(\mathbf{x}) + \epsilon\mathbf{x}.\mathbf{v} - \mathbf{x}.\mathbf{p}\} = \arg\max_{\mathbf{x} \in A}\{u(\mathbf{x}) - \mathbf{x}.(\mathbf{p} - \epsilon\mathbf{v})\} = D_u(\mathbf{p} - \epsilon\mathbf{v}).$$

The remainder of the lemma follows by definition of $\mathcal{T}_u$, and the Cauchy-Schwarz inequality. ☐

Now we relate this material to the question of competitive equilibrium:

Lemma A.19. *Suppose we have agents* $1, 2$ *with valuations* u^1, u^2 *and supply* $\mathbf{x} \in A_1 + A_2$ *for which competitive equilibrium does not exist: there does not exist* $\mathbf{p}$ *such that* $\mathbf{x} \in D_U(\mathbf{p})$*. Then for any* $\mathbf{v} \in \mathbb{R}^n$*, competitive equilibrium also fails when the agent's valuations are* u^1 *and* u^2_ϵ*, for all sufficiently small* ϵ*.*

Proof. Let $\mathbf{p}$ be a price such that $\mathbf{x} \in \operatorname{Conv} D_U(\mathbf{p})$, $\mathbf{x} \notin D_U(\mathbf{p})$. (Such a price exists since the "SNP" subdivision subdivides the whole of $\operatorname{Conv} A$).

Use indices β to label the elements of $D_U(\mathbf{p})$. By assumption there exist $\lambda_\beta \in [0, 1]$ with $\sum_\beta \lambda_\beta = 1$ such that $\mathbf{x} = \sum_\beta \lambda_\beta \mathbf{y}^\beta$. Following the logic of Section 2.4 it must be that $U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta)$. So pick $\eta > 0$ such that

$$U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \eta. \tag{1}$$

Now, fix any $\mathbf{v} \in \mathbb{R}^n$. Let $\epsilon > 0$ be any real satisfying $\epsilon \leq \frac{\eta}{2R\|\mathbf{v}\|}$ where $R > \|\mathbf{x}\|$ for all $\mathbf{x} \in A_2$, the domain of u^2. Let U_ϵ be the aggregate valuation of u^1 and u^2_ϵ. Now, there exist $\mathbf{x}^1 \in A_1$, $\mathbf{x}^2 \in A_2$ such that

$$U_\epsilon(\mathbf{x}) = u^1(\mathbf{x}^1) + u^2_\epsilon(\mathbf{x}^2) = u^1(\mathbf{x}^1) + u^2(\mathbf{x}^2) + \epsilon \mathbf{v}.\mathbf{x}^2 \leq U(\mathbf{x}) + \epsilon \mathbf{v}.\mathbf{x}^2 \leq U(\mathbf{x}) + \frac{1}{2}\eta$$

by Lemma A.18 and definition of ϵ. Since we can apply Lemma A.18 again, reversing the roles of u^2 and u^2_ϵ (and reversing the sign of $\mathbf{v}$), it follows that $|U_\epsilon(\mathbf{x}) - U(\mathbf{x})| \leq \frac{1}{2}\eta$. Moreover, we can apply the same argument for every $\mathbf{y}^\beta \in D_U(\mathbf{p})$, so $|U_\epsilon(\mathbf{y}^\beta) - U(\mathbf{y}^\beta)| \leq \frac{1}{2}\eta$ for all β. It follows from Equation (1) that $U_\epsilon(\mathbf{x}) < \sum_\beta \lambda_\beta U_\epsilon(\mathbf{y}^\beta)$. But recall that $\mathbf{x} = \sum_\beta \lambda_\beta \mathbf{y}^\beta$, so this illustrates the failure of U_ϵ to be concave at $\mathbf{x}$, and, by Lemma 2.5, the impossibility of $\mathbf{x}$ being aggregate demand when the aggregate valuation is U_ϵ, which completes the proof. □

Proposition A.20. *If competitive equilibrium fails for a set of agents, then there exists a small perturbation of their valuations such that equilibrium still fails and such that all intersections are transverse.*

Proof. This follows from repeated application of Proposition 4.6 and Lemma A.19. □

Thus, to analyse when competitive equilibrium exists for all agents of a given type, we need only be concerned with those whose THs meet transversally wherever they intersect. Hence:

Proof of Theorem 4.2. For necessity of unimodularity of the demand type we use Proposition A.16: if $\mathcal{D}$ is not unimodular, we can (Remark A.15) choose a linearly independent subset of $\mathcal{D}$ which is also not unimodular, and then apply Proposition A.16 using this set of vectors. Sufficiency is given by Proposition A.17 and the contrapositive of Proposition A.20. □

A.3.3 Proofs for Section 4.3.2: Basis changes

Proposition A.21 (cf. Gorman, 1976, p. 219-20)**.** *For* $A \subsetneq \mathbb{Z}^n$ *and* $u : A \to \mathbb{R}$ *and a unimodular* $n \times n$ *matrix* G*, define the ("pullback") basis change of* u *by* G *to be* $G^*u : G^{-1}A \to \mathbb{R}$ *via* $G^*u(\mathbf{y}) := u(G\mathbf{y})$*. Then*

1. G^*u *is concave iff* u *is concave.*
2. *A bundle is demanded under the original demand at a certain price iff an associated bundle is demanded under the transformed demand at an associated price; specifically:* $\mathbf{x} \in D_u(\mathbf{p}) \iff G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p})$*.*
3. *The TH of the transformed demand is given by a linear transformation of the original demand:* $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$*;*
4. *The inverse transformation to* G *applies to demand types:* $u(\cdot)$ *is of demand type* $\mathcal{D}$ *iff* $G^*u(\cdot)$ *is of demand type* $G^{-1}\mathcal{D} = \{G^{-1}\mathbf{v} \mid \mathbf{v} \in \mathcal{D}\}$*.*

Proof. (1) is trivial. (2): by definition, $\mathbf{x} \in D_u(\mathbf{p})$ if $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}' \in A$, with equality iff $\mathbf{x}' \in D_u(\mathbf{p})$ also. For any invertible matrix G, we may re-write

$$\mathbf{p}^T(\mathbf{x} - \mathbf{x}') = \mathbf{p}^T GG^{-1}(\mathbf{x} - \mathbf{x}') = (G^T\mathbf{p})^T(G^{-1}\mathbf{x} - G^{-1}\mathbf{x}').$$

If G is additionally unimodular, then $G^{-1}\mathbf{x}$ and $G^{-1}\mathbf{x}' \in \mathbb{Z}^n$. If we write $\mathbf{y} = G^{-1}\mathbf{x}$ and $\mathbf{y}' = G^{-1}\mathbf{x}'$ then $(G^T\mathbf{p})^T(\mathbf{y}-\mathbf{y}') \leq G^*u(\mathbf{y}) - G^*u(\mathbf{y}')$ holds iff $\mathbf{p}^T(\mathbf{x}-\mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$. So we have

$$\mathbf{x} \in D_u(\mathbf{p}) \;\Leftrightarrow\; \mathbf{y} = G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p}),$$

as required.

(3): since the underlying set of $\mathcal{T}_u$ is those $\mathbf{p}$ for which $\#D_u(\mathbf{p}) > 1$ it follows immediately from (2) that $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$, as required.

(4): suppose $\mathbf{v}$ is normal to a facet F of $\mathcal{T}_u$. It follows from (3) that the facet corresponding to F in $\mathcal{T}_{G^*u}$ has the form $G^TF = \{G^T\mathbf{p} \mid \mathbf{p} \in F\}$. We know $\mathbf{p}^T\mathbf{v}$ is constant for $\mathbf{p} \in F$, from which it follows that $(G^T\mathbf{p})^TG^{-1}\mathbf{v} = \mathbf{p}^TGG^{-1}\mathbf{v}$ is constant for $G^T\mathbf{p} \in G^TF$: we see $G^{-1}\mathbf{v}$ is normal to a facet of $\mathcal{T}_{G^*u}$. As G has an integer inverse, the converse is also true. □

Proof of Proposition 4.7. Suppose there is always an equilibrium for every finite set of agents with concave valuations of type $G^{-1}\mathcal{D}$, and any supply bundle in the domain of their aggregate valuation.

Let $u^1, \ldots, u^k$ be concave valuations of type $\mathcal{D}$ and let $\mathbf{x}$ be in the domain of their aggregate valuation. Then, by Proposition A.21(1) and (4), the valuations $G^*u^1, \ldots, G^*u^k$ are concave and have type $G^{-1}\mathcal{D}$. By definition of pullback, $\mathbf{y} := G^{-1}\mathbf{x}$ is in the domain of their aggregate valuation. By assumption competitive equilibrium exists in the latter case: there exists a price $\mathbf{p}$ at which the agent with valuation G^*u^i demands $\mathbf{y}^i$ and $\sum_i \mathbf{y}^i = \mathbf{y}$. But then in each case we may define $\mathbf{x}^i := G\mathbf{y}^i \in D_{u^i}(G^{-T}\mathbf{p})$ (see Proposition A.21(2)). So at price $G^{-T}\mathbf{p}$ the market clears for $\mathbf{x} := \sum_i \mathbf{x}^i$: we have a competitive equilibrium for our agents with concave valuations of type $\mathcal{D}$.

As G is invertible, the converse is shown similarly. □

A.4 Proofs for Section 5: Application of the tropical Bézout-Kouchnirenko-Bernshtein theorem

To provide proofs for the material in Section 5, we need to associate additional mathematical structure to the TH and SNP. In particular, it is possible to re-interpret our previous results on equilibrium (Propositions A.16 and A.17) in terms of subgroup indices of lattices associated with SNPs (see Proposition A.28 below). These subgroup indices are used in the definition of intersection multiplicity in tropical geometry, and thus allow us to apply results from tropical intersection theory to the question of competitive equilibrium.

In Appendices A.4.1 and A.4.2 we consider intersections of more than two THs, which is greater generality than we require. (We will not use this generality when applying Theorems 5.3, 5.5 and 5.6 to multiple agents; we will instead first consider the intersection of the first two agents, then the intersection of the third with the aggregation of the first two, etc.) It is easy to state the results in full generality, and the relationships between subgroup indices, TH intersections and equilibrium go beyond Theorems 5.3, 5.5 and 5.6. In particular Proposition A.28 provides the key link between economics and geometry, in this general case.

This appendix section is followed by Appendix A.5, which explains mathematical ideas that may be unfamiliar to some readers, on lattices, volumes and and subgroup indices.[101]

We use the following notation throughout this appendix:

Notation We suppose that the THs of r agents intersect at some price $\mathbf{p}$ (although there may be additional agents in the economy). Label these r agents as $1, \ldots, r$ and let $\mathcal{T}_U$ be the aggregate TH of $\mathcal{T}_{u^1}, \ldots, \mathcal{T}_{u^r}$. Label as C a cell of $\mathcal{T}_U$ in the intersection of these individual THs at $\mathbf{p}$. Label the smallest cells containing C of $\mathcal{T}_{u^j}$ as C^j, and let σ, σ^j be the respective corresponding SNP cells, for $j = 1, \ldots, r$. Set $d^j := \dim \sigma^j$ and $d := \dim \sigma$. Note that $d^j \geq 1$ for all j because C^j is a cell of the TH (see Proposition A.10(4)).

A.4.1 Parallel linear spaces for an SNP

To define the relevant lattices, we first provide definitions and results on their associated linear (vector) spaces.

Definition A.22. Given an SNP cell ρ, the *parallel linear space* L_ρ is the vector space parallel to the affine span of ρ: L_ρ consists of all linear combinations of the vectors $\{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in \rho\}$.

Clearly, the dimension of ρ is equal to the dimension of L_ρ.

Lemma A.23. *In the notation of Appendix A.4,* $\sigma = \sigma^1 + \cdots + \sigma^r$.

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[101] Note that our use of "lattice" is in its group-theoretic meaning; see Appendix A.5 below. "Lattices" are also used economics in their *order-theoretic* sense, particularly in work on comparative statics (see e.g. Milgrom and Shannon, 1994). We emphasise that mathematics has unfortunately used the same word with two *entirely* different meanings.

Proof. By definition, for any $\mathbf{p}$ in the relative interior of C (and hence of C^j for relevant j) we have $D_U(\mathbf{p}) = D_{u^1}(\mathbf{p}) + \cdots + D_{u^r}(\mathbf{p})$. So by the properties of polytopes[102] we know $\sigma = \sigma^1 + \cdots + \sigma^r$. ☐

Corollary A.24. *In the notation introduced at the beginning of Appendix A.4,* $L_\sigma = L_{\sigma^1} + \cdots + L_{\sigma^r}$.

Proof. Follows from Definition A.22 and Lemma A.23. ☐

Lemma A.25. *In the notation introduced at the beginning of Appendix A.4:*

(1) If $r = 2$, *the intersection of* $\mathcal{T}_{u^1}$ *and* $\mathcal{T}_{u^2}$ *is transverse at* C *iff* $L_{\sigma^1} \cap L_{\sigma^2} = \{0\}$.

(2) For any $r \geq 1$, *the intersection of* $\mathcal{T}_{u^1}, \ldots, \mathcal{T}_{u^r}$ *is* transverse *at* C *iff* $L_\sigma = L_{\sigma^1} \oplus \cdots \oplus L_{\sigma^r}$, *which holds iff* $d^1 + \cdots + d^r = d$.[103]

Proof. (1) By definition of transversality (Definition 4.5),

$$\begin{aligned} n = \dim(C^1 + C^2) &= \dim C^1 + \dim C^2 - \dim C \\ &= (n - d^1) + (n - d^2) - (n - d) \\ \iff d^1 + d^2 = d \end{aligned}$$

On the other hand, we know from Lemma A.24 that $L_\sigma = L_{\sigma^1} + L_{\sigma^2}$ and so $\dim L_\sigma = \dim L_{\sigma^1} + \dim L_{\sigma^2} - \dim(L_{\sigma^1} \cap L_{\sigma^2})$. Since the dimension of a polytope is defined to be the dimension of its affine span, which is naturally equal to the dimension of the parallel linear space, we conclude that $L_{\sigma^1} \cap L_{\sigma^2} = \{0\}$ iff $\dim(C^1 + C^2) = n$, i.e. iff the intersection is transverse at C.

(2) Suppose $r = 2$. Then, by part (1), we know $L_{\sigma^1} \cap L_{\sigma^2} = \{0\}$; we also know that $L_\sigma = L_{\sigma^1} + L_{\sigma^2}$ by Corollary A.24. So $L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2}$. For $r \geq 3$, recall from Definition 4.5 that we check transversality incrementally, taking the aggregate THs of agents $1, \ldots, j$ and checking transversality of the intersection of this with the $(j+1)^{\text{th}}$ TH. Applying the $r = 2$ case each time yields us the first result, from which $d = d^1 + \cdots + d^r$ follows. ☐

A.4.2 Lattices associated with an SNP

Here we give the definitions and results we will need to prove Theorems 5.3, 5.5 and 5.6. Basic material on lattices in this context is presented in Appendix A.5.

Definition A.26. Given an SNP cell ρ, the *parallel lattice* N_ρ is the set of integer vectors parallel to the cell: $N_\rho = L_\rho \cap \mathbb{Z}^n$.

Lemma A.27. *In the notation introduced at the beginning of Appendix A.4, the lattice* $N_{\sigma^1} + \cdots + N_{\sigma^r}$ *is a sublattice of* N_σ, *and the linear spans coincide.*

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[102] See e.g. Cox et al 2005, Section 7.4, Exercise 3.

[103] In linear algebra, "$L = L_1 \oplus L_2$" is shorthand for saying that the (Minkowski) sum of vector spaces $L_1 + L_2 = L$ and also that $L_1 \cap L_2 = \{0\}$. It extends naturally across several vector spaces.

Proof. By Corollary A.24, for $j = 1, \dots, r$, we know $L_{\sigma^j} \subseteq L_\sigma$. Take intersections with $\mathbb{Z}^n$ to see $N_{\sigma^j} \subseteq N_\sigma$. Thus, as N_σ is additively closed, $N_{\sigma^1} + \cdots + N_{\sigma^r} \subseteq N_\sigma$. It is a sublattice as it is a lattice.

We know $L_\sigma = L_{\sigma^1} + \cdots + L_{\sigma^r}$, so, to show the linear spans coincide, it is sufficient to show that $L_{\sigma^1} + \cdots + L_{\sigma^r}$ is the linear span of $N_{\sigma^1} + \cdots + N_{\sigma^r}$. But the latter must certainly contain L_{σ^j} for $j = 1, \dots, r$ and so it contains their sum; on the other hand $N_{\sigma^1} + \cdots + N_{\sigma^r} \subseteq L_{\sigma^1} + \cdots + L_{\sigma^r}$ and the latter is linear; so $L_{\sigma^1} + \cdots + L_{\sigma^r}$ is indeed the minimal linear vector subspace of $\mathbb{R}^n$ containing $N_{\sigma^1} + \cdots + N_{\sigma^r}$. □

Importantly, we thus have a well-defined subgroup index $[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^r}]$ (see Appendix A.5.2). Note from Fact A.45 that this is greater than 1 *precisely* when the parallelepiped we have discussed before contains a non-vertex point.

Having understood this definition, we now re-write and slightly generalise Propositions A.16 and A.17:

Proposition A.28. *Use the notation introduced at the beginning of Appendix A.4. Suppose* $\mathbf{p}$ *is in the relative interior of* C*.*

(1) If $[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^r}] = 1$ *then* $D_U(\mathbf{p})$ *is discrete-convex.*
(2) If $[N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^r}] > 1$ *and if* $\dim \sigma^1 \leq 2$ *and* $\dim \sigma^j = 1$ *for* $j = 2, \dots, r$ *then* $D_U(\mathbf{p})$ *is not discrete-convex.*

Proof. Part (1) slightly extends Proposition A.17, to cover cases in which there is no basis for N_{σ^j} consisting of edges of σ^j. However, in the proof of that proposition, we only used the fact that the set of vectors we assigned to agent j was a set of integer vectors and was a basis for the linear span for that agent's demand set. Thus we need only make the weaker assumption that some integer basis exists for each agent's parallel lattice so that the combination of the bases is unimodular, to obtain the result in the same way.

Part (2) is follows easily from Proposition A.16 when $\dim \sigma^1 = 1$. It may also be understood in more detail by following the argument for $\dim \sigma^1 = 2$ below, ignoring the role of σ^1.

So suppose $\dim \sigma^1 = 2$. Without loss of generality we may assume that $\mathbf{0} \in \sigma^j$ for $j = 1 \dots, r$ (otherwise the following arguments are simply augmented by a fixed shift). For $j = 2, \dots, r$, fix an minimal integer non-zero vector $\mathbf{v}^j \in \sigma^j$. In each case this vector then forms a basis for the corresponding lattice N_{σ^j}.

For $j = 1$ we will need find a basis for N_{σ^1} consisting of vectors $\mathbf{v}^0, \mathbf{v}^1$ which are actually contained inside σ^1 (note that this is not immediate). Start by taking $\mathbf{w}^0$,$\mathbf{w}^1 \in \sigma^1$ which are linearly independent integer vectors. If these are a basis for N_{σ^1}, we are done. If not, they span a sublattice M_1 of N_{σ^1}, such that $[N_{\sigma^1} : M_1] > 1$, and so there must exist $\mathbf{w} \in \mathbb{Z}^n$ which is a non-vertex point of the parallelepiped they span. Then $\mathbf{w} = \alpha^0 \mathbf{w}^0 + \alpha^1 \mathbf{w}^1$ with $\alpha^0, \alpha^1 \in [0, 1]$. If $\alpha^0 + \alpha^1 \leq 1$ then we fix $\mathbf{w}^2 := \mathbf{w}$; if $\alpha^0 + \alpha^1 > 1$ then let $\mathbf{w}^2 = \mathbf{w}^0 + \mathbf{w}^1 - \mathbf{w}$. In either case now $\mathbf{w}^2 = \beta^0 \mathbf{w}^0 + \beta^1 \mathbf{w}^1$ with $\beta^0 + \beta^1 \leq 1$. As σ^1 is convex we conclude that $\mathbf{w}^2 \in \sigma$.

Recalling that $\mathbf{w}$ was a non-vertex point of the parallelepiped spanned by $\mathbf{w}^0, \mathbf{w}^1$, we know $\mathbf{w}^2$ is distinct from $\mathbf{w}^0, \mathbf{w}^1, \mathbf{0}$. So $\mathbf{w}^2$ is a non-vertex point of the convex hull Δ^0 of $\mathbf{0}, \mathbf{w}^0, \mathbf{w}^1$. Hence the convex hull Δ^1 of $\mathbf{0}, \mathbf{w}^1, \mathbf{w}^2$ has strictly smaller area than Δ^0. Moreover, the parallelepipeds spanned by $\mathbf{w}^0, \mathbf{w}^1$ and by $\mathbf{w}^1, \mathbf{w}^2$ have areas equal to twice the areas of Δ^0, Δ^1, respectively. So, if M_2 is the sublattice of N_{σ^1} spanned by $\mathbf{w}^1, \mathbf{w}^2$, then $[N_{\sigma^1} : M_2] < [N_{\sigma^1} : M_1]$.

As all subgroup indices are positive-integer-valued, after a finite number of repetitions of this process, the subgroup index will be 1, and hence (by Fact A.45(3)) we will have obtained vectors $\mathbf{v}^0, \mathbf{v}^1 \in \sigma^1$ which are a basis of N_{σ^1}, as required.

Now, by Fact A.45(3), there exists a bundle $\mathbf{x} \in N_\sigma$, $\mathbf{x} \notin N_{\sigma^1} + \cdots + N_{\sigma^r}$. Our identified vectors $\mathbf{v}^0, \mathbf{v}^1, \ldots, \mathbf{v}^r$ are an integer basis for this sublattice of N_σ. Because the linear spans coincide (by Lemma A.27), they are thus a basis for L_σ, so we can write $\mathbf{x}$ as a (real-valued) linear combination of these vectors. Moreover, since subtracting integer multiples of the $\mathbf{v}^j$ from $\mathbf{x}$ yields a new element of N_σ, we can assume that $\mathbf{x}$ *is* in the fundamental parallelepiped of the sublattice with respect to this basis. So we can write $\mathbf{x} = \sum_{j=0}^{r} \alpha^j \mathbf{v}^j$ with $\alpha^j \in [0,1]$ for $j = 0, \ldots, r$. Additionally, we can assume that $\alpha^0 + \alpha^1 \leq 1$: if $\alpha^0 + \alpha^1 > 1$ then replace $\mathbf{x}$ with $\sum_{j=0}^{r} \mathbf{v}^j - \mathbf{x} \in N_\sigma$. Now $\alpha^0\mathbf{v}^0 + \alpha^1\mathbf{v}^1 \in \sigma^1$ and, for all j, also $\alpha^j\mathbf{v}^j \in \sigma^j$. So, $\mathbf{x} \in \sigma^1 + \cdots + \sigma^r = \sigma = \operatorname{Conv} D_U(\mathbf{p})$. Moreover, $\mathbf{x} \in N_\sigma \subseteq \mathbb{Z}^n$. But, by assumption, $\mathbf{x} \notin N_{\sigma^1} + \cdots + N_{\sigma^r}$, and so $\mathbf{x} \notin D_{u^1}(\mathbf{p}) + \cdots + D_{u^r}(\mathbf{p}) = D_U(\mathbf{p})$. ☐

To show the tightness of this result, we now give an example where $[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] > 1$, where $\dim \sigma^1 = \dim \sigma^2 = 2$ and where $D_U(\mathbf{p})$ *is* discrete-convex. Although there will be a non-vertex lattice point in the fundamental parallelepiped of $N_{\sigma^1} + N_{\sigma^2}$, the key question for us is whether that lattice point is actually in $\sigma^1 + \sigma^2$. If both these SNP cells are small enough, for example if they are both simplices, the answer can be "no". But the answer can also be "yes", without any change in the volumes or subgroup indices, and so it seems that our method of analysis reaches its limit here.

Example A.29. In this example $n = 4$.

Agent 1 has valuation $u^1(0,0,0,0) = 0$, $u^1(1,1,0,0) = 6$, $u^1(0,0,1,1) = 6$. So Agent 1 is indifferent between these three bundles at prices $\mathbf{p}$ such that $p_1 + p_2 = 6$, $p_3 + p_4 = 6$. There are three facets emanating from this 2-cell. Facets are 3-dimensional; recall that on each the agent is indifferent between a pair of these bundles.

Agent 2 has valuation $u^2(0,0,0,0) = 0$, $u^2(0,1,1,0) = 9$, $u^2(4,0,0,1) = 6$, and so is indifferent between these bundles at prices $\mathbf{p}$ such that $p_2 + p_3 = 9$ and $4p_1 + p_4 = 6$. Again, there are three facets, on which the agent is indifferent between pairs of these bundles, emanating from this 2-cell.

These conditions for indifference are all satisfied iff $\mathbf{p} = (1,5,4,2)$. At this price, we have $\sigma^1 = \operatorname{Conv}((0,0,0,0),(1,1,0,0),(0,0,1,1))$ and $\sigma^2 = \operatorname{Conv}((0,0,0,0),(0,1,1,0),(4,0,0,1))$. The aggregate SNP cell σ is the Minkowski sum of all these; it is of course 4-dimensional and so $N_\sigma = \mathbb{Z}^4$. Meanwhile N_{σ^1} and N_{σ^2} are 2-dimensional lattices, and we check that the non-zero vectors we already know in each lattice do give a basis in each case, by checking that the sets $\{(1,1,0,0),(0,0,1,1)\}$ and $\{(0,1,1,0),(4,0,0,1)\}$ are unimodular (use Remark A.15(4)).

Thus, $[N_\sigma : N_{\sigma^1} + N_{\sigma^2}]$ is given by the absolute value of the determinant of these four vectors; it is clearest to see the value of this determinant by re-ordering them:

$$\det \begin{pmatrix} 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = -3$$

Using Fact A.45(1) we now know that there are exactly 2 interior points to the fundamental parallelepiped of $N_{\sigma^1} + N_{\sigma^2}$. We express them explicitly in terms of the basis vectors:

$$\frac{2}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\1\\1\end{pmatrix} \tag{2}$$

and

$$\frac{1}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\\1\end{pmatrix} \tag{3}$$

These expressions show clearly that $(2,1,1,1)$ can be decomposed to give a part in σ^2 and a part *not* in σ^1, whereas $(3,1,1,1)$ can be decomposed to give a part in σ^1 and a part *not* in σ^2. Moreover, by linear independence of this set of four vectors, these are the *only* possible decompositions into sums of bundles in the affine spans of σ^1, σ^2. So neither is in $\sigma^1 + \sigma^2 = \sigma = \operatorname{Conv} D_U(1,5,4,2)$. But this means that the only integer vectors in $\operatorname{Conv} D_U(1,5,4,2)$ are in fact in $D_U(1,5,4,2)$ itself: it is discrete-convex.

However, by changing which combinations of these four vectors pertain to each agent, we change the situation:

Example A.30. Now suppose Agent 1 has valuation $u^1(0,0,0,0) = 0$, $u^1(1,1,0,0) = 6$, $u^1(0,1,1,0) = 9$ and Agent 2 has valuation $u^2(0,0,0,0) = 0$, $u^1(0,0,1,1) = 6$, $u^2(4,0,0,1) = 6$. Now $\sigma^1 = \operatorname{Conv}((0,0,0,0),(1,1,0,0),(0,1,1,0))$ and $\sigma^2 = \operatorname{Conv}((0,0,0,0),(0,0,1,1),(4,0,0,1))$. The analysis proceeds very similarly to before, but in this case the decompositions (2) and (3) show clearly that both $(2,1,1,1)$ and $(3,1,1,1)$ *are* in $\sigma^1 + \sigma^2$. Thus they are in $\operatorname{Conv} D_U(1,5,4,2)$ but not in $D_U(1,5,4,2)$: this set is in this case *not* discrete-convex.

In both Example A.29 and A.30 we can calculate the mixed volume relevant at the intersection.[104] In both cases $MV_4(\sigma^1, \sigma^2, (2,2)) = 3$. In both cases, the weights of the individual's 2-cells that meet at $(1,5,4,2)$ are both 1. There appears to be no way to distinguish Examples A.29 and A.30 using the tools developed here, and so we reiterate: the condition of Theorem 5.3 is sufficient, but *not* necessary, for existence of equilibrium when $n \geq 4$.

A.4.3 Full Statement of the Tropical Bézout-Kouchnirenko-Bernshtein theorem

The classical theorems of Bézout, Kouchnirenko and Bernshtein count intersections of curves defined by polynomials, weighting these counts with the "multiplicities" of the intersections. Bézout's (1779) theorem counts intersections of polynomial curves of "pure degree" in complex "projective" space, as being the product of their degrees. In practice

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[104] This is not too hard, using Definition 5.1, as we only need consider combinations involving both σ^1 and σ^2 (the 4-dimensional volume of an object of fewer than 4 dimensions is 0) and as σ^1 and σ^2 lie in linearly independent subspaces of $\mathbb{R}^4$.

this means that one must put polynomials into a particular form and potentially count many intersections "at infinity". Bernshtein's (1975) theorem (see also Kouchnirenko, 1976) extended this by showing that the number of intersections of *general* curves in *affine* complex space is given by the mixed volume of their Newton polytopes.

Many authors have developed how one should think about tropical intersection multiplicities in order to translate such results in classical algebraic geometry across to the tropical world (see e.g. Sturmfels, 2002, Mikhalkin, 2005, Gathmann and Markwig, 2008, Osserman and Payne, 2013). A 2-dimensional tropical Bézout-Bernshtein theorem dates back to Sturmfels (2002) at the very beginning of the discipline of tropical geometry. But indeed, Bézout-Bernshtein theorems for polytopes, working in n-dimensional space, were already well known in one form or another: see McMullen (1993), Huber and Sturmfels (1995) and Fulton and Sturmfels (1996). In this paper, we appeal to the conventions and definitions of Bertrand and Bihan, who provide a nice tropical exposition in their 2007 preprint, the relevant parts of which are published in their (otherwise different) 2013 book chapter.

Recall that we defined the mixed volume in the text (Definition 5.1) as being a linear combination of volumes of sums of convex sets. We use this definition because it requires the least additional background in geometry. However, there are many equivalent definitions, and ours is not the most intuitive for those with more familiarity with the subject. Cox et al (2005, Section 7.4) give much more explanation.[105]

Also note that n-dimensional mixed volume takes n arguments and calculates the volume in dimension n–which is zero on any object which has fewer than n dimensions itself.

We define the weight of a general TH cell using the *lattice-volume* $\mathrm{Vol}_{N_\sigma}(\sigma)$ of the corresponding SNP face; see Definition A.43 in Appendix A.5.2.

Definition A.31. The *weight* w_u of a k-cell C_σ of a TH, associated to the $(n-k)$-cell σ of the SNP, is given by

$$w_u(C_\sigma) = (n-k)!\mathrm{Vol}_{N_\sigma}(\sigma)$$

We use again the notation introduced at the beginning of Appendix A.4. Now we can define the tropical intersection multiplicity:

Definition A.32 (Bertrand and Bihan, 2013, Definition 5.2)**.** Using the notation introduced at the beginning of Appendix A.4, the multiplicity of cell C in the intersection of THs $\mathcal{T}_{u^1}, \dots, \mathcal{T}_{u^r}$ is defined as follows:

1. If the intersection is transverse at $\mathbf{p}$ then $\mathrm{mult}(C) := [N_\sigma : N_{\sigma^1} + \cdots + N_{\sigma^r}] \cdot \prod_{j=1}^{r} w_{u^j}(C^j)$
2. If the intersection is not transverse at $\mathbf{p}$, translate the THs by small generic vectors (as in Lemma 4.6) so that all intersections emerging from C are transverse. Define $\mathrm{mult}(C)$ as the sum of the multiplicities of the transverse intersections emerging from C which are cells of dimension $n-d$.

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[105] Note that there are two competing conventions for the mixed volume; some writers divide the form given here by $n!$. We use the same convention as Cox et al (2005) and Bertrand and Bihan (2007, 2013).

It is useful to recall from Lemma A.25 that, in our notation, an intersection is transverse iff $d^1 + \cdots + d^r = d$.

Theorem A.33 (Bertrand and Bihan, 2013, Thm. 6.1)**.** *Use the notation of Appendix A.4.*

(1) If the intersection of $\mathcal{T}_{u^1}, \ldots, \mathcal{T}_{u^r}$ is transverse at C, then

$$\mathrm{mult}(C) = MV_d(\sigma^1, \ldots, \sigma^r; (d^1, \ldots, d^r)).$$

(2) In general, when $d \leq d^1 + \cdots + d^r$, we have

$$\mathrm{mult}(C) = \sum_{t^1+\cdots+t^r=d;\ t^j \geq 1} MV_d(\sigma^1, \ldots, \sigma^r; (t^1, \ldots, t^r)),$$

where the sum is over all r-tuples $(t^1, \ldots, t^r)$ such that $t^1 + \cdots + t^r = d$ and $t^j \geq 1$ for all j. In particular, if $d = r$, then $\mathrm{mult}(C) = MV_d(\sigma^1, \ldots, \sigma^r)$*.*

We state the following result in less generality than do Bertrand and Bihan, and in different language, so that its use for our purposes is clearer.

Lemma A.34 (Bertrand and Bihan, 2013, Lemma 6.7)**.** *Suppose the intersection of $\mathcal{T}_{u^1}$, $\mathcal{T}_{u^2}$ is transverse.*[106]

$$MV_n(\tilde{A}^1, \tilde{A}^2, (d^1, d^2)) = \sum_{\dim \sigma^j = d^j} MV_n(\sigma^1, \sigma^2, (d^1, d^2))$$

where the sum is taken over all cells $\sigma = \sigma^1 + \sigma^2$ of the SNP of U, such that $\dim \sigma^j = d^j$ *for $j = 1, 2$, and such that σ^1, σ^2 correspond to TH cells which intersect along an aggregate TH cell corresponding to σ.*

A.4.4 Proofs of results in Appendix 5.2

We can finally prove the results stated in the text.

Lemma A.35. *Suppose $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ intersect transversally at the respective cells C^1, C^2. Then $w_{u^1}(C^1) w_{u^2}(C^2) \leq MV_n(\sigma^1, \sigma^2; (d^1, d^2))$, with equality holding iff $[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 1$.*

Proof. By Theorem A.33(1) we know that $\mathrm{mult}(C) = w_{u^1}(C^1) w_{u^2}(C^2) [N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = MV_n(\sigma^1, \sigma^2; (d^1, d^2))$, from which the result follows. ☐

Proof of Lemma 5.2. We have THs $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$. Let C be a 0-cell of their intersection, and use the notation of Appendix A.4. In particular, the d from that notation is equal to n and, since the intersection is transverse, we know $n = d^1 + d^2$ (recalling that $d^j = \dim \sigma^j = n - \dim C^j$).

Now we take the sum over all 0-cells of the intersection which are the intersection of a $(n - d^1)$-cell of $\mathcal{T}_{u^1}$ and an $(n - d^2)$-cell of $\mathcal{T}_{u^2}$, and apply Lemmas A.34 and A.35,

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[106] The assumption of transversality here is equivalent to the condition, in the language of Bertrand and Bihan, 2013, Lemma 6.7, that the "convex mixed subdivision" is "pure".

to see that the sum of such 0-cells, weighted only by the products of weights of the intersecting cells, is bounded above by $MV_n(\tilde{A}^1, \tilde{A}^2, (d^1, d^2))$. Moreover, equality holds iff $[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] = 1$ in every case. The result now follows by Proposition A.28. In particular, if $n = 2$ then every transverse intersection of cells is an intersection of two facets, and if $n = 3$ then every transverse intersection of cells is an intersection of two facets or a facet and a 1-cell. So in these cases there is at most one corresponding SNP cell with dimension > 1, and hence (by Proposition A.28) $[N_\sigma : N_{\sigma^1} + N_{\sigma^2}] > 1$ demonstrates *failure* of equilibrium. $\square$

Proof of Theorem 5.3. That the upper bound holds is obvious from Lemma 5.2.

If the naïvely-weighted count is equal to the upper bound, then by Lemma 5.2, the demand set is discrete-convex for every 0-cell in the intersection of the THs. Now suppose that equilibrium fails for some supply. By Lemma 4.4, the demand set is not discrete-convex at some price in the intersection, and hence it is not discrete-convex for any price in the interior of the corresponding cell in the TH of aggregate demand. But by Corollary A.11, because we assume the domain of aggregate demand to be in n dimensions, there is a 0-cell in the boundary of this cell, at which price the demand set is *also* not discrete-convex. Since a TH intersection is closed, this 0-cell is also in the intersection of the individual THs: we have a contradiction. So in this case, equilibrium exists for every supply.

If the naïvely-weighted count is strictly below the upper bound, and if $n \leq 3$, then it is immediate by Lemma 5.2 that equilibrium must fail for some supply. $\square$

A.4.5 Proofs for Section 5.4: results when the intersection is not transverse

First, an interesting example of a non-transverse intersection, such that competitive equilibrium does exist for all supplies in the domain of aggregate demand, *only* if we do *not* make any small shift of the valuations.

Example A.36. Consider two identical agents whose demand sets at price $(2, 2)$ are the bundles $(0, 0), (1, 2), (2, 1)$ and $(1, 1)$. (For example, this is consistent with a valuation $u(x, 0) = x; u(0, y) = y; u(1, 1) = 4; u(1, 2) = u(2, 1) = 6; u(2, 2) = 7$). Then the bundle $(2, 2)$ is in the aggregate demand set at this price: we assign bundle $(1, 1)$ to both agents. But observe that bundle $(1, 1)$ is an interior point of each agent's SNP cell with the vertices $(0, 0), (1, 2)$, and $(2, 1)$, so if we make a small perturbation to either agent's valuation so their THs intersect transversally, then for any prices close to $(2, 2)$ the perturbed agent's demand must be some subset of these vertices, and it is easy to see that $(2, 2)$ cannot be an aggregate demand.

We now state the full version of Maclagan and Sturmfels (2015) Proposition 3.6.12.

Proposition A.37 (Maclagan and Sturmfels, 2015, Proposition 3.6.12)**.** *For any THs $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$, and generic $v \in \mathbb{R}^n$, the limit $\lim_{\epsilon\to 0} \mathcal{T}_{u^1} \cap (\epsilon \mathbf{v} + \mathcal{T}_{u^2})$ exists and equals the stable intersection of $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$.*[107]

Additionally, the proof of the following is clear from the proof of Proposition 3.6.12 of Maclagan and Sturmfels, 2015.

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[107] The limit is taken in the Hausdorff metric.

Corollary A.38. *For sufficiently small $\epsilon > 0$ the combinatorial type of $\mathcal{T}_{U_\epsilon}$ is independent of ϵ.* ☐

This enables us to prove Theorem 5.5

Proof of Theorem 5.5. For first necessity and then sufficiency, we prove the result by contradiction.

If there exists any price at which the aggregate demand set is not discrete-convex, then clearly equilibrium fails for supply bundles exhibiting failure of discrete-convexity (see Lemma 2.9). In particular this holds if the price is at a 0-cell of the stable intersection.

Conversely, suppose equilibrium does not exist for $\mathbf{x}$ in the domain of aggregate valuation. Then, combining Proposition 4.6 with Lemmas A.18 and A.19, there exists $\mathbf{w}$ such that, for all $\epsilon > 0$ and beneath some upper bound, the THs $\mathbf{T}_{u^1}$ and $\epsilon\mathbf{w} + \mathcal{T}_{u^2}$ meet transversally everywhere, and equilibrium fails for supply $\mathbf{x}$ when the agents have the corresponding valuations. Fix suitable ϵ, and write the corresponding aggregate valuation as U_ϵ. Note by Corollary A.38 that the combinatorial type of $\mathcal{T}_{U_\epsilon}$ is independent of ϵ for such ϵ.

As $\mathbf{x}$ is in the domain of the aggregate valuation, we have $\mathbf{x} \in \sigma_\epsilon$ for some SNP cell σ_ϵ of the SNP corresponding to U_ϵ. Moreover, we assumed that the domain of U has dimension n, from which it follows that σ_ϵ has dimension n, and corresponds to a 0-cell of $\mathcal{T}_{U_\epsilon}$. Let $\mathbf{p}_\epsilon$ be the price at this 0-cell. Then $\mathbf{x} \in \operatorname{Conv} D_{U_\epsilon}(\mathbf{p}_\epsilon)$. However, we know that $\mathbf{x} \notin D_{U_\epsilon}(\mathbf{p})$ for any $\mathbf{p} \in \mathbb{R}^n$. We conclude, as argued at Lemma 4.4, that $\mathbf{p}_\epsilon$ is at the intersection of the individual tropical hypersurfaces, that is, $\mathbf{p}_\epsilon \in \mathcal{T}_{u^1} \cap (\epsilon\mathbf{w} + \mathcal{T}_{u^2})$.

Identify minimal cells C^1 of $\mathcal{T}_{u^1}$ and C^2_ϵ of $\epsilon\mathbf{w} + \mathcal{T}_{u^2} (= \mathcal{T}_{u^2_\epsilon})$ such that $\mathbf{p}_\epsilon \in C^1 \cap C^2_\epsilon$. Since $\mathbf{p}_\epsilon$ is at a 0-cell of U_ϵ, by minimality of C^1 and C^2_ϵ, this intersection must be at only one point: $\{\mathbf{p}_\epsilon\} = C^1 \cap C^2_\epsilon$.

Now note that $C^2_\epsilon = \epsilon\mathbf{v} + C^2$, where C^2 is a cell of $\mathcal{T}_{u^2}$, by Lemma A.18. So $\mathbf{p}_\epsilon \in C^1 \cap (\epsilon\mathbf{v} + C^2)$, and hence in particular the latter is non-empty. As we chose sufficiently small ϵ that the combinatorial type of $\mathcal{T}_{U_\epsilon}$ is independent of ϵ, it follows that $C^1 \cap (\epsilon'\mathbf{v} + C^2)$ contains one point, $\mathbf{p}_{\epsilon'}$, for all ϵ' with $0 < \epsilon' < \epsilon$. Hence, since cells are closed, the limit as $\epsilon \to 0$ is a 0-cell of the stable intersection. Let $\mathbf{p}$ be the price at this 0-cell. Obviously $\mathbf{p} \in C^1 \cap C^2$.

Now let σ^1 and σ^2 be the SNP cells of the individual valuations corresponding to C^1 and C^2. Of course σ^2 is *also* the SNP cell for agent $2'$ corresponding to cell $C^2_\epsilon = \epsilon\mathbf{v} + C^2$. So $\sigma_\epsilon = \sigma^1 + \sigma^2$. Since $\mathbf{p} \in C^1 \cap C^2$ we know $\sigma^1 \subseteq D_{u^1}(\mathbf{p})$ and $\sigma^2 \subseteq D_{u^2}(\mathbf{p})$. Hence $\mathbf{x} \in \sigma_\epsilon = \sigma^1 + \sigma^2 \subseteq \operatorname{Conv} D_{u^1}(\mathbf{p}) + \operatorname{Conv} D_{u^1}(\mathbf{p}) = \operatorname{Conv} D_U(\mathbf{p})$. ☐

Our remaining theorem is also easy to prove from Bertrand and Bihan's results and definitions.

Proof of Theorem 5.6. Take a small translation of $\mathcal{T}_{u^2}$ so that the intersections are all transverse. By Definition A.31(2), the weight of any 0-cell in the stable intersection (before this translation) is given by the sum of the weights of the 0-cells that emerged from it (after this translation); by Proposition A.37, such 0-cells will indeed emerge. As these weights are all positive integers, then, the number of 0-cells in the stable intersection is bounded above by the weighted sum of 0-cells after the translation, which, by Lemma A.34, is the number stated. ☐

A.5 Mathematical background for the proofs in Appendix A.4: Lattices and subgroup indices

The following material is intended as a companion to Appendix A.4.

A.5.1 Lattices

The key to the question of equilibrium with indivisible goods, is that available bundles form a subset of a lattice, and individual agents' demands aggregate as lattices. So, before considering the parallel lattice associated with an SNP cell, we cover some preliminary material on lattices. We will only be interested in lattices within $\mathbb{Z}^n$. Readers familiar with group theory will recognise them as additive subgroups of $\mathbb{Z}^n$. Familiarity with group theory is not essential to understand this section, but may deepen understanding a little.

Definition A.39.

(1) A *lattice* is a set $N \subseteq \mathbb{Z}^n$ such that $\mathbf{0} \in N$ and if $\mathbf{n}, \mathbf{n}' \in N$ then $\mathbf{n} - \mathbf{n}' \in N$.
(2) M is a *sublattice* of N if $M \subseteq N$ and M has the structure of a lattice.
(3) The *linear span* of a lattice is the minimal vector subspace of $\mathbb{R}^n$ containing N.
(4) The *rank* of a lattice is the dimension of its linear span.
(5) An *integer basis* for a lattice N is a set $\{\mathbf{n}^1, \ldots, \mathbf{n}^r\}$ such that any $\mathbf{n} \in N$ can be uniquely presented as $\mathbf{n} = \sum_i \alpha_i \mathbf{n}^i$ for $\alpha_i \in \mathbb{Z}$.

We refer to *integer bases* rather than just bases for lattices to retain clarity that these are bases for lattices and not just the linear space they span. We will be particularly interested in sublattices of equal rank.

We now group some important results.

Fact A.40.

(1) If $N, M \subseteq \mathbb{Z}^n$ are lattices, then the Minkowski sum $N + M$ is a lattice.
(2) Any lattice has an integer basis.[108]
(3) If M, N are lattices whose linear spans have zero intersection and if $\{\mathbf{m}^1, \ldots \mathbf{m}^r\}$, $\{\mathbf{n}^1, \ldots, \mathbf{n}^s\}$ are integer bases for them respectively, then $\{\mathbf{m}^1, \ldots \mathbf{m}^r, \mathbf{n}^1, \ldots, \mathbf{n}^s\}$ is an integer basis for $M + N$.
(4) Suppose the rank k lattice N is equal to $\mathbb{Z}^n \cap L_N$ where L_N is its linear span. A set $\{\mathbf{n}^1, \ldots, \mathbf{n}^k\}$ of linearly independent vectors in N is a basis iff it is unimodular.

We emphasise in particular Fact A.40(4), an important result which was also mentioned in Remark A.15.

We can therefore define:

Definition A.41. A *fundamental parallelepiped* of a lattice N is the set $\{\sum_i \lambda_i \mathbf{n}^i \mid 0 \leq \lambda_i \leq 1\}$, where $\{\mathbf{n}^1, \ldots, \mathbf{n}^r\}$ are a basis for N.

Note that a different basis will give a different parallelepiped, but they will always be related by a unimodular basis change.

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[108] This is the "fundamental theorem on discrete subgroups of Euclidean spaces", see Cox et al., 2005, p334.

A.5.2 Volumes of Lattice Polytopes, and subgroup indices

Definition A.42. The *n-dimensional volume* of a convex set X in $\mathbb{R}^n$ is:

$$\mathrm{Vol}_n(X) := \int \cdots \int_X 1 dp_1 \dots dp_n.$$

If the dimension k of X is less than n then this will always be zero. We need, however, some measure of the k-dimensional volume of a k-dimensional polytope which lives in $\mathbb{R}^n$.[109] Moreover, we wish to normalise so that, for example, an SNP edge comprising only one copy of a primitive integer vector has "length 1". Similarly, will consider the fundamental parallelepiped of a rank k sublattice $N \subseteq \mathbb{Z}^n$ to have k-dimensional volume equal to 1 *with respect to this lattice.* Thus the appropriate change of basis is not orthogonal, but defined by a basis for the lattice in question.

Specifically, take such a basis $\{\mathbf{n}^1, \dots, \mathbf{n}^k\}$ for N and extend it to a basis for $\mathbb{R}^n$, for example by appending suitable coordinate vectors (the choice of these vectors will not be relevant).[110] Let G_N be the inverse of the matrix with these vectors as its columns, so $G_N \mathbf{n}^i = \mathbf{e}^i$ for $i = 1, \dots, k$. Then G_N restricts to an isomorphism between N and $\mathbb{Z}^k$: any polytope X with vertices in N is mapped under G_N to a polytope in $\mathbb{R}^k \subseteq \mathbb{R}^n$ with vertices in $\mathbb{Z}^k \subseteq \mathbb{Z}^n$. With these conventions:

Definition A.43. The *lattice-volume* of a polytope X with vertices in a lattice $N \subseteq \mathbb{Z}^n$ is $\mathrm{Vol}_N(X) := \mathrm{Vol}_k(G_N X)$.

It is easy to see that this is independent of the choice of basis for N (and its extension to $\mathbb{R}^n$) and that the volume of the fundamental parallelepiped in N is 1.

Now the sublattice index easy to define.

Definition A.44. Let $M \subseteq N$ be a sublattice of equal rank, and let Δ_M be a fundamental parallelepiped of M. The *subgroup index* $[N : M]$ is $\mathrm{Vol}_N(\Delta_M)$.

The following points are standard:[111]

Fact A.45. Using the notation defined above, and in particular using the *same* vectors in $\mathbb{R}^n$ to extend a basis for either N or M to $\mathbb{R}^n$, we have

(1) $[N : M] - 1$ is equal to the number points of N in Δ_M which are not vertices.
(2) $[N : M] = |\det(G_N G_M^{-1})|$.
(3) $[N : M] = 1$ iff $N = M$.

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[109] One would usually determine the k-dimensional volume of a k-dimensional subset X of $\mathbb{R}^n$ by using an orthonormal change of basis matrix so that $X \subseteq \mathbb{R}^k$ for some fixed subset of coordinates of $\mathbb{R}^n$. Then the volume as defined above can be taken.

[110] Those familiar with abstract linear transformations will see that choosing such vectors is unnecessary; we include this step so readers unfamiliar with such material can think entirely in terms of square matrices.

[111] Those familiar with group theory will recognise that $[N : M]$ is the subgroup index in the ordinary sense: it is the number of cosets of M in N, that is, the number of disjoint sets $\mathbf{n} + M$ where $\mathbf{n} \in N$. It is standard group theory that each such coset may be represented by some $\mathbf{n}$ in the fundamental parallelepiped of M, and if $\mathbf{n} + M \neq M$ then such $\mathbf{n}$ is not a vertex of this parallelepiped, and is unique; these points both follow from the simple observation that $\mathbf{n} + M = \mathbf{n}' + M \Leftrightarrow \mathbf{n} - \mathbf{n}' \in M$. This shows part 1. In part 2, the fundamental point is that $[N : M]$ is the determinant of the $(k \times k)$ change of basis from M to N, but again we present an explicit $n \times n$ matrix with the requisite property so readers need not concern themselves unnecessarily with unfamiliar material.