SUPPLEMENTARY MATERIAL FOR: A UNIFORM LAW FOR CONVERGENCE TO THE LOCAL TIMES OF LINEAR FRACTIONAL STABLE MOTIONS

A. Verifications for examples from Section 3.

Verification of Example 3.2. Let $\epsilon > 0$ be given, and $\{\theta_k\}_{k=1}^K$ be the centres of a collection of $\epsilon^{1/\tau}$-balls that cover Θ. Then for every $\theta \in B(\theta_k, \epsilon^{1/\tau})$,

$$l_k(x) \coloneqq g(x, \theta_k) - \epsilon \dot{g}(x) \leq g(x, \theta) \leq g(x, \theta_k) + \epsilon \dot{g}(x) \eqqcolon u_k(x)$$

whence the continuous brackets $\{l_k, u_k\}_{k=1}^K$ have size $2\epsilon \|\dot{g}\|_1$, and cover $\mathscr{F}$. A suitable envelope for $\mathscr{F}$ is given by

$$F(x) \coloneqq |g(x, \theta_1)| + (\operatorname{diam} \Theta)^{1/\tau} \dot{g}(x).$$

$\square$

Verification of Example 3.3. Let $\epsilon > 0$ be given, and $M < \infty$ chosen such that

$$\sup_{|x| \geq M} F(x) < \epsilon \qquad\qquad \int_{[-M,M]^c} F(x)\, \mathrm{d}x < \epsilon,$$

which is possible by Lemma B.1. Let $\mathscr{F}_{|M}$ denote the set formed by restricting each $f \in \mathscr{F}$ to the domain $[-M, M]$. In view of the proof of Theorem 2.7.1 in van der Vaart and Wellner (1996), for any given $\delta > 0$, there exist continuous functions $\{f_k\}_{k=1}^K$ such that the balls

$$B(f_k, \delta) \coloneqq \{g : [-M, M] \to \mathbb{R} \mid \|g - f_k\|_\infty < \delta\}$$

cover $\mathscr{F}_{|M}$. Thence the brackets formed by

$$\begin{aligned} l_k(x) &= \{[f_k(x) - \delta] \vee [-F(x)]\} \mathbf{1}\{x \in [-M, M\} - F(x) \mathbf{1}\{x \notin [-M, M]\} \\ u_k(x) &= \{[f_k(x) + \delta] \wedge F(x)\} \mathbf{1}\{x \in [-M, M\} + F(x) \mathbf{1}\{x \notin [-M, M]\} \end{aligned}$$

cover $\mathscr{F}$, and have size $\|u_k - l_k\|_1 \leq 2M\delta + \epsilon = 2\epsilon$, where the final equality follows by taking $\delta = \epsilon/2M$. Since u_k is piecewise continuous (with possible discontinuities at $-M$ and M), and agrees with $F(x)$ for all $|x| > M$, there clearly exists a $\tilde{u}_k \in \mathrm{BIL}_\beta$ with $\tilde{u}_k \geq u_k$ and $\|\tilde{u}_k - u_k\|_1 < \epsilon$. Constructing $\tilde{l}_k$ from l_k in an analogous manner, we thus obtain a collection of continuous brackets $\{\tilde{l}_k, \tilde{u}_k\}_{k=1}^K$ of size 4ϵ, which cover $\mathscr{F}$. $\square$

B. Modifications required for the proof Theorem 3.1(ii). We must first strengthen (6.1) to weak convergence in $\ell_\infty(\mathbb{R})$. To that end, define

$$\mathfrak{U}_n \coloneqq \frac{1}{d_n}\max_{t\leq n}|x_t| \qquad\qquad \mathfrak{U} \coloneqq \sup_{r\in[0,1]}|X(r)|,$$

so that $\mathfrak{U}_n \rightsquigarrow \mathfrak{U}$ under (2.7). Noting that the supports of $\mathcal{L}_n^\varphi$ and $\mathcal{L}^\varphi$ are contained in $[-\mathfrak{U}_n - 1, \mathfrak{U}_n + 1]$ and $[-\mathfrak{U}, \mathfrak{U}]$ respectively – in the first case, because φ is compactly supported – we may choose $M < \infty$ sufficiently large such that

$$\limsup_{n\to\infty}\mathbb{P}\left\{\sup_{a\in[-M,M]^c}\mathcal{L}_n^\varphi(a) > 0\right\} + \mathbb{P}\left\{\sup_{a\in[-M,M]^c}\mathcal{L}(a) > 0\right\} < \frac{\epsilon}{2}$$

for any given $\epsilon > 0$. By the result of part (i) and Theorem 1.10.3 in van der Vaart and Wellner (1996), there exists a distributionally equivalent sequence $\mathcal{L}_n^* =_d \mathcal{L}_n^\varphi$ such that $\mathcal{L}_n^* \overset{\text{a.s.}}{\to} \mathcal{L}^* =_d \mathcal{L}$ in $\ell_{\text{ucc}}(\mathbb{R})$. Hence

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{a\in\mathbb{R}}|\mathcal{L}_n^*(a) - \mathcal{L}^*(a)| > \epsilon\right\} \\
&\qquad \leq \mathbb{P}\left\{\sup_{a\in[-M,M]}|\mathcal{L}_n^*(a) - \mathcal{L}^*(a)| > \epsilon\right\} \\
&\qquad\qquad + \mathbb{P}\left\{\sup_{a\in[-M,M]^c}\mathcal{L}_n^*(a) > 0\right\} + \mathbb{P}\left\{\sup_{a\in[-M,M]^c}\mathcal{L}^*(a) > 0\right\} \\
&\qquad < \epsilon
\end{aligned}$$

for all n sufficiently large. Deduce that $\mathcal{L}_n^* \overset{p}{\to} \mathcal{L}^*$ in $\ell_\infty(\mathbb{R})$, whence (6.1) holds in $\ell_\infty(\mathbb{R})$.

To extend (6.4) to weak convergence on $\ell_\infty(\mathbb{R})$, it suffices to show that

$$\sup_{(a,b)\in[-M_n,M_n]\times\mathscr{B}_n}|\mathcal{L}_n^f(a,b^{-1}) - \mathcal{L}_n^\varphi(a)\mu_f| = o_p(1) \tag{B.1}$$

$$\sup_{(a,b)\in[-M_n,M_n]^c\times\mathscr{B}_n}|\mathcal{L}_n^f(a,b^{-1})| = o_p(1). \tag{B.2}$$

where $\mu_f \coloneqq \int_\mathbb{R} f$, and $M_n \coloneqq n^\tau$ for some $\tau > 0$. In view of Lemma 6.1, (B.1) may be proved via precisely the same arguments as which established the asymptotic negligibility of (6.2) above – albeit with a different choice of γ and δ (depending on τ). Regarding (B.2), we have the following (see the end of this section for the proof).

LEMMA B.1. *Suppose $f \in \mathrm{BIL}_\gamma$ for some $\gamma > 0$. Then $|f(x)| = o(|x|^{-\gamma/2})$ as $x \to \pm\infty$.*

Since $\max_{t\leq n}|x_t| \lesssim_p d_n$, we have w.p.a.1 that

$$\inf_{t\leq n}\inf_{|a|\geq M_n}|x_t - d_n a| \geq d_n n^\tau\left(1 - n^{-\tau}d_n^{-1}\max_{t\leq n}|x_t|\right) = d_n n^\tau(1 + o_p(1)).$$

In view of Lemma B.1, we may choose $\beta > 0$ such that $|f(x)| = o(|x|^{-\beta})$ as $x \to \pm\infty$. Then

$$\begin{aligned}\max_{t\leq n}\sup_{(a,b)\in[-M_n,M_n]^c\times\mathscr{B}_n} & bf[b(x_t - d_n a)] \\ &\lesssim \max_{t\leq n}\sup_{(a,b)\in[-M_n,M_n]^c\times\mathscr{B}_n} b^{1-\beta}|x_t - d_n a|^{-\beta} \\ &\lesssim_p e_n(e_n d_n^{-1}n^{-\tau})^\beta = o(ne_n^{-1})\end{aligned}$$

if τ is chosen sufficiently large. Thus (B.2) holds, whence (6.4) obtains in $\ell_\infty(\mathbb{R})$. An identical bracketing argument to that given above now establishes that (6.5) holds with $\mathbb{R}$ in place of $[-M, M]$.

PROOF OF LEMMA B.1. For simplicity, suppose f has Lipschitz constant $C_f = 1$. Suppose for a contradiction that the claim is false (for $x \to +\infty$). Then

[A] there exists a $\delta \in (0, 1)$, and a positive, increasing sequence $x_k \to \infty$ with $x_k - x_{k-1} \geq 2$, such that $f(x_k)x_k^{\gamma/2} \geq \delta$ for all $k \in \mathbb{N}$.

Since f is Lipschitz (with $C_f = 1$) and $f(x_k) \geq \delta x_k^{-\gamma/2}$, we can bound the integral

$$\int_{x_k-1}^{x_k+1}|f(x)|\,\mathrm{d}x$$

from below by the area of a triangle having height $\delta x_k^{-\gamma/2}$ and base $2\delta x_k^{-\gamma/2}$. Hence

$$\begin{aligned}\int|f(x)||x|^\gamma\,\mathrm{d}x &\geq \sum_{k=1}^{\infty}\int_{x_k-1}^{x_k+1}|f(x)||x|^\gamma\,\mathrm{d}x \\ &\geq \frac{1}{2}\sum_{k=2}^{\infty}x_k^\gamma\int_{x_k-1}^{x_k+1}|f(x)|\,\mathrm{d}x \geq \frac{\delta^2}{2}\sum_{k=2}^{\infty}1.\end{aligned}$$

But the RHS diverges, contradicting that $\int|f(x)||x|^\gamma\,\mathrm{d}x < \infty$. Hence [A] is false, and thus the claim must be true. $\square$

C. Proofs of Lemmas 6.1 and 6.2.

Proof of Lemma 6.1. By the Lipschitz continuity of f, straightforward calculations yield that

$$|f_{(a_1,b_1)}(x) - f_{(a_2,b_2)}(x)| \le |b_1 - b_2|[1 + b_2(|x| + d_n|a_2|)] + b_1 b_2 d_n |a_1 - a_2|.$$

In particular, taking $(a_1, b_1) = (a, b) \in C_n$ and $(a_2, b_2) = p_n(a, b)$, and noting that $b \lesssim e_n \lesssim n$ and $d_n \lesssim n$, we have

$$\begin{aligned} |f_{(a,b)}(x) - f_{p_n(a,b)}(x)| &\le n^{-\delta}[1 + n(|x| + d_n n^{\gamma})] + d_n n^{2-\delta} \\ &\lesssim n^{2+\gamma-\delta} + n^{1-\delta}|x| \end{aligned}$$

whence

$$\sup_{(a,b)\in C_n} \frac{1}{e_n} \sum_{t=1}^{n} |f_{(a,b)}(x_t) - f_{p_n(a,b)}(x_t)| \le n^{3+\gamma-\delta} + n^{1-\delta} \sum_{t=1}^{n} |x_t|. \tag{C.1}$$

To control the final term, note that by Chebyshev's inequality,

$$\mathbb{P}\left\{ n^{1-\delta} \sum_{t=1}^{n} |x_t| \ge M \right\} \le n^2 \mathbb{P}\left\{ |v_0| \ge \frac{M}{n^{3-\delta}} \right\} \le \frac{n^{2+(3-\delta)p}}{M^p} \mathbb{E}|v_0|^p,$$

for any $p > 0$ such that $\mathbb{E}|v_0|^p < \infty$. Thus, we need only to show that such a $p > 0$ always exists, in order to deduce that δ may be chosen sufficiently large such that the right side of (C.1) is $o_p(1)$.

To that end, we note that for every $p \in (0, 2]$,

$$\mathbb{E}|v_0|^p \lesssim \sum_{k=0}^{\infty} |\phi_k|^p \mathbb{E}|\epsilon_0|^p, \tag{C.2}$$

using Theorem 3 in von Bahr and Esseen (1965) when $p \in (1, 2]$, and the elementary inequality $|x + y|^p \le |x|^p + |y|^p$ when $p \in (0, 1]$. Now $\mathbb{E}|\epsilon_0|^p < \infty$ for every $p \in (0, \alpha)$, by Theorem 2.6.4 in Ibragimov and Linnik (1971), while when $H \ne 1/\alpha$, $\sum_{k=0}^{\infty} |\phi_k|^p < \infty$ for any p such that

$$p(H - 1 - 1/\alpha) < -1 \iff p > \frac{1}{1 - (H - 1/\alpha)} =: \underline{p}.$$

Importantly,

$$\alpha - \underline{p} = \frac{\alpha(1 - H)}{1 - (H - 1/\alpha)} > 0$$

since $H - 1/\alpha < 1$. Thus when $H \ne 1/\alpha$, we may take a $p \in (\underline{p}, \alpha)$ such that the right side of (C.2) is finite; when $H = 1/\alpha$ – in which case $\alpha \in (1, 2]$ – it suffices to take $p = 1$. □

PROOF OF LEMMA 6.2. Let $\{l_k, u_k\}_{k=1}^K$ denote a collection of continuous ϵ-brackets for $\mathscr{F}$; we may certainly take

$$-F(x) \le l_k(x) \le u_k(x) \le F(x)$$

without loss of generality. Indeed, since F is integrable and continuous, we may choose brackets having the property that, for some $M < \infty$

$$l_k(x) = -F(x) \qquad\qquad u_k(x) = F(x)$$

for all $|x| > M$, where M is chosen to be sufficiently large that

$$\int_{[-M,M]^c} F(x) < \epsilon. \tag{C.3}$$

Let $\delta > 0$. Since l_k is continuous on $[-M, M]$, there exists a polynomial l'_k on $[-M, M]$ such that $l'_k(-M) = F(-M)$, $l'_k(M) = F(M)$ and

$$\sup_{x\in[-M,M]} |l_k(x) - l'_k(x)| < \delta.$$

Thus, setting

$$\tilde{l}_k(x) \coloneqq \begin{cases} [l'_k(x) - \delta] \vee [-F(x)] & \text{if } x \in [-M, M], \\ -F(x) & \text{otherwise,} \end{cases}$$

ensures that $\tilde{l}_k(x) \le l_k(x)$ for all $x \in \mathbb{R}$, $\tilde{l}_k \in \mathrm{BIL}_\beta$, and – in view of (C.3) –

$$\int_{\mathbb{R}} [l_k(x) - \tilde{l}_k(x)]\,\mathrm{d}x \le 4M\delta + \epsilon = 2\epsilon$$

where the final equality follows by taking $\delta = \epsilon/4M$.

Constructing $\tilde{u}_k$ in an analogous manner from u_k, we thus obtain a collection $\{\tilde{l}_k, \tilde{u}_k\}_{k=1}^K \subset \mathrm{BIL}_\beta$ of 5ϵ-brackets for $\mathscr{F}$. □

D. Proofs of Lemmas 7.1, 7.2 and 7.5.

PROOF OF LEMMA 7.1. Let $S_n(\theta) \coloneqq \sum_{k=1}^{K_n} M_{nk}(\theta)$ and $\Omega_n \coloneqq \sum_{k=1}^{K_n} \omega_{nk}$. It is easily verified that

$$\left\{\max_{\theta\in\Theta_n} |S_n(\theta)| \ge x\Omega_n\right\} \subseteq \bigcup_{k,\theta} \{|M_{nk}(\theta)| \ge x\omega_{nk}\} \tag{D.1}$$

for every $x \in \mathbb{R}_+$, where $\bigcup_{k,\theta} := \bigcup_{k=1}^{K_n} \bigcup_{\theta \in \Theta_n}$. Define

$$E_n(x) := \bigcup_{k,\theta} \{[M_{nk}(\theta)] \vee \langle M_{nk}(\theta) \rangle \le x\omega_{nk}^2\},$$

and note that by (7.2) and Lemma 2.2.2 in van der Vaart and Wellner (1996),

$$\left\| \max_{k,\theta} \omega_{nk}^{-2} \{[M_{nk}(\theta)] \vee \langle M_{nk}(\theta) \rangle\} \right\|_1 \lesssim \log(K_n \cdot \#\Theta_n) \lesssim \log n$$

where $\max_{k,\theta} := \max_{1 \le k \le K_n} \max_{\theta \in \Theta_n}$, whence by Chebyshev's inequality

$$\mathbb{P}E_n^c(x) = \mathbb{P}\left\{ \max_{k,\theta} \omega_{nk}^{-2} \{[M_{nk}(\theta)] \vee \langle M_{nk}(\theta) \rangle\} > x \right\} \lesssim \frac{\log n}{x}. \tag{D.2}$$

It follows from (D.1) that

$$\left\{ \max_{\theta \in \Theta_n} |S_n(\theta)| \ge x\Omega_n \right\} \cap E_n(x)$$
$$\subseteq \bigcup_{k,\theta} \{|M_{nk}(\theta)| \ge x\omega_{nk},\ [M_{nk}(\theta)] \vee \langle M_{nk}(\theta) \rangle \le x\omega_{nk}^2\},$$

and so by Theorem 2.1 in Bercu and Touati (2008),

$$\begin{aligned} \mathbb{P}&\left\{ \max_{\theta \in \Theta_n} |S_n(\theta)| \ge x\Omega_n \right\} \cap E_n(x) \\ &\lesssim (K_n \cdot \#\Theta_n) \exp\left(-\frac{(x\omega_{nk})^2}{4x\omega_{nk}^2} \right) \lesssim n^C \exp\left(-\frac{x}{4} \right). \end{aligned} \tag{D.3}$$

Together, (D.2) and (D.3) yield

$$\begin{aligned} \mathbb{P}\left\{ \max_{\theta \in \Theta_n} |S_n(\theta)| \ge x\Omega_n \right\} &\le \mathbb{P}\left\{ \max_{\theta \in \Theta_n} |S_n(\theta)| \ge x\Omega_n \right\} \cap E_n(x) + \mathbb{P}E_n^c(x) \\ &\lesssim n^C \exp\left(-\frac{x}{4} \right) + \frac{\log n}{x}. \end{aligned}$$

Setting $x = a \log n$ for $a > 0$ sufficiently large, we thus have

$$\mathbb{P}\left\{ \max_{\theta \in \Theta_n} |S_n(\theta)| \ge x\Omega_n \right\} \lesssim n^{C - a/4} + a^{-1} \to 0$$

as $n \to \infty$ and then $a \to \infty$. □

Proof of Lemma 7.2. In both cases, the reverse implication is trivial. Regarding the forward implication, in case (i) this follows immediately from the fact that

$$\mathbb{E}\tau_1\left(\frac{|X|}{q\sigma}\right) = \mathbb{E}\sum_{p=1}^{\infty}\frac{|X|^p}{p!\cdot(q\sigma)^p} = \sum_{p=1}^{\infty}\frac{\mathbb{E}|X|^p}{p!\cdot(q\sigma)^p} \leq \sum_{p=1}^{\infty}\left(\frac{C}{q}\right)^p \leq 1$$

for $q > 0$ sufficiently large. In order to prove (ii), note that by Hölder's inequality, for any $p \in \mathbb{N}$,

$$\mathbb{E}|X|^{2p/3} \leq (\mathbb{E}|X|^{2p})^{1/3}$$

and that by Stirling's formula (Rudin, 1976, 8.22),

$$\frac{(3p)!}{(p!)^3} \asymp 3^{3p}\frac{(6\pi p)^{1/2}}{(2\pi p)^{3/2}} \lesssim 3^{3p}.$$

Hence

$$\begin{aligned}
\mathbb{E}\tau_{2/3}\left(\frac{|X|}{q\sigma}\right) &\leq \frac{\mathbb{E}|X|}{q\sigma} + \sum_{p=1}^{\infty}\frac{\mathbb{E}|X|^{2p/3}}{p!\cdot(q\sigma)^{2p/3}} \\
&\leq \frac{(\mathbb{E}|X|^2)^{1/2}}{q\sigma} + \sum_{p=1}^{\infty}\frac{(\mathbb{E}|X|^{2p})^{1/3}}{p!\cdot(q\sigma)^{2p/3}} \\
&\lesssim \frac{1}{q} + \sum_{p=1}^{\infty}\left(\frac{C}{q^{2/3}}\right)^p \\
&\leq 1
\end{aligned}$$

for $q > 0$ sufficiently large. □

Proof of Lemma 7.5. Recalling the definitions given at the start of Section 7.2, it is clear that

$$\sup_{f\in\mathscr{G}}\varsigma_n(\beta,f) + \sum_{k=0}^{n-1}\sup_{f\in\mathscr{G}}\sigma_{nk}(\beta,f)$$

may be bounded by

$$\|\mathscr{G}\|_\infty + e_n^{1/2}(\|\mathscr{G}\|_1 + \|\mathscr{G}\|_2) + \|\mathscr{G}\|_{[\beta]}\left[\sum_{k=1}^{n} d_k^{-(1+\beta)} + e_n^{1/2}\sum_{k=1}^{n-1} k^{-1/2} d_k^{-(1+2\beta)/2}\right].$$

The claimed bound follows since, for some $C < \infty$ depending on β,

$$\begin{aligned}\sum_{k=1}^{n} d_k^{-(1+\beta)} + e_n^{1/2} \sum_{k=1}^{n-1} k^{-1/2} d_k^{-(1+2\beta)/2} &\le C(e_n d_n^{-\beta} + e_n^{1/2} n^{1/2} d_n^{-(1+2\beta)/2}) \\ &\le C e_n d_n^{-\beta}\end{aligned}$$

by Karamata's theorem, noting in particular that $\{k^{-1/2} d_k^{-(1+2\beta)/2}\}$ is regularly varying with index

$$-\frac{1}{2} - \left(\frac{1}{2} + \beta\right) H = -1 + H\left(\frac{1-H}{2H} - \beta\right) > -1,$$

since $\beta < \overline{\beta}_H \le \frac{1-H}{2H}$. □

E. Proof of (8.1). For $-\infty < a < b < \infty$, the same argument as appears in the proof of Lemma 8 in Jeganathan (2008) yields

$$\begin{aligned}\mu_n(b) - \mu_n(a) &= \frac{1}{n} \sum_{t=1}^{n} \mathbf{1}\{a < d_n^{-1} x_t \le b\} \\ &\rightsquigarrow \int_0^1 \mathbf{1}\{a < X(r) \le b\} \, \mathrm{d}r \\ &= \int_a^b \mathcal{L}(x) \, \mathrm{d}x \\ &= \mu(b) - \mu(a)\end{aligned}$$

where the penultimate equality follows by (2.8). In consequence, for any $a > 0$,

$$\mu_n(-a) + [1 - \mu_n(a)] = 1 - [\mu_n(a) - \mu_n(-a)] \rightsquigarrow 1 - [\mu(a) - \mu(-a)] \overset{\text{a.s.}}{\to} 0$$

as $n \to \infty$ and then $a \to \infty$. Hence $\mu_n(-a) \overset{p}{\to} 0$ as $n \to \infty$ and then $a \to \infty$. Similarly,

$$\mu_n(b) - \mu_n(-a) \rightsquigarrow \mu(b) - \mu(-a) \overset{\text{a.s.}}{\to} \mu(b)$$

as $n \to \infty$ and then $a \to \infty$. Since weak convergence is metrisable, it follows that we may choose $a = a_n \to \infty$ sufficiently slowly such that

$$\mu_n(b) = [\mu(b) - \mu(-a_n)] + \mu(-a_n) \rightsquigarrow \mu(b)$$

as $n \to \infty$. Thus $\mu_n \rightsquigarrow_{\text{fdd}} \mu$; because μ and μ_n are monotone and continuous, with a uniformly bounded range, weak convergence on $\ell_\infty(\mathbb{R})$ follows automatically (see the proof of Lemma 2.11 in van der Vaart, 1998).

F. Proofs of results from Section 9.

Verification of (9.1). It suffices to prove the result when $y_0 = 0$. Since the Fourier transform is an isometry on L^2 (Stein and Weiss, 1971, Thm. I.2.3), $f_k \to f$ on L^2, where

$$f_k(x) := \frac{1}{2\pi}\int_{-k}^{k} \mathrm{e}^{-\mathrm{i}\lambda x}\hat{f}(\lambda)\,\mathrm{d}\lambda.$$

Since $\mathbf{1}_{[-k,k]}(\lambda)\hat{f}(\lambda) \in L^1$, it follows that for every $k \in \mathbb{N}$

$$\mathbb{E}f_k(Y) = \frac{1}{2\pi}\int_{-k}^{k} \hat{f}(\lambda)\mathbb{E}\Big[\mathrm{e}^{-\mathrm{i}\lambda' Y}\Big]\,\mathrm{d}\lambda. \tag{F.1}$$

By assumption, Y has an integrable characteristic function, and thus a bounded density π_Y, by the inversion formula (Feller, 1971, Thm XV.3.3). Hence, by the Cauchy-Schwarz inequality,

$$\begin{aligned}|\mathbb{E}f_k(Y) - \mathbb{E}f(Y)| &\le \big(\mathbb{E}|f_k(Y) - f(Y)|^2\big)^{1/2}\\ &\le \|\pi_Y\|_\infty^{1/2}\left(\int_{\mathbb{R}}|f_k(y) - f(y)|^2\,\mathrm{d}y\right)^{1/2} \to 0,\end{aligned}$$

as $k \to \infty$. A further application of the Cauchy-Schwarz inequality (noting $\hat{f} \in L^2$) yields

$$\begin{aligned}&\left|\int_{\{|\lambda|>k\}} \hat{f}(\lambda)\mathbb{E}[\mathrm{e}^{-\mathrm{i}\lambda' Y}]\,\mathrm{d}\lambda\right|\\ &\qquad\le \left(\int_{\{|\lambda|>k\}}|\hat{f}(\lambda)|^2\,\mathrm{d}\lambda\right)^{1/2}\left(\int_{\{|\lambda|>k\}}|\psi_Y(\lambda)|^2\,\mathrm{d}\lambda\right)^{1/2} \to 0\end{aligned}$$

as $k \to \infty$. Letting $k \to \infty$ on both sides of (F.1) then gives the result. □

Proof of Lemma 9.1. (i) is immediate from $|\hat{f}(\lambda)| \le \|f\|_1$ and the definition of $\|\cdot\|_{[\beta]}$. Regarding (ii), in this case $\hat{f}(0) = \int f = 0$. Therefore, using the elementary inequality $|\mathrm{e}^{\mathrm{i}z} - 1| \le 2^{1-\beta}|z|^\beta$ (for $z \in \mathbb{R}$), we find that

$$\begin{aligned}|\hat{f}(\lambda)| &= |\hat{f}(\lambda) - \hat{f}(0)\mathrm{e}^{-\mathrm{i}\lambda y}|\\ &\le \int_{\mathbb{R}}|f(x)||\mathrm{e}^{\mathrm{i}\lambda(x+y)} - 1|\,\mathrm{d}x \le 2^{1-\beta}|\lambda|^\beta\int_{\mathbb{R}}|f(x-y)||x|^\beta\,\mathrm{d}x.\end{aligned}$$

for every $y \in \mathbb{R}$. Finally, for f as in (iii)

$$
\begin{aligned}
|\hat{f}(\lambda)| &= |g(\hat{\lambda})||\mathrm{e}^{\mathrm{i}\lambda a_1} - \mathrm{e}^{\mathrm{i}\lambda a_2}| \\
&= |g(\hat{\lambda})||1 - \mathrm{e}^{\mathrm{i}\lambda(a_1 - a_2)}| \leq 2^{1-\beta}\|g\|_1 |\lambda|^\beta |a_1 - a_2|^\beta.
\end{aligned}
$$

□

VERIFICATION OF (9.4). When $H = 1/\alpha$, the result follows from arguments given in Wang and Phillips (2009): see their (7.14), in particular. Otherwise, first note that by Karamata's theorem,

$$a_k = \sum_{l=0}^{k} \phi_l \sim \sum_{l=1}^{k} l^{H-1-1/\alpha}\pi_l \asymp k^{H-1/\alpha}\pi_k = c_k$$

when $H > 1/\alpha$, and

$$a_k = \sum_{l=0}^{k} \phi_l = -\sum_{l=k+1}^{\infty} \phi_l \sim \sum_{l=k+1}^{\infty} l^{H-1-1/\alpha}\pi_l \asymp k^{H-1/\alpha}\pi_k = c_k$$

when $H < 1/\alpha$, since $\sum_{l=0}^{\infty} \phi_l = 0$. In the first case, setting $\delta \coloneqq \frac{1}{2}(H - 1/\alpha)$, it follows from Potter's inequality that we may choose k_0 sufficiently large that

$$2^{-3\delta} \lesssim \left(\frac{l}{k}\right)^{3\delta} \lesssim \frac{c_l}{c_k} \lesssim \left(\frac{l}{k}\right)^{\delta} \leq 1$$

for all $k \geq k_0$ and $\lfloor k/2 \rfloor \leq l \leq k$. Since $a_k \asymp c_k$, this yields the stated result, which follows also when $H < 1/\alpha$ by a strictly analogous argument. □

The proof of Lemma 9.2 requires the following two results. The first is an immediate consequence of (9.4), and the fact that ϵ_0 is in the domain of attraction of a stable distribution, with $\psi \in L^{p_0}$.

LEMMA F.1. *There exist $\eta_0, \gamma_0 \in (0, \infty)$ such that*

$$\sup_{k \geq k_0+1} \sup_{\lfloor k/2 \rfloor \leq l \leq k} |\psi(c_k^{-1} a_l \lambda)| \leq \begin{cases} \mathrm{e}^{-\gamma_0 |\lambda|^\alpha G(\lambda)} & \text{if } |\lambda| \leq \eta_0, \\ \mathrm{e}^{-\gamma_0} & \text{if } |\lambda| > \eta_0. \end{cases}$$

LEMMA F.2. *Let $k \geq k_0 + 1$, $p \in [0, 5]$, $q \in (0, 2]$ and $z_1, z_2 \in \mathbb{R}_+$. Then there exists a $\gamma_1 > 0$ such that*

$$\int_{\mathbb{R}} (z_1 |\lambda|^p \wedge z_2) \prod_{l \in \mathcal{K}} |\psi(a_l \lambda)| \,\mathrm{d}\lambda \lesssim z_1 d_k^{-(p+1)} + z_2 \mathrm{e}^{-\gamma_1 k} \tag{F.2}$$

and if $F(u) \asymp G^{p/\alpha}(u)$ *as* $u \to 0$,

$$\int_{\mathbb{R}} (z_1|a_k|^p|\lambda|^{p+q}F(a_k\lambda) \wedge z_2) \prod_{l\in\mathcal{K}} |\psi(a_l\lambda)| \,\mathrm{d}\lambda \lesssim z_1 k^{-p/\alpha} d_k^{-(1+q)} + z_2 \mathrm{e}^{-\gamma_1 k} \tag{F.3}$$

uniformly over all $\mathcal{K} \subseteq \{\lfloor k/2\rfloor + 1, \ldots, k\}$ *with* $\#\mathcal{K} \geq \lceil k/4\rceil$.

Proof of Lemma F.1. As noted in (9.5) above, there exist $\eta, \gamma \in (0,\infty)$ such that

$$|\psi(\lambda)| \leq \mathrm{e}^{-\gamma|\lambda|^\alpha G(\lambda)}$$

whenever $|\lambda| \leq \eta$. Defining

$$\mathcal{I} \coloneqq \{(k,l) \mid k \geq k_0 + 1, \lfloor k/2\rfloor \leq l \leq k\}$$

it follows from (9.4) that

$$|\lambda| \leq \overline{a}^{-1}\eta \implies \sup_{(k,l)\in\mathcal{I}} |c_k^{-1}a_l\lambda| \leq \eta.$$

Let $\eta_0 \coloneqq \overline{a}^{-1}\eta$ and $r(\lambda) \coloneqq |\lambda|^\alpha G(\lambda)$. Then whenever $|\lambda| \leq \eta_0$,

$$\sup_{(k,l)\in\mathcal{I}} |\psi(c_k^{-1}a_l\lambda)| \leq \exp\biggl(-\inf_{(k,l)\in\mathcal{I}} r(c_k^{-1}a_l\lambda)\biggr) \leq \exp\biggl(-\inf_{a\in[\underline{a},\overline{a}]} r(a\lambda)\biggr),$$

using (9.4) again. Since r is regularly varying at zero,

$$\inf_{a\in[\underline{a},\overline{a}]} r(a\lambda) = r(\lambda) \inf_{a\in[\underline{a},\overline{a}]} \frac{r(a\lambda)}{r(\lambda)} \leq C_0 r(\lambda)$$

for some $C_0 \in (0,\infty)$, for all $|\lambda| \leq \eta_0$. Hence

$$\sup_{(k,l)\in\mathcal{I}} |\psi(c_k^{-1}a_l\lambda)| \leq \exp(-\gamma C_0|\lambda|^\alpha G(\lambda))$$

for all $|\lambda| \leq \eta_0$.

Next, note that since $\psi \in L^{p_0}$ and $\|\psi\|_\infty \leq 1$, we have $\varphi \coloneqq |\psi|^{2^k} \in L^1$ for a $k \in \mathbb{N}$ chosen such that $2^k \geq p_0$. Thus φ is the characteristic function of a random variable having bounded continuous density (Feller, 1971, corollaries to Lem. XV.1.2 and Thm XV.3.3), and so by the Riemann-Lebesgue lemma

$$\limsup_{|\lambda|\to\infty} |\psi(\lambda)| = \biggl(\limsup_{|\lambda|\to\infty} |\varphi(\lambda)|\biggr)^{2^{-k}} = 0$$

(Feller, 1971, Lem. XV.3.3). Further, $\varphi \in L^1$ cannot be periodic, and so $|\varphi(\lambda)| < 1$ for all $\lambda \neq 0$ (Feller, 1971, Lem. XV.1.4). Since φ is necessarily continuous, it follows that

$$\sup_{|\lambda|>\delta} |\psi(\lambda)| = \left(\sup_{|\lambda|>\delta} |\varphi(\lambda)| \right)^{2^{-k}} < 1$$

for every $\delta > 0$. Noting that

$$|\lambda| > \eta_0 \implies \inf_{(k,l)\in\mathcal{I}} |c_k^{-1} a_l \lambda| > \underline{a}\eta_0$$

it follows that

$$\sup_{|\lambda|>\eta_0} \sup_{(k,l)\in\mathcal{I}} |\psi(c_k^{-1} a_l \lambda)| \leq \sup_{|\lambda|>\underline{a}\eta_0} |\psi(\lambda)| \leq \mathrm{e}^{-C_1}$$

for some $C_1 \in (0, \infty)$. Setting $\gamma_0 \coloneqq \gamma C_0 \wedge C_1$ thus yields the result. □

Proof of Lemma F.2. We shall only give the proof of (F.3): the proof of (F.2) is strictly analogous, albeit somewhat simpler. Letting $K \coloneqq \#\mathcal{K}$ and $h_k(\lambda) \coloneqq (z_1 |a_k|^p |\lambda|^{p+q} F(a_k\lambda) \wedge z_2)$, we first note that by repeated applications of Hölder's inequality (see Jeganathan, 2008, Lem. 7) and then a change of variables,

$$\begin{aligned}
\text{(F.4)} \quad \int_{\mathbb{R}} h_k(\lambda) \prod_{l\in\mathcal{K}} |\psi(a_l\lambda)| \,\mathrm{d}\lambda &\leq \prod_{l\in\mathcal{K}} \left(\int_{\mathbb{R}} h_k(\lambda) |\psi(a_l\lambda)|^K \,\mathrm{d}\lambda \right)^{1/K} \\
&\leq \max_{l\in\mathcal{K}} \int_{\mathbb{R}} h_k(\lambda) |\psi(a_l\lambda)|^K \,\mathrm{d}\lambda \\
\text{(F.5)} \quad &= c_k^{-1} \max_{l\in\mathcal{K}} \int_{\mathbb{R}} h_k(c_k^{-1}\lambda) |\psi(c_k^{-1} a_l \lambda)|^K \,\mathrm{d}\lambda.
\end{aligned}$$

We proceed by handling this integral separately on the domains $[-\eta_0, \eta_0]$ and $[-\eta_0, \eta_0]^c$. In the first case, we use $h_k(\lambda) \leq z_1 |a_k|^p |\lambda|^{p+q} F(a_k\lambda)$, and are thus led to consider

$$\begin{aligned}
\text{(F.6)} \quad & c_k^{-1} \max_{l\in\mathcal{K}} \int_{[-\eta_0,\eta_0]} h_k(c_k^{-1}\lambda) |\psi(c_k^{-1} a_l \lambda)|^K \,\mathrm{d}\lambda \\
&= c_k^{-(1+p+q)} |a_k|^p \int_{[-\eta_0,\eta_0]} |\lambda|^{p+q} F(c_k^{-1} a_k \lambda) |\psi(c_k^{-1} a_l \lambda)|^K \,\mathrm{d}\lambda \\
&\lesssim c_k^{-(1+q)} \int_{[-\eta_0,\eta_0]} |\lambda|^{p+q} F(c_k^{-1} a_k \lambda) \mathrm{e}^{-\gamma_0 K |\lambda|^\alpha G(\lambda)} \,\mathrm{d}\lambda,
\end{aligned}$$

using (9.4) and Lemma F.1. Now let $r(\lambda) \coloneqq |\lambda|^{\alpha} G(\lambda)$; as noted in Jeganathan (2004, p. 1774), the sequence $b_n \coloneqq n^{1/\alpha}\varrho_n$ satisfies

$$r(b_n^{-1}) = b_n^{-\alpha} G(b_n^{-1}) \sim n^{-1} \tag{F.7}$$

as $n \to \infty$. Therefore, setting $\mu = \lambda b_K$, we obtain

$$K \cdot r(\lambda) = K \cdot r(\mu b_K^{-1}) \gtrsim \frac{r(\mu b_K^{-1})}{r(b_K^{-1})} \gtrsim |\mu|^{\alpha/2}$$

since r is regularly varying at zero, with index α. Further, recalling (9.4), we have

$$\begin{aligned} F(c_k^{-1} a_k b_K^{-1} \mu) &= F(c_k^{-1} a_k b_K^{-1}) \frac{F(c_k^{-1} a_k b_K^{-1} \mu)}{F(c_k^{-1} a_k b_K^{-1})} \\ &\lesssim G^{p/\alpha}(b_K^{-1}) |\mu|^{-\epsilon} \\ &\lesssim K^{-p/\alpha} b_K^{p} |\mu|^{-\epsilon} \end{aligned}$$

for any $\epsilon > 0$, using the fact that F is slowly varying, $F(u) \asymp G^{p/\alpha}(u)$ as $u \to 0$, and (F.7). Hence, by a change of variables, the right side of (F.6) may be bounded by

$$\begin{aligned} c_k^{-(1+q)} b_K^{-(1+p+q)} &\int_{[-\eta_0 b_K, \eta_0 b_K]} |\mu|^{p+q} F(c_k^{-1} a_k b_K^{-1} \mu) \mathrm{e}^{-\gamma_0 K \cdot r(\mu b_K^{-1})} \,\mathrm{d}\mu \\ &\lesssim c_k^{-(1+q)} K^{-p/\alpha} b_K^{-(1+q)} \int_{\mathbb{R}} |\mu|^{p+q-\epsilon} \mathrm{e}^{-C|\mu|^{\alpha/2}} \,\mathrm{d}\mu \\ &\lesssim c_k^{-(1+q)} k^{-p/\alpha} b_k^{-(1+q)} \\ &= k^{-p/\alpha} d_k^{-(1+q)} \end{aligned} \tag{F.8}$$

since $\lceil k/4 \rceil \leq K \leq k$, and $b_k c_k = n^{1/\alpha} c_k \varrho_k = d_k$.

Since $h_k(\lambda) \leq z_2$, to complete the proof we need only to consider

$$c_k^{-1} \int_{[-\eta_0, \eta_0]^c} |\psi(c_k^{-1} a_l \lambda)|^K \,\mathrm{d}\lambda.$$

Thence, taking a $\mathcal{K}' \subseteq \mathcal{K}$ with $\#\mathcal{K}' = \lceil k/8 \rceil$,

$$\begin{aligned} c_k^{-1} \int_{[-\eta_0, \eta_0]^c} |\psi(c_k^{-1} a_l \lambda)|^K \,\mathrm{d}\lambda &\leq c_k^{-1} \mathrm{e}^{-\gamma_0 (K - \lceil k/8 \rceil)} \int_{\mathbb{R}} |\psi(c_k^{-1} a_l \lambda)|^{\lceil k/8 \rceil} \,\mathrm{d}\lambda \\ &\lesssim \mathrm{e}^{-\gamma_1 k} \end{aligned} \tag{F.9}$$

for any $\gamma_1 \in (0, \gamma_0/8)$; note that the right hand integral is finite because $\psi \in L^{p_0}$, and $\lceil k/8 \rceil \geq k_0/8 \geq p_0$, and again the uniform boundedness of $c_k^{-1} a_l$ follows from (9.4). Thus (F.5), (F.8) and the preceding yield

$$\int_{\mathbb{R}} h_k(\lambda) \prod_{l \in \mathcal{K}} |\psi(a_l \lambda)| \, \mathrm{d}\lambda \lesssim z_1 k^{-p/\alpha} d_k^{-(1+q)} + z_2 \mathrm{e}^{-\gamma_1 k}. \tag{F.10}$$

□

Proof of Lemma 9.2. Recall from (9.3) the decompositions

$$x'_{t+1,t+k,t+k} = \sum_{l=0}^{k-1} a_l \epsilon_{t+k-l} \qquad x'_{t-s+1,t-1,t+k} = \sum_{l=k+1}^{k+s-1} a_l \epsilon_{t+k-l}.$$

Thence

$$|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \leq \prod_{l=\lfloor k/2 \rfloor+1}^{k-1} |\psi(-\lambda a_l)|$$

whereupon (i) follows immediately from Lemma F.2. The proof of (ii) requires a slight modification of the arguments used to prove Lemma F.2. Since

$$|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t-s+1,t-1,t+k}}| \leq \prod_{l=k+\lfloor s/2 \rfloor}^{k+s-1} |\psi(-\lambda a_l)|,$$

the problem reduces to one of controlling

$$c_{k+s}^{-1} \max_{l \in \mathcal{K}} \int_{\mathbb{R}} |\psi(c_{k+s}^{-1} a_l \lambda)|^K \, \mathrm{d}\lambda,$$

as per (F.5) above, where $K \coloneqq \#\mathcal{K}$ for

$$\mathcal{K} \coloneqq \{l \in \mathbb{N} \mid k + \lfloor s/2 \rfloor \leq l \leq k+s-1\}.$$

Thus in this case, the same arguments as which led to (F.8) and (F.9) now yield

$$c_{k+s}^{-1} \max_{l \in \mathcal{K}} \int_{\mathbb{R}} |\psi(c_{k+s}^{-1} a_l \lambda)|^K \, \mathrm{d}\lambda \lesssim c_{k+s}^{-1} (b_K^{-1} + \mathrm{e}^{-\gamma_1 K}) \lesssim \frac{c_K}{c_{k+s}} d_K^{-1} \lesssim \frac{c_s}{c_{k+s}} d_s^{-1},$$

since $\{c_k\}$ and $\{d_k\}$ are regularly varying, and $s/3 \leq K \leq 2s/3$. □

PROOF OF LEMMA 9.3.
(i). Recall from (9.2) the decomposition

$$x_{t+k} = x^*_{t,t+k} + x'_{t+1,t+k,t+k}. \tag{F.11}$$

Let $\tilde{f}$ denote the Fourier transform of $x \mapsto |f(x)|$, noting that $|\tilde{f}(\lambda)| \leq \|f\|_1$. Thence by Fourier inversion and Lemma 9.2(i),

$$\begin{aligned}
\mathbb{E}_t|f(x_{t+k})| &= \left| \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{f}(\lambda) \mathrm{e}^{-\mathrm{i}\lambda x^*_{t,t+k}} \mathbb{E}[\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}] \,\mathrm{d}\lambda \right| \\
&\lesssim \|f\|_1 \int_{\mathbb{R}} |\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda \\
&\lesssim \|f\|_1 d_k^{-1}.
\end{aligned}$$

(ii). By (F.11), Fourier inversion, Lemma 9.1(i) and then Lemma 9.2(i),

$$\begin{aligned}
|\mathbb{E}_t f(x_{t+k})| &\lesssim \int_{\mathbb{R}} |\hat{f}(\lambda)||\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda \\
&\leq \int_{\mathbb{R}} (\|f\|_{[\beta]}|\lambda|^\beta \wedge \|f\|_1)|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda \\
&\lesssim \|f\|_{[\beta]} d_k^{-(1+\beta)} + \|f\|_1 \mathrm{e}^{-\gamma_1 k}.
\end{aligned}$$

□

PROOF OF LEMMA 9.4. Using Jensen's inequality and $|\mathrm{e}^{\mathrm{i}x} - 1| \lesssim |x| \wedge 1$, we obtain that for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}|\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0}|^2 &= \mathbb{E}|(\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0} - 1) - \mathbb{E}(\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0} - 1)|^2 \\
&\leq 2\mathbb{E}\Big[|\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0} - 1|^2 + (\mathbb{E}|\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0} - 1|)^2\Big] \\
&\leq 2\mathbb{E}|\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0} - 1|^2 \\
&\lesssim \mathbb{E}[|\lambda\epsilon_0|^2 \wedge 1].
\end{aligned} \tag{F.12}$$

To obtain a bound for the final term, let F denote the distribution function of ϵ_0; following Ibragimov and Linnik (1971, Sec. 2.6), we define

$$\chi(x) \coloneqq 1 - F(x) + F(-x) \sim x^\alpha l(x)$$

for $x > 0$, where l is slowly varying at infinity, and

$$L(x) \coloneqq -\int_0^x u^2 \,\mathrm{d}\chi(u).$$

Then

$$\begin{aligned}
\mathbb{E}[(\lambda\epsilon_0)^2 \wedge 1] &= \left[\int_{[-\lambda^{-1},\lambda^{-1}]} + \int_{[-\lambda^{-1},\lambda^{-1}]^c}\right]((\lambda\epsilon_0)^2 \wedge 1)\,\mathrm{d}F(\epsilon) \\
&= \lambda^2 \int_{[-\lambda^{-1},\lambda^{-1}]} \epsilon^2\,\mathrm{d}F(\epsilon) + \int_{[-\lambda^{-1},\lambda^{-1}]^c} \mathrm{d}F(\epsilon) \\
&= -\lambda^2 \int_0^{\lambda^{-1}} \epsilon^2\,\mathrm{d}\chi(\epsilon) + 1 - F(\lambda^{-1}) + F(-\lambda^{-1}) \\
&= \lambda^2 L(\lambda^{-1}) + \chi(\lambda^{-1}).
\end{aligned}$$

Now by Theorem 2.6.3 and (2.6.24) in Ibragimov and Linnik (1971), we have

$$\chi(\lambda^{-1}) = \lambda^2 \cdot \lambda^{-2}\chi(\lambda^{-1}) \asymp \lambda^2 L(\lambda^{-1}) \tag{F.13}$$

when $\alpha \in (0,2)$, and

$$\chi(\lambda^{-1}) \lesssim \lambda^2 L(\lambda^{-1})$$

when $\alpha = 2$, for λ in a neighbourhood of zero. Thus, defining

$$\tilde{G}(\lambda) \coloneqq |\lambda|^{2-\alpha} L(\lambda^{-1})$$

it follows that

$$\mathbb{E}[(\lambda\epsilon_0)^2 \wedge 1] \lesssim |\lambda|^\alpha \tilde{G}(\lambda).$$

That $\tilde{G}(\lambda) \asymp G(\lambda)$ as $\lambda \to 0$ is evident from (F.13) and the proof of Theorem 2.6.5 in Ibragimov and Linnik (1971): see their (2.6.38) and (2.6.39), in particular.

Since the left side of (F.12) is also bounded by 4, we thus have

$$\mathbb{E}|\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda\epsilon_0}|^2 \lesssim |\lambda|^\alpha \tilde{G}(\lambda) \wedge 1.$$

Hence, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\vartheta(z_1,z_2) &\le \left(\mathbb{E}|\mathrm{e}^{-\mathrm{i}z_1\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}z_1\epsilon_0}|^2\right)^{1/2}\left(\mathbb{E}|\mathrm{e}^{-\mathrm{i}z_2\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}z_2\epsilon_0}|^2\right)^{1/2} \\
&\lesssim [|z_1|^\alpha \tilde{G}(z_1) \wedge 1]^{1/2}[|z_2|^\alpha \tilde{G}(z_2) \wedge 1]^{1/2}.
\end{aligned}$$

□

G. Proof of (10.1). Note first that

$$\begin{aligned}\mathbb{E}|\mathcal{V}_{nk}f|^p &= \mathbb{E}\left(\sum_{t=1}^{n-k}\mathbb{E}_{t-1}\xi_{kt}^2 f\right)^p \\ &\le p! \cdot \sum_{t_1=1}^{n-k}\cdots\sum_{t_{p-1}=t_{p-2}}^{n-k}\sum_{t_p=t_{p-1}}^{n-k} \\ &\qquad \mathbb{E}\Big[\mathbb{E}_{t_1-1}(\xi_{kt_1}^2 f)\cdots\mathbb{E}_{t_{p-1}-1}(\xi_{kt_{p-1}}^2 f)\cdot\mathbb{E}_{t_p-1}(\xi_{kt_p}^2 f)\Big]\end{aligned}$$

and that by the law of iterated expectations, when $t_{p-1} < t_p$,

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{E}_{t_1-1}(\xi_{kt_1}^2 f)\cdots\mathbb{E}_{t_{p-1}-1}(\xi_{kt_{p-1}}^2 f)\cdot\mathbb{E}_{t_p-1}(\xi_{kt_p}^2 f)\Big] \\ &\quad = \mathbb{E}\Big[\mathbb{E}_{t_1-1}(\xi_{kt_1}^2 f)\cdots\mathbb{E}_{t_{p-1}-1}(\xi_{kt_{p-1}}^2 f)\cdot\mathbb{E}_{t_{p-1}-1}(\xi_{kt_p}^2 f)\Big] \\ &\qquad \le \|\mathbb{E}_{t_{p-1}-1}\xi_{kt_p}^2 f\|_\infty \mathbb{E}\Big[\mathbb{E}_{t_1-1}(\xi_{kt_1}^2 f)\cdots\mathbb{E}_{t_{p-1}-1}(\xi_{kt_{p-1}}^2 f)\Big].\end{aligned}$$

When $t_p = t_{p-1}$, we may instead use

$$\begin{aligned}(\mathbb{E}_{t_p-1}\xi_{kt}^2 f)^2 &\le \|\mathbb{E}_{t_{p-1}-1}\xi_{kt_{p-1}}^2 f\|_\infty \mathbb{E}_{t_{p-1}-1}\xi_{kt_{p-1}}^2 f \\ &\le \|\xi_{kt_{p-1}}^2 f\|_\infty \mathbb{E}_{t_{p-1}-1}\xi_{kt_{p-1}}^2 f.\end{aligned}$$

Thus $\mathbb{E}|\mathcal{V}_{nk}f|^p$ is bounded by

$$\begin{aligned}&p! \cdot \sum_{t_1=1}^{n-k}\cdots\sum_{t_{p-1}=t_{p-2}}^{n-k}\mathbb{E}\Big[\mathbb{E}_{t_1-1}(\xi_{kt_1}^2 f)\cdots\mathbb{E}_{t_{p-1}-1}(\xi_{kt_{p-1}}^2 f)\Big] \\ &\qquad \cdot\left(\|\xi_{kt_{p-1}}^2 f\|_\infty + \sum_{s=1}^{n-k-t_{p-1}}\|\mathbb{E}_{t_{p-1}-1}\xi_{k,t_{p-1}+s}^2 f\|_\infty\right).\end{aligned}$$

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H. List of notation.

Greek and Roman symbols. Listed in (Roman) alphabetical order. Greek symbols are listed according to their English names: thus Ω, as 'omega', appears before ξ, as 'xi'.

a_i	partial sum of $\{\phi_i\}$, $a_i \coloneqq \sum_{j=0}^{i} \phi_j$	Sec. 9
α	index of domain of attraction of ϵ_0	Ass. 1(i)
$\overline{\beta}_H$	upper bound for β, depends on H	(4.6)
BI	bounded and integrable functions on $\mathbb{R}$	Sec. 1.1
BI_β	$f \in \mathrm{BI}$ with $\int \lvert f(x)\rvert \lvert x\rvert^\beta \,\mathrm{d}x < \infty$	(3.2)
$\mathrm{BI}_{[\beta]}$	$f \in \mathrm{BI}$ with $\lVert f\rVert_{[\beta]} < \infty$	Sec. 4.2
BIL_β	Lipschitz functions in BI_β	Sec. 3
c_n	norming sequence	(2.3)
C	generic constant	Sec. 1.1
d_n	norming sequence used to define X_n	(2.4)
$\delta_n(\beta, \mathscr{F})$	appears in Proposition 4.2	(4.8)
e_n	norming sequence used to define $\mathcal{L}_n^f$	(2.4)
ϵ_t	i.i.d. sequence	Ass. 1(i)
$\mathbb{E}_t$	expectation conditional on $\mathcal{F}_{-\infty}^t$	Sec. 7.1
$\mathcal{F}_s^t$	σ-field generated by $\{\epsilon_r\}_{r=s}^t$	Sec. 7.1
$\mathscr{F}$	subset of BI	Ass. 3
G	specific slowly varying function	(9.5)
h, h_n	bandwidth parameter (or sequence)	(3.1), (5.1)
$\underline{h}_n, \overline{h}_n$	lower and upper bounds defining $\mathscr{H}_n$	Ass. 2
H	sets the decay rate of ϕ_k as $k \to \infty$	Ass. 1(ii)
$\mathscr{H}_n$	set of allowable bandwidths	Ass. 2
$\ell_{\mathrm{ucc}}(Q)$	bounded functions on Q, with ucc topology	Sec. 1.1
$\ell_\infty(Q)$	bounded functions on Q, with uniform topology	Sec. 1.1
$\mathcal{L}$	local time of X	Rem. 2.5
$\mathcal{L}_n^f$	sample estimate of local time	(3.1)
$\mathcal{M}_{nk} f$	martingale components in decomposition of $\mathcal{S}_n f$	(7.4)
$\mathcal{N}_n f$	remainder from decomposition of $\mathcal{S}_n f$	(7.4)
$N_{[]}^*(\epsilon, \mathscr{F})$	number of continuous ϵ-brackets to cover $\mathscr{F}$	Sec. 3
Ω	sample space	Sec. 8
ϕ_k	coefficients defining the linear process v_t	Ass. 1(ii)

φ	triangular kernel function	(4.2)
ψ	characteristic function of ϵ_0	Ass. 1(i)
ϱ_n	norming sequence	(2.2)
$\mathcal{S}_n$	summation operator, $\mathcal{S}_n f \coloneqq \sum_{t=1}^n f(x_t)$	(4.4)
$\tau_{2/3}$	specific convex and increasing function	(4.7)
τ_1	function $x \mapsto \mathrm{e}^x - 1$	Sec. 7
v_t	linear process built from $\{\epsilon_t\}$	(2.1)
x_t	partial sum of $\{v_t\}$	(2.1)
$x^*_{s,t}$	$\mathcal{F}^s_{-\infty}$-measurable component of x_t	(9.2)
$x'_{s,r,t}$	$\mathcal{F}^r_s$-measurable component of x_t	(9.3)
X	finite-dimensional limit of X_n, an LFSM	(2.6)
X_n	process constructed from $\{x_t\}$	(2.5)
$\xi_{kt} f$	martingale difference components of $\mathcal{M}_{nk} f$	(7.3)
Z_α	α-stable Lévy motion	Rem. 2.1

Symbols not connected to Greek or Roman letters. Ordered alphabetically by their description.

$=_d$	both sides have the same distribution	Rem. 3.2
$\lceil \cdot \rceil$	ceiling function	Sec. 1.1
$\xrightarrow{p}$	converges in probability to	Sec. 7.1
$\rightsquigarrow_{\mathrm{fdd}}$	finite-dimensional convergence	Sec. 4.2
$\lfloor \cdot \rfloor$	floor function (integer part)	Sec. 1.1
$\lVert \cdot \rVert_{[\beta]}$	fourier transform modulus (at origin) norm	(4.5)
$\hat{f}$	fourier transform of f	Sec. 4.2
$\lesssim$	left side bounded by a constant times the right side	Sec. 1.1
$\lesssim_p$	left side bounded in probability by the right side ($a_n \lesssim_p b_n$ if $a_n = O_p(b_n)$)	Sec. 4.2
$\lVert f \rVert_p$	L^p norm, $(\int \lvert f \rvert^p)^{1/p}$, for function f denotes $\sup_{x\in\mathbb{R}} \lvert f(x) \rvert$ when $p = \infty$	Sec. 1.1
$\lVert X \rVert_p$	L^p norm, $(\mathbb{E}\lvert X \rvert^p)^{1/p}$, for random variable X	Sec. 1.1
$\langle M \rangle$	martingale conditional variance	(7.1)
$[M]$	martingale sum of squares	(7.1)
$\lVert X \rVert_\tau$	Orlicz norm associated to function τ	Sec. 4.2

$\sim$	strong asymptotic equivalence . . . . ($a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$)	Sec. 4.2
$\|\mathscr{F}\|$	supremum of norm $\|\cdot\|$ over $\mathscr{F}$: $\sup_{f\in\mathscr{F}}\|f\|$	Sec. 4.2
$\asymp$	weak asymptotic equivalence . . . . ($a_n \asymp b_n$ if $\lim_{n\to\infty} a_n/b_n \in (-\infty,\infty)\backslash\{0\}$)	Sec. 4.2
$\rightsquigarrow$	weak convergence (van der Vaart and Wellner, 1996)	Sec. 1.1