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Proof that the Product-Mix Auction Bidding Language can represent *any* Substitutes Preferences

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Please send corrections and questions

Abstract

We show all substitutes (respectively, strong substitutes) preferences can be represented, and no other preferences can be represented, by appropriate sets of permitted bids in the corresponding version of the Product-Mix Auction bidding language. The Product-Mix Auction languages thus also provide new characterizations of both ordinary substitutes and strong substitutes. This material will form part of "Implementing Walrasian Equilibrium–the Language of Product-Mix Auctions" (Baldwin and Klemperer, in preparation).

1 Introduction

1.1 Product-Mix Auctions (PMAs)

This note shows that the Product-Mix Auction (PMA) bidding language[1] permits the specification of precisely the set of preferences that are substitutes.[2] That is, any concave substitutes preferences can be represented by an appropriate set of bids of the kind permitted by the PMA bidding language. Furthermore, since our proof is constructive, it can be used to provide an algorithm to generate the set of bids representing the

*This note generalises and subsumes Baldwin and Klemperer (2021).

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[1]Free open-source software to run several versions of the PMA is at http://pma.nuff.ox.ac.uk/.

[2]Baldwin and Klemperer (2016) presented the argument for the proof for strong substitutes. Full details are in Baldwin and Klemperer (2021). Lin and Tran (2017) showed how any valuation can be analysed tropical-geometrically and decomposed into a combination of simpler pieces, but if the valuation is not strong substitutes, these simpler pieces do not correspond to positive and negative bids. Klemperer (2010) stated the result for strong substitutes with multiple units of each of two goods.

valuation.[3] Moreover, the representation of any valuation is essentially unique,[4] and no permitted combination of PMA bids represents any other form of preferences.[5]

PMAs are sealed-bid mechanisms for trading multiple units of multiple differentiated goods. They implement competitive-equilibrium allocations based on the preferences that participants express in an easy-to-use-and-understand geometric language. The PMA was originally developed by Klemperer (2008) for the Bank of England, which implemented a simplified version of it;[6] the results shown in this note are important for the usefulness of the auction in other contexts.[7]

A revised version of the material in this note will form a central part of "Implementing Walrasian Equilibrium—the Language of Product-Mix Auctions" (Baldwin and Klemperer, in preparation).

1.2 Strong substitutes

The results are especially significant for the important special case of strong substitutes. It follows from our more general results that the strong-substitutes PMA bidding language permits the specification of precisely the set of preferences that are strong substitutes. We know of no other language with these properties. For example, neither Hatfield and Milgrom (2005)'s endowed assignment messages nor Milgrom (2009)'s (integer) assignment messages can express all strong substitute valuations (see Ostrovsky and Paes Leme (2015), and Fichtl (2021b), respectively).[8]

Strong-substitutes preferences are those that would be ordinary-substitutes preferences if we treated every unit of every good as a separate good. These preferences have many attractive theoretical properties.[9] They also naturally arise in practical contexts. For example, bidders' preferences in the Bank of England's liquidity auctions seem to be well-represented by strong substitutes.[10]

[3]The algorithm would use knowledge of the valuation's "weighted LIP" (see Section 2.3). However, standard software can find this (see Baldwin and Klemperer (2019), Remark A2). Moreover, if the explicit valuation is not available, it and its weighted LIP can be found using a natural extension of the methodology described in Goldberg et al. (2020) if either a value oracle or a demand oracle is available.

[4]That is, the representation is unique after removing redundancies, such as a pair of bids that exactly cancel each other out.

[5]Permitted or "valid" combinations of bids are those that satisfy the "law of demand" that the demand for a good cannot decrease if its price falls while no other price changes. See Definition 2.4 below.

[6]Klemperer (2008) responded to the Governor of the Bank of England's 2007 request for a mechanism to allocate central-bank funds to bidders who would be permitted to offer different qualities of collateral; the UK suffered its first bank run for 140 years in Sept. 2007 in an early sign of the financial crisis. Over £200 billion in indexed long-term repos have now been auctioned using the Bank's auction; it is currently run weekly.

[7]See Klemperer (2018) for discussion of variants of the PMA language. Fichtl (2021a) details an algorithm for solving, and Finster et al. (2021) provides further discussion of, "Arctic" (Budget-Constrained) PMAs.

[8]Furthermore, Tran (2020) shows that it is not possible to express all strong substitute valuations as combinations of weighted ranks of matroids on a ground set bounded by the number of goods.

[9]These preferences mean, for example, that if the price of any one good increases, and the demand for it decreases, then the demand for all other goods can increase by at most the amount of that decrease. Strong substitutability is the terminology coined by Milgrom and Strulovici (2009). It is equivalent to $M^\natural$-concavity (see Murota and Shioura (1999), Murota (2003) and Shioura and Tamura (2015)).

[10]See Klemperer (2018), Appendix I(A2a)–strong substitutability allows an agent's preferences to

1.3 Outline

We proceed as follows. Section 2 explains the bidding language and states the main results. It also reviews some key concepts from Baldwin and Klemperer (2019).

Section 3 provides intuition for our results. In particular, Section 3.2 explains the main ideas by considering a simple example of an agent's valuation over just two varieties of goods. It shows how to construct the collection of bids that expresses the agent's preferences, and explains why our construction works in general. The remaining subsections of Section 3 explain the generalisation to multiple varieties of goods, and give intuition for our other main results.

Section 4 develops an "arithmetic" on "pseudo-LIPs", which are generalisations of the "locuses of indifference prices (LIPs)" that we introduced in Baldwin and Klemperer (2019), and relates them to the bidding language.

Section 5 provides the structure of the proofs, and some of the details; further details are in Appendices.

2 Conventions, Definitions and Summary of Results

2.1 Valuations and Substitutes

This paper concerns the representation of substitutes preferences for indivisible goods, when utility is quasilinear.

That is, an agent has a *valuation* $v : A_v \to \mathbb{R}$ on a finite set of bundles $\mathbf{x} \in A_v \subsetneq \mathbb{Z}^n_{\geq 0}$. The bundles are formed of n distinct indivisible *goods*, and each good is available in multiple units. We write $[n] = \{1, \ldots, n\}$ for the set of all the goods, and $[n]_0$ for the set $\{0, 1, \ldots, n\}$. The 0th good refers to "nothing"; this allows us to refer to the case in which no good is demanded.

An important special case is when the *domain*, A_v, is a *discrete simplex*. For $I \subseteq [n]_0$ write $\Delta_I := \{\mathbf{e}^i \mid i \in I\}$, in which $\mathbf{e}^i$ is the ith coordinate vector, and we write $\mathbf{e}^0 := \mathbf{0}$. For $m \in \mathbb{Z}$ we slightly abuse notation by writing $m\Delta_I := \{m\mathbf{x} \in \mathbb{Z}^n \mid \mathbf{x} \in \operatorname{conv}(\Delta_I)\}$; note that we include the case $m < 0$ here. For $\mathbf{t} \in \mathbb{Z}^n_{>0}$ we write $\mathbf{t} \odot \Delta_I := \{(t_1x_1, \ldots, t_nx_n) \in \mathbb{Z}^n \mid \mathbf{x} \in \operatorname{conv}(\Delta_I)\}$.

Prices $\mathbf{p} \in \mathbb{R}^n$ are linear on the n goods, and there are no budget constraints, so the agent's quasilinear utility is $v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$ for all $\mathbf{x} \in A_v$. The agent demands any bundle that maximises its utility, so that its *demand set* at price $\mathbf{p}$ is $D_v(\mathbf{p}) = \arg\max_{\mathbf{x} \in A_v}\{v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}\}$. Observe that the demand set need not be a single bundle, although it will be at a single bundle at a dense set of prices in $\mathbb{R}^n$.

The valuations v considered in this paper will always be *concave*: the set A_v satisfies $\operatorname{conv}(A_v) \cap \mathbb{Z}^n = A_v$, and, if $\operatorname{conc}(v) : \operatorname{conv}(A_v) \to \mathbb{R}$ is the concave majorant[11] of v then $v(\mathbf{x}) = \operatorname{conc}(v)(\mathbf{x})$ for all $\mathbf{x} \in A_v$. Equivalently, every demand set $D_v(\mathbf{p})$ is discrete-convex.[12] And valuations will always be *ordinary substitutes*, and may also be *strong substitutes*, in the following senses:

exhibit one-for-one substitution between goods, but not more general tradeoffs. Baldwin et al. (2019) and Baldwin et al. (2021a) provide algorithms to solve the strong-substitutes PMA, i.e., find competitive equilibrium prices and allocations, given any valid sets of bids.

[11] That is, the minimal weakly-concave function that is everywhere greater than v.

[12] See, for example, Baldwin and Klemperer (2019), Lemma 2.11.

Definition 2.1 (See e.g. Ausubel and Milgrom (2002) and Milgrom and Strulovici (2009))**.** Let $v : A_v \to \mathbb{R}$ be a valuation.

(1) v is *ordinary substitutes* if, for any prices $\mathbf{p}' \geq \mathbf{p}$ with $D_v(\mathbf{p}) = \{\mathbf{x}\}$ and $D_v(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \geq x_k$ for all k such that $p'_k = p_k$.
(2) v is *strong substitutes* if, when we consider every unit of every good to be a separate good, it is a valuation for ordinary substitutes.

Both ordinary and strong substitutes valuations are natural extensions of Kelso and Crawford (1982)'s "gross substitutes" to the multi-unit case. Strong substitutes, which are automatically concave, guarantee existence of competitive equilibrium, and are also known as "$M^\natural$-concave functions" in the literature on discrete convex analysis (see, e.g. Murota, 2003).

The simplest examples of strong substitutes valuations are the "unit demands" of Gul and Stacchetti (1999):

Example 2.2. A unit demand valuation is a valuation v with domain $\Delta_{[n]_0} = \{\mathbf{e}^i \mid i \in [n]_0\}$ and such that, for some $\mathbf{r} \in \mathbb{R}^n$, $v(\mathbf{x}) = \mathbf{r} \cdot \mathbf{x}$ for all $\mathbf{x} \in \Delta_{[n]_0}$.

When $\mathbf{p} = \mathbf{r}$, the agent is indifferent between all the bundles $\mathbf{x} \in \Delta_{[n]_0}$, as all deliver utility $v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = 0$ at that price.

We will show that all strong substitutes valuations can be built up from unit demand valuations, and that all concave ordinary substitutes valuations can be built up from unit demand valuations that have undergone a simple deformation, in a sense that will be made clear below.

2.2 PMA Bids and the Representation Theorems

This section lays out the key concepts and results of the paper. Proofs of all results stated here will be provided later in the paper.

The bids in our languages are defined as follows. First, recall that an integer vector $\mathbf{t}$ is "primitive" if the greatest common divisor of its entries is 1. Now write:

Definition 2.3. A *positive PMA bid* $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ with *root* $\mathbf{r} = \mathbf{r}(\mathbf{b}) \in \mathbb{R}^n$, *trade-off* $\mathbf{t} = \mathbf{t}(\mathbf{b}) \in \mathbb{Z}^n_{>0}$ which is a primitive integer vector, and *multiplicity* $m = m(\mathbf{b}) \in \mathbb{Z}_{>0}$, represents valuation $v_{\mathbf{b}}$ with domain $m\mathbf{t} \odot \Delta_{[n]_0} = \{(mt_1x_1, \ldots, mt_nx_n) \in \mathbb{Z}^n \mid \mathbf{x} \in \operatorname{conv}\left(\Delta_{[n]_0}\right)\}$ and such that $v_{\mathbf{b}}(\mathbf{x}) = \mathbf{r} \cdot \mathbf{x}$ for $\mathbf{x}$ in this domain.

A *positive SSPMA bid* is a positive PMA bid $\mathbf{b}$ such that $\mathbf{t}(\mathbf{b}) = \mathbf{1}$.

Note that the coordinates of the "root" give the bid's location in $\mathbb{R}^n$, and also give the per-unit values for the goods. We call this parameter the root, notated $\mathbf{r}$, rather than the "value" because we will identify collections of bids with valuations (notated v) and wish to avoid confusion. Figure 1 shows an example of a positive PMA bid with $n = 2$.

We write $D_{\mathbf{b}}(\mathbf{p}) := D_{v_{\mathbf{b}}}(\mathbf{p})$ to simplify notation. It is easy to see that a SSPMA bid with multiplicity m is an aggregate of m identical unit demand valuations as in Example 2.2. A PMA bid is a simple generalisation of this, allowing for more general trade-offs between units of goods: at price $\mathbf{r}(\mathbf{b})$ the bidder is indifferent between receiving nothing, and receiving (up to) t_i units of good $i \in [n]$ (and so also indifferent between receiving

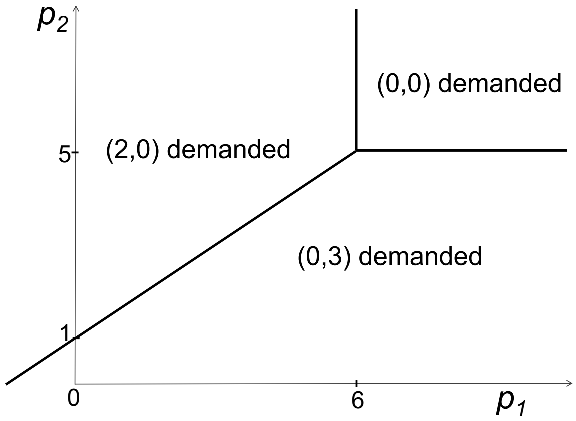


Figure 1: The bid $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m) = [6, 5; 2, 3; 1]$, expressing the valuation $v(0, 0) = 0$, $v(1, 0) = 6$, $v(2, 0) = 12$, $v(0, 1) = 5$, $v(0, 2) = 10$, $v(0, 3) = 15$, $v(1, 1) = 11$. The bundles demanded at prices in the open regions separated by line segments are labelled; the bundles $(0, 0)$, $(1, 0)$, and $(2, 0)$ are all demanded on the vertical line segment; the bundles $(0, 0)$, $(0, 1)$, $(0, 2)$, and $(0, 3)$ are all demanded on the horizontal line segment; the bundles $(2, 0)$ and $(0, 3)$ are both demanded on the diagonal line segment; and all these bundles together with $(1, 1)$ are demanded at the "root" of the bid, $\mathbf{r} = (6, 5)$. A vector normal to the diagonal line segment is $(-t_1(\mathbf{b}), t_2(\mathbf{b})) = (-2, 3)$.

(up to) t_i units of good i, and receiving up to t_j units of good $j \in [n]$). For brevity we will refer to PMA bids as "bids", and specify SSPMA bids only in that case.

We also write $r_0(\mathbf{b}) := 0$, $p_0 := 0$ and $t_0(\mathbf{b}) := 1$; these 0th coordinates are understood not to be generally included in $\mathbf{r}, \mathbf{p}$ and $\mathbf{t}$ respectively, unless $i \in [n]_0$ is explicitly stated, but allow us to conveniently refer to the case in which the 0th good $\mathbf{0}$ is demanded.

So $I(\mathbf{b}, \mathbf{p}) := \arg\max_{i \in [n]_0} t_i(\mathbf{b})(r_i(\mathbf{b}) - p_i)$ is the set of goods optimal for a bid, and includes the possibility that the optimal outcome is an assignment of nothing. And

$$D_{\mathbf{b}}(\mathbf{p}) = m(\mathbf{b})\mathbf{t}(\mathbf{b}) \odot \Delta_{I(\mathbf{b},\mathbf{p})}. \tag{1}$$

We define the demand $D_{\mathcal{B}}$ of a finite collection (i.e., multiset) $\mathcal{B}$ of positive bids to be

$$D_{\mathcal{B}}(\mathbf{p}) := \operatorname{conv}\left(\sum_{\mathbf{b} \in \mathcal{B}} D_{\mathbf{b}}(\mathbf{p})\right) \cap \mathbb{Z}^n, \tag{2}$$

that is, the convex hull of the aggregate demand set of these bids. We do this because we assume that the individual agent's valuation is concave, so its demand set at any price is convex.[13]

It is *not* true that, for every concave ordinary substitutes valuation v, there is a collection $\mathcal{B}$ of positive bids such that $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$. So we also introduce *negative bids*, for which $m(\mathbf{b}) \in \mathbb{Z}_{<0}$.

A negative bid $\mathbf{b}$ does not correspond to a valuation $v_{\mathbf{b}}$. However, we may, nonetheless, define $D_{\mathbf{b}}(\mathbf{p})$ for such a bid as in Equation (1) above. Write, for convenience,

[13]Note the Minkowski sum of individual demand sets with ordinary substitutes need not be convex (see Example B.1 in Appendix B.1).

$|\mathbf{b}| = (\mathbf{r}(\mathbf{b}); \mathbf{t}(\mathbf{b}); |m(\mathbf{b})|)$, and observe that if $m(\mathbf{b}) < 0$ then

$$D_{\mathbf{b}}(\mathbf{p}) = -D_{|\mathbf{b}|}(\mathbf{p}) = \{-\mathbf{x} \mid \mathbf{x} \in D_{|\mathbf{b}|}(\mathbf{p})\}. \tag{3}$$

That is, an increase in the price of good i leads to a weak *increase* in demand $D_{\mathbf{b}}$ for good i, and so $D_{\mathbf{b}}$ does not satisfy the "law of demand" in the following sense:

Definition 2.4. For any demand correspondence D, say that D *satisfies the law of demand* if, given $\mathbf{p}$ and $\mathbf{p}' = \mathbf{p} + \lambda \mathbf{e}^i \in \mathbb{R}^n$, where $i \in [n]$ and $\lambda > 0$, such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(\mathbf{p}') = \{\mathbf{x}'\}$, it holds that $x'_i \leq x_i$, with equality if and only if $\mathbf{x}' = \mathbf{x}$.

By contrast, the demand correspondence D_v of any quasilinear valuation v, does satisfy the law of demand.[14]

To assign a demand correspondence to a finite collection $\mathcal{B}$ of bids (of either sign), first recall that, for each $\mathbf{b} \in \mathcal{B}$, demand $D_{\mathbf{b}}(\mathbf{p})$ is single-valued at a dense set of prices in $\mathbb{R}^n$. So, since there are finitely many $\mathbf{b} \in \mathcal{B}$, we can identify the set Q of all price vectors $\mathbf{q}$ in a small open neighbourhood of $\mathbf{p}$, such that $D_{\mathbf{b}}(\mathbf{p})$ is single-valued for all $\mathbf{q} \in Q$. Now define

$$D_{\mathcal{B}}(\mathbf{p}) := \operatorname{conv} \left\{ \sum_{\mathbf{b} \in \mathcal{B}} D_{\mathbf{b}}(\mathbf{q}) \mid \mathbf{q} \in Q \right\} \cap \mathbb{Z}^n \tag{4}$$

We are interested in combinations of positive and negative bids whose demand correspondences, $D_{\mathcal{B}}$, do satisfy the law of demand, so we define:

Definition 2.5. Let U be a convex open subset of $\mathbb{R}^n$. The finite collection $\mathcal{B}$ of bids is *valid in* U if $D_{\mathcal{B}}$ satisfies the law of demand restricted to prices $\mathbf{p}, \mathbf{p}' \in U$.

$\mathcal{B}$ is *valid* if it is valid in $\mathbb{R}^n$.

Our next proposition tells us that validity (in $\mathbb{R}^n$) is the only condition needed for bids $\mathcal{B}$ to correspond to a concave ordinary substitutes valuation. (We will provide further characterisations of validity later.[15])

Proposition 2.6 (Cf. Baldwin et al. 2019, Theorem 1, for the strong substitutes case)**.** *Let $\mathcal{B}$ be a finite collection of bids. The following are equivalent:*

(1) $\mathcal{B}$ is valid;
(2) there exists a concave ordinary substitutes valuation $v_{\mathcal{B}}$ such that $D_{v_{\mathcal{B}}}(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.

Indeed, the valuation $v_{\mathcal{B}}$ is easy to describe. For those bundles $\mathbf{x}$ such that there exists a price $\mathbf{p}^{\mathbf{x}}$ at which $D_{\mathcal{B}}(\mathbf{p}^{\mathbf{x}}) = \{\mathbf{x}\}$ and $|I(\mathbf{b}, \mathbf{p}^{\mathbf{x}})| = 1$ for all $\mathbf{b} \in \mathcal{B}$,[16] we write

[14] See e.g., Mas-Colell et al. 1995, Proposition 3.E.4. We also show that any valuation satisfies a stricter property in Lemma B.13.

[15] See Proposition 4.14 which subsumes Propositions 2.6 and 2.8 of this section. We provide the proof in Appendix B.3, after developing the necessary technical machinery.

[16] Such a price $\mathbf{p}^{\mathbf{x}}$ exists for any bundle $\mathbf{x}$ that is uniquely demanded at any price, because being uniquely demanded is an open condition and the condition that $|I(\mathbf{b}, \mathbf{p}^{\mathbf{x}})| = 1$ for all $\mathbf{b} \in \mathcal{B}$ is generic. However, the latter condition can fail at some prices at which $\mathbf{x}$ is uniquely demanded, because positive and negative marginal bids may cancel each other out.

$I(\mathbf{b}, \mathbf{p}^{\mathbf{x}}) = \{i_{\mathbf{b},\mathbf{x}}\}$ and set

$$\widehat{v}_{\mathcal{B}}(\mathbf{x}) = \sum_{\mathbf{b}\in\mathcal{B}} r_{i_{\mathbf{b},\mathbf{x}}}(\mathbf{b}) m(\mathbf{b}) t_{i_{\mathbf{b},\mathbf{x}}}(\mathbf{b}).$$

We can then define the valuation $v_{\mathcal{B}}$ as follows:

Definition 2.7. $v_{\mathcal{B}} : A_{\mathcal{B}} \to \mathbb{R}$ is the valuation defined via $v_{\mathcal{B}}(\mathbf{x}) = \operatorname{conc}(\widehat{v}_{\mathcal{B}})(\mathbf{x})$ for all $\mathbf{x} \in A_{\mathcal{B}}$, where $A_{\mathcal{B}} = \bigcup\{D_{\mathcal{B}}(\mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^n\}$.

It is not immediate from this definition that the valuation $v_{\mathcal{B}}$ is well-defined, because it is not clear that $\widehat{v}_{\mathcal{B}}(\mathbf{x})$ is independent of the choice of $\mathbf{p}^{\mathbf{x}}$. However, we show that this is the case when the bids are valid, and:

Proposition 2.8. *When $\mathcal{B}$ is valid, the valuation $v_{\mathcal{B}}$ is well defined and concave, and satisfies $D_{v_{\mathcal{B}}}(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.*

In fact the domain $A_{\mathcal{B}}$ of the concave ordinary substitutes valuation corresponding to $\mathcal{B}$ satisfies the following description:

Definition 2.9. A is a *Finite Bids Domain (FBD)* if:
(1) A is discrete-convex and $\mathbf{0} \in A \subsetneq \mathbb{Z}^n_{\geq 0}$;
(2) if $\mathbf{x} \in A$ then $\mathbf{x} - x_i\mathbf{e}^i \in A$;
(3) for all $i \in [n]$ there exists $W_i \in \mathbb{Z}_{>0}$ such that $\arg\max_{\mathbf{x}\in A}\{x_i\} = \{W_i\mathbf{e}^i\}$.

Any discrete simplex Δ_I such that $0 \in I$ satisfies this condition, as does, for example, the discrete-convex hull of $\{(0,0), (3,0), (2,1), (0,2)\}$.

Lemma 2.10 (cf. Baldwin et al. 2021a, Proposition 3)**.** *Let $\mathcal{B}$ be a finite valid collection of bids, and let $A_{\mathcal{B}} = \bigcup\{D_{\mathcal{B}}(\mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^n\}$. Then $A_{\mathcal{B}}$ is an FBD.*

If $\mathcal{B}$ is a finite valid collection of SSPMA bids then $A_{\mathcal{B}} = W\Delta_{[n]_0}$, where $W = \sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b})$.

The SSPMA case is presented by Baldwin et al. (2021a); the general case can be shown in a similar way.[17]

Thus it is easiest to state our representation for valuations whose domain is an FBD. And our central result is that the PMA bidding language can indeed represent all such valuations; moreover, the SSPMA bidding language can represent all strong substitutes valuations whose domain is a discrete simplex.

Theorem 2.11. *If v is a concave ordinary substitutes valuation whose domain is an FBD, and such that $v(\mathbf{0}) = 0$, then there exists a valid PMA bid collection $\mathcal{B}$ such that $v_{\mathcal{B}} = v$.*

If v is a strong substitutes valuation with discrete simplex domain then there exists a valid SSPMA bid collection $\mathcal{B}$ such that $v_{\mathcal{B}} = v$.[18]

[17] The proof of Lemma 2.10 is provided in Appendix B.3.

[18] Our results can also be understood in "tropical algebra" (see, e.g., Maclagan and Sturmfels, 2015). Each of our positive bids corresponds to a simple "tropical polynomial", a collection of positive bids corresponds to the "tropical product" of these tropical polynomials, and negative bids introduce a form of "tropical division". So these presentations are a "rational factorisation". Moreover, SSPMA bids correspond to "tropical lines" so the case of strong substitutes is a kind of Fundamental Theorem of Algebra for tropical geometry. Building on our presentation of this case (Baldwin and Klemperer (2016)), Lin and Tran (2017) have developed and generalised this observation from a mathematical perspective. Our ordinary substitutes case does not follow from their generalisation.

If the domain of v is not an FBD, Lemma 2.10 shows us that we cannot globally represent D_v using a collection of bids $\mathcal{B}$, i.e., $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ will not hold for all $\mathbf{p} \in \mathbb{R}^n$. However, we can represent D_v in any sufficiently large bounded region of $\mathbb{R}^n$, as we now show.

Definition 2.12. The *Bounding Box* is $\mathbf{H} := [\underline{H}, \overline{H}]^n$ for some $\underline{H} < \overline{H}$, such that $\underline{H}, \overline{H} \in \mathbb{R}$.

As is standard, write also $\mathbf{H}^\circ = (\underline{H}, \overline{H})^n$.

Theorem 2.13. *If v is a concave ordinary substitutes valuation then for any $\underline{H} < \overline{H}$ there exists a bid collection $\mathcal{B}$, valid in $\mathbf{H}^\circ$, such that $\mathbf{r}(\mathbf{b}) \in \mathbf{H}$ for all $\mathbf{b} \in \mathcal{B}$ and $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$.*[19]

If v is a strong substitutes valuation then $\mathcal{B}$ is a collection of SSPMA bids valid in $\mathbf{H}^\circ$.[20]

So by choosing values of $\underline{H}$ and $\overline{H}$, that are, respectively, lower than, and higher than, any prices we might ever be interested in for any good, we can find a bid collection $\mathcal{B}$ that represents the demand set at all prices of interest.

The bounding box also makes it easy to describe bids such as, for example, that of a bidder who is interested in units of good 1, but who has no interest in any other good; this bid will be on the lower faces of the bounding box in every dimension except the first, that is, it will be rooted at $(r_1, \underline{H}, \underline{H}, \ldots, \underline{H})$ for some r_1.

We now consider the extent to which these bid collections are unique. Allowing multiple bids with the same combination $(\mathbf{r}; \mathbf{t})$ of root and trade-off would pose problems for uniqueness, but only in ways that are not interesting, so uniqueness of the valid bid collection is straightforward to state for Theorem 2.11. However, for Theorem 2.13, uniqueness requires a little more care: we need to remove potentially "redundant" failures of uniqueness in the following ways:

Definition 2.14. A bid collection $\mathcal{B}$ has *no redundancies* if it contains at most one bid with any combination $(\mathbf{r}; \mathbf{t})$ of root and trade-off. It has *no redundancies relative to* $\mathbf{H}$ if it has no redundancies, and also, for all $\mathbf{b} = (\mathbf{r}, \mathbf{t}, m) \in \mathcal{B}$:

(1) if $r_i = \underline{H}$ then $t_i = 1$;
(2) $\sum_{r_i \neq \underline{H}} t_i \mathbf{e}^i$ is a primitive integer vector;
(3) $\mathbf{r} \neq (\underline{H}, \ldots, \underline{H})$.

To see why the additional cases must be excluded, recall that a positive bid $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ represents the possibility of buying mt_i units of good i when p_i is sufficiently low relative to r_i, but if $r_i = \underline{H}$, then Theorem 2.13 explicitly excludes such prices from consideration, so t_i is not uniquely defined. Stipulation (1) fixes such t_i to be 1, but leaves ambiguity in the remaining part of $\mathbf{t}$, which could be multiplied by any positive integer, absent requirement (2). Finally, any bid rooted at $(\underline{H}, \ldots, \underline{H})$ is also redundant, since $D_{\mathbf{b}}(\mathbf{p}) = \{\mathbf{0}\}$ for such a bid for all $\mathbf{p} \in \mathbf{H}^\circ$.

[19] It can also be shown that there exists a valid PMA bid collection $\mathcal{B}$ such that $v_{\mathcal{B}}$ equals v in the following sense: if $\underline{H}$ and $\overline{H}$ are sufficiently low and high, respectively, that every bundle in A_v is demanded at some price in $\mathbf{H}^\circ$ then, if we define our valuation $v_{\mathcal{B}}$ using only prices in $\mathbf{H}^\circ$, and so only on the domain A_v, then it is well-defined and equal to v.

[20] In the strong substitutes case, it can be shown (see Baldwin and Klemperer (2021), Theorem 2.7) that the collection $\mathcal{B}$ of SSPMA bids is valid (in $\mathbb{R}^n$).

Theorem 2.15. *There is a unique bid collection for Theorem 2.11 with no redundancies.*

There is a unique bid collection for Theorem 2.13 with no redundancies relative to $\mathbf{H}$.

Note that there are alternative conventions that provide uniqueness. Baldwin et al. (2019), working in the context of SSPMA bids, restrict bid multiplicities to ± 1, not allowing otherwise identical positive and negative bids to coincide. It is straightforward that Theorem 2.15 implies uniqueness in this case also.[21]

2.3 LIPs, Balancing, Hods and Fins, and the Valuation-Complex Equivalence Theorem

We recall several definitions that Baldwin and Klemperer (2019) introduced from the literature on convex and "tropical" geometry.

Definition 2.16 (**The Locus of Indifference Prices (LIP)**, see Baldwin and Klemperer 2019, Definitions 2.1, 2.2 and 2.3, and for the geometric definitions that Baldwin and Klemperer 2019 repurposed, see, e.g., Maclagan and Sturmfels 2015)**.** Let $v : A_v \to \mathbb{R}$ be a valuation on a finite set of bundles $A_v \subseteq \mathbb{Z}^n$.

(1) The *Locus of Indifference Prices (LIP)* is $\mathcal{L}_v := \{\mathbf{p} \in \mathbb{R}^n \mid |D_v(\mathbf{p})| > 1\}$.
(2) A *unique demand region (UDR)* of a valuation v is the set of all prices at which a given bundle in A_v is uniquely demanded. That is, it has the form $\{\mathbf{p} \in \mathbb{R}^n \mid \{\mathbf{x}\} = D_v(\mathbf{p})\}$ for some $\mathbf{x} \in A_v$.
(3) A *price complex cell* of v is a non-empty set $C \subseteq \mathbb{R}^n$ such that there exist $\mathbf{x}^1, \ldots, \mathbf{x}^k \in A_v$, with $k \geq 1$, satisfying $C = \left\{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{x}^1, \ldots, \mathbf{x}^k \in D_v(\mathbf{p})\right\}$.
(4) The *price complex* is the set of all price complex cells.
(5) The *cells of the LIP* are the price complex cells contained in the LIP.
(6) A *facet* of the LIP is a price complex cell of dimension $n - 1$.[22]
(7) Let $\mathbf{x}, \mathbf{x}'$ be the bundles demanded in the UDRs on either side of facet F of a LIP. The *weight* of F, $w_v(F)$, is the greatest common divisor of the entries of $\mathbf{x}' - \mathbf{x}$.

Our second collection of recalled definitions relate to polyhedral complexes. A modification made here, relative to Baldwin and Klemperer (2019), is to allow more general weightings.

Definition 2.17 (standard)**.**

(1) A *rational polyhedron* in $\mathbb{R}^n$ is the intersection of a finite set of half-spaces $\{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{d} \leq \alpha\}$ for some $\mathbf{d} \in \mathbb{Z}^n$ and $\alpha \in \mathbb{R}$.
(2) A *face* of a polyhedron C maximises $\mathbf{p} \cdot \mathbf{d}$ over $\mathbf{p} \in C$, for some fixed $\mathbf{d} \in \mathbb{R}^n$.
(3) The *interior* of polyhedron C is $C^\circ := \{\mathbf{p} \in C \mid \mathbf{p} \notin C' \text{ for any face } C' \subsetneq C\}$.
(4) A *rational polyhedral complex* Π is a finite collection of *cells* $C \subseteq \mathbb{R}^n$ such that:
   (i) if $C \in \Pi$ then C is a rational polyhedron and any face of C is also in Π;
   (ii) if $C, C' \in \Pi$ then either $C \cap C' = \emptyset$ or $C \cap C'$ is a face of both C and C'.

[21] See Lemma 4.18 to see how Theorem 2.15 extends to this case.

[22] The dimension of a set $F \subseteq \mathbb{R}^n$ is the dimension of its affine span, i.e. the dimension of the smallest linear subspace $U \subseteq \mathbb{R}^n$ such that $F \subseteq \{\mathbf{c}\} + U$ for some fixed vector $\mathbf{c}$.

(5) A *k-cell* is a cell of dimension k. A *facet* is a cell of dimension $n-1$.
(6) A polyhedral complex is *k-dimensional* if all its cells are contained in its k-cells.
(7) The *support* of a polyhedral complex Π is the set $\bigcup \Pi$.
(8) For any set X, an *X-weighted* polyhedral complex is a pair (Π, w) where Π is a polyhedral complex and w is a function assigning a weight $w(F) \in X$ to each facet $F \in \Pi$.

In this paper we will be interested in both $\mathbb{Z}$-weighted and $\mathbb{Z}_{>0}$-weighted polyhedral complexes.

The notions from Definition 2.16 are related to those from Definition 2.17 via the following two results.

Fact 2.18 (See, e.g. Baldwin and Klemperer 2019, Proposition 2.7)**.** The price complex paired with the facet weights is an n-dimensional weighted rational polyhedral complex.

Particularly important is the relationship between facet normals of LIPs and changes in demand:

Fact 2.19 (Baldwin and Klemperer 2019 Proposition 2.4)**.**
(1) If $\mathbf{x}$, $\mathbf{x}'$ are uniquely demanded on either side of facet F of a LIP, then $\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})$ is constant for all $\mathbf{p} \in F$.
(2) The change in demand as price changes between the UDRs on either side of F is $w_v(F)$ times the primitive integer vector that is normal to F, and that points in the opposite direction to the change in price.

Definition 2.20 (**The Balancing Condition**, Mikhalkin 2004, Definition 3)**.** An $(n-1)$- or n-dimensional $\mathbb{Z}_{>0}$-weighted rational polyhedral complex Π is *balanced* if, for every $(n-2)$-cell $C \in \Pi$, the weights $w(F^j)$ on the facets $F^1 \ldots F^l$ that contain C, and primitive integer normal vectors $\mathbf{d}_{F^j}$ for these facets that are defined by a fixed rotational direction about C, satisfy $\sum_{j=1}^{l} w(F^j)\mathbf{d}_{F^j} = \mathbf{0}$.[23]

Fact 2.21 (**The Valuation-Complex Equivalence Theorem**, Mikhalkin 2004, Remark 2.3 and Prop. 2.4; Baldwin and Klemperer 2019, Theorem 2.14)**.** Suppose that (Π, w) is an $(n-1)$-dimensional $\mathbb{Z}_{>0}$-weighted rational polyhedral complex in $\mathbb{R}^n$, that $\mathcal{L}$ is the support of Π, and $\mathbf{p}$ is any price not contained in $\mathcal{L}$.

(1) There exists a finite set $A_v \subsetneq \mathbb{Z}^n$ and a function $v : A_v \to \mathbb{R}$ such that $\mathcal{L}_v = \mathcal{L}$ and $w_v = w$, if and only if (Π, w) is balanced.
(2) If (Π, w) is balanced then there exists a finite set $A_v \subsetneq \mathbb{Z}^n$ and a unique *concave* valuation $v : A_v \to \mathbb{R}$ such that $D_v(\mathbf{p}) = \{\mathbf{0}\}$, $v(\mathbf{0}) = 0$, $\mathcal{L}_v = \mathcal{L}$ and $w_v = w$.

Baldwin and Klemperer (2019) introduced "demand types" to classify economic properties of valuations via the shapes of their facets. First, *demand type vector sets* $\mathcal{D} \subseteq \mathbb{Z}^n$ consist of primitive integer vectors and satisfy $\mathbf{d} \in \mathcal{D} \Rightarrow -\mathbf{d} \in \mathcal{D}$. Then, for any demand type vector set, the *demand type* is the set of valuations v such that every facet of $\mathcal{L}_v$ has normal vector in $\mathcal{D}$. It is not hard to see that the vectors in $\mathcal{D}$ describe how the agent's demand changes in response to a small generic price change.

[23]For example, that the balancing condition holds for the example shown in Figure 2 can be seen by observing 1.(-2,3)+2.(1,0)+3.(0,-1)=$\mathbf{0}$. See note 25 for further examples of checking the balancing condition.

The cases of interest for this paper are the *ordinary substitutes demand type vector set* given in dimension n by $\{\pm\mathbf{e}^i, t_i\mathbf{e}^i - t_j\mathbf{e}^j \mid i,j \in [n],\ t_i, t_j \in \mathbb{Z}_{>0},\ t_i, t_j \text{ coprime},\ i \neq j\}$; and the *strong substitutes demand type vector set*, given in dimension n by $\{\pm\mathbf{e}^i, \mathbf{e}^i - \mathbf{e}^j \mid i,j \in [n], i \neq j\}$. Then:

Fact 2.22 (See Baldwin and Klemperer 2014, 2019, Shioura and Tamura 2015)**.**
(1) A valuation is ordinary substitutes if and only if it is of the ordinary substitutes demand type.
(2) A valuation is strong substitutes if and only if it is concave and of the strong substitutes demand type.

Observe that, if $t_i\mathbf{e}^i - t_j\mathbf{e}^j$ is normal to a facet then so is, for example, $\frac{t_i}{t_j}\mathbf{e}^i - \mathbf{e}^j$. The demand type vector set consists of the primitive integer vectors in the relevant directions, but when we consider normal vectors more generally, it will not always be necessary or convenient to assume that they have this form.

Limiting our study to concave ordinary substitutes valuations therefore restricts the possible forms for the facets that will concern us, and we introduce the following terminology:

Definition 2.23 (Hods and Fins).
(1) For $i \in [n]$, a facet of $\mathcal{L}_v$ is a *i-hod* if $\mathbf{e}^i$ is a normal vector.
(2) For $i \in [n]$ with $i \neq j$, and $\alpha \in \mathbb{Q}$, a facet of $\mathcal{L}_v$ is a *$(i,j;\alpha)$-fin* if $\alpha\mathbf{e}^i - \mathbf{e}^j$ is a normal vector.

We use these terms because, when $n = 3$, the "hods" of a bid appear to form a builder's hod, and the "fins" of a bid resemble the fins or blades of a turbine (see Figures 3a and 3b). Observe that if $t_i\mathbf{e}^i - t_j\mathbf{e}^j$ is a normal vector to a facet, then that facet is an $(i,j;t_i/t_j)$-fin. We will refer to *hods*, to *(i,j)-fins*, and to *fins*, where it is not necessary or convenient to specify i, j or α.

We will see (Lemma 4.9 Part(2)(i)) that all the fins of a bid intersect in a single 1-cell:

Definition 2.24 (Strong Diagonal). The *strong diagonal* of a bid $\mathbf{b} = (\mathbf{r};\mathbf{t};m)$ is the set $\{\mathbf{r} - \lambda\sum_{i=1}^{n}\frac{\mathbf{e}^i}{t_i} \mid \lambda \geq 0\}$, which is the intersection of all the fins of the bid.

Figure 2 shows the facet weights and normal vectors for the two-dimensional bid $\mathbf{b} = (\mathbf{r};\mathbf{t};m) = [6,5;2,3;1]$ that was illustrated in Figure 1. Figure 3 shows the same information for the hods (panel (a)) and fins (panel (b)) of the three-dimensional bid $\mathbf{b} = (\mathbf{r};\mathbf{t};m) = [6,6,5;2,2,3;1]$.

3 Intuition for our Main Results

We now describe the intuition for our main results, and defer further technical detail to later sections. Section 3.1 introduces "pseudo-LIPs", that extend the notion of Locuses of Indifference Prices (LIPs), to allow us to analyse negative bids. Section 3.2 then illustrates the proof of our most challenging result–that any concave ordinary substitutes valuation can be represented by PMA bids–in the case of just two goods. We describe the generalisation to any number of goods in Section 3.3. Sections 3.4 and 3.5 give the intuition for the converse result, and for the result that the representation is unique.

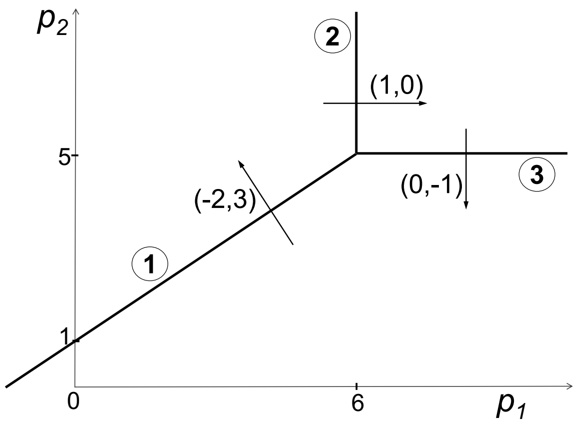
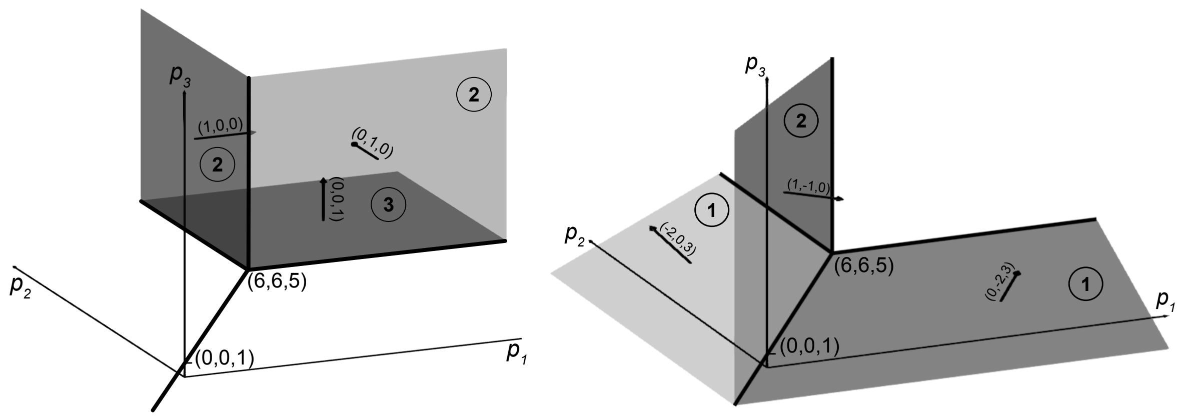


Figure 2: The facets of the bid $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m) = [6, 5; 2, 3; 1]$ (also shown in Figure 1), together with their weights (shown in the circles), and normal vectors (shown in the directions of the arrows). The "strong diagonal", which for $n = 2$ is also the "fin", emanates in direction $(-1/2, -1/3)$ from the "root" at (6,5). The horizontal and vertical facets are the "hods". The bundles demanded in the different cells are listed in Figure 1.



(a) (Parts of) the "hods" of bid $\mathbf{b} = [6, 6, 5; 2, 2, 3; 1]$.

(b) (Parts of) the "fins" of bid $\mathbf{b} = [6, 6, 5; 2, 2, 3; 1]$.

Figure 3: The four 1-cells of the bid $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m) = [6, 6, 5; 2, 2, 3; 1]$ are shown as thick lines (the axes are shown as thin lines), together with (panel (a)) (parts of) the bid's hods, and (panel (b)) (parts of) its fins. The facets' weights are shown in the circles, and their primitive integer normal vectors are shown in the directions of the arrows. The strong diagonal emanates in direction $(-1/2, -1/2, -1/3)$ from the root, $\mathbf{r} = (6, 6, 5)$. Demand is $(2, 0, 0), (0, 2, 0), (0, 0, 3)$, and $(0, 0, 0)$, in the unique demand regions (UDRs) that are to the left of the left vertical hod, in front of the right vertical hod, below the horizontal hod, and at higher prices than any of the three hods, respectively. (This reflects the tradeoff, $\mathbf{t} = (2, 2, 3)$.) Demand in every cell (each hod, each fin, each 1-cell, and the root) is the discrete convex hull of the demands in the cell's adjacent UDRs.[24]

3.1 Pseudo-LIPs

As described in Section 2.3 above, any concave ordinary substitutes valuation induces a weighted LIP. We will want to show how to build up the identical (weighted) LIP from a collection of individual PMA bids. The Valuation Complex Equivalence Theorem (Fact 2.21) will then guarantee that our bid collection represents the valuation we want to express.

To allow for negative bids we will need to allow negative weights in the weighted rational polyhedral complexes we analyse. So we call a union of (weighted) facets of a rational polyhedral complex whose weights need not all be positive but are non-zero integers, and which *does* still satisfy the standard Balancing Condition (Definition 2.20) that LIPs satisfy, a weighted *pseudo-LIP*. (Note that a (weighted) pseudo-LIP does not in general represent a valid valuation, just as a negative bid does not represent a valid valuation. Note also that LIPs are also pseudo-LIPs.)

Moreover, we will describe an "arithmetic" of (weighted) pseudo-LIPs: the sum of two (weighted) pseudo-LIPs is just their superposition (that is, the union of their facets, with the weight of coincident facets being the sum of their weights), with any resulting zero-weighted facets then being eliminated. Subtracting one (weighted) pseudo-LIP from another corresponds to adding them after reversing the signs of the weights of the one to be subtracted; again, any resulting zero-weighted facets are then eliminated. It is not hard to see that these operations preserve the balancing property, so that both the sum and the difference of (weighted) pseudo-LIPs are also (weighted) pseudo-LIPs.

3.2 Any Concave Ordinary Substitutes Valuation can be Represented by PMA Bids: the Two-Goods Case

With just two goods, finding the bids that represent a concave ordinary substitutes valuation is relatively straightforward. We illustrate with a specific example.

Consider the valuation whose LIP's facets are shown as line segments in Figure 4. The facet weights (in circles) show that, starting from the top right unique demand region (UDR) (Definition 2.16(2)) and moving clockwise, the bundles the agent demands in the UDRs are (0, 0), (0, 2), (1, 2), (2, 2), and (2, 0), respectively. (The valuation is $v(0,0) = 0, v(1,0) = 5, v(2,0) = 10, v(0,1) = 4, v(0,2) = 8, v(1,1) = 9, v(1,2) = 12, v(2,1) = 12, v(2,2) = 14.$[25]) We want to find the collection of bids, the "sum" of

[24] In this example, the demand in the hods and fins includes the union of the demands in their adjacent UDRs and, additionally, $(1,0,0)$ is demanded in the left vertical hod, $(0,1,0)$ is demanded in the right vertical hod, $(0,0,1)$ and $(0,0,2)$ are demanded in the horizontal hod, and $(1,1,0)$ is demanded in the vertical fin, respectively. (Observe that the total number of bundles in each facet is 1 more than the facet's weight, reflecting the fact that a facet's weight times its primitive integer normal vector is the difference between the bundles demanded on either side of it.) Demand in the 1-cells includes the union of the demands in their intersecting facets, and $(0,1,1)$ and $(1,0,1)$ are additionally demanded in the 1-cells in the direction of the 1- and 2-coordinates, respectively. Demand at the root is the union of all the previously mentioned demands. (In this example, by contrast to that of Figure 2, there are no additional bundles.)

[25] The (weighted) facets of the LIP are identified by considering the agent's marginal values for additional units. (See Klemperer (2018), Appendix I(A2a) for an example.) It can easily be checked that the (weighted) LIP is balanced (Definition 2.20). The normal vectors of the fins with weights 1 and 2 are $(1,-2)$ and $(1,-1)$, respectively, so for example, going clockwise around $(2,2)$, the facets satisfy $1.(1,-2) + 1.(-1,0) + 2.(0,1) = \mathbf{0}$; going anti-clockwise around $(4,3)$, the facets satisfy $1.(1,-2) +$

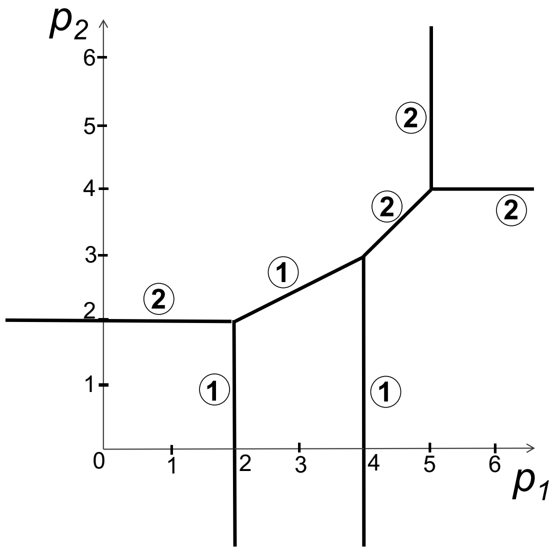


Figure 4: The LIP of a valuation, with its facet weights shown in circles.

whose (weighted) facets matches this (weighted) LIP. With just two goods, bids have a single "fin" (Definition 2.23(2)) and two "hods" (Definition 2.23(1)), as illustrated in Figure 2.

We proceed by first finding bids whose fins "cover" the facets of the LIP[26] that are fins, starting from the "top right". We will then be able to create any facets that are still needed (these will be hods), by adding bids on the edges of a "bounding box" (Definition 2.12)). (We choose a bounding box large enough to include all prices we might ever be interested in.)

We begin by creating a grid from (i) the affine spans of the fins of the LIP, (ii) the affine spans of the horizontal hods, and (iii) the edges of the bounding box, here chosen as $[-1,6]^2$.[27] We show these grid lines as dashed lines in Figure 5. We then *also* add any additional horizontal lines that pass through any 0-cell of this grid, to this grid—in our current example, we need to add three lines, which we show as dotted lines in Figure 5. Next, along each distinct affine-span-of-fin in the grid, we define a "candidate bid" at every 0-cell of the grid on that affine-span-of-fin.[28] The "tradeoff", $\mathbf{t}$, of a candidate bid is chosen so that the fin of the bid covers part of that affine-span-of-fin.[29] So the candidate bids for the (affine span of the) fin with normal $(1,-2)$ have tradeoff $\mathbf{t}=(1,2)$, and the candidate bids for the (affine span of the) fin with normal $(1,-1)$ have tradeoff $\mathbf{t}=(1,1)$.

We now take all candidate bids that are located in a fin (not merely in the fin's

$1.(1,0)+2.(-1,1)=\mathbf{0}$.

[26]For ease of reading, we largely omit the adjective "weighted" on LIPs and pseudo-LIPs, where this cannot result in any ambiguity.

[27]The bounding box (in two dimensions) is any square that contains part or all of every cell of the LIP (see Definition 2.12 and Assumption 4.15). So we could choose a smaller (or larger) box, but wish our figure to be clear.

[28]Our proof does not, in fact, define candidate bids at the 0-cells on the bottom edges of the bounding box, since we will never need such candidates.

[29]That is, the bid's pseudo-LIP's "strong diagonal" (Definition 2.24) is in that affine-span-of-fin.

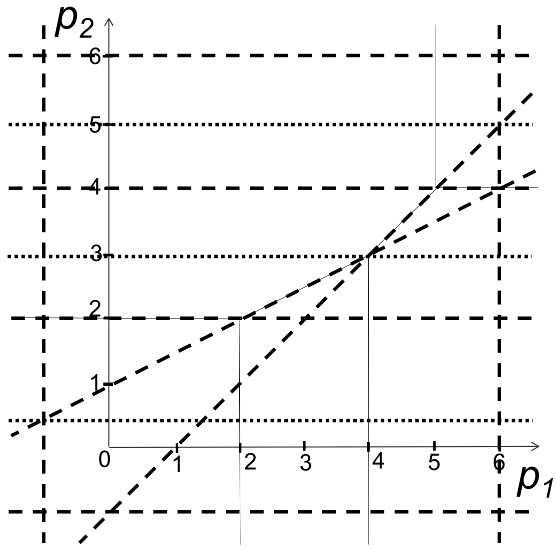


Figure 5: The "grid". "Candidate bids" are located at the intersections of the dotted and dashed horizontal and vertical lines (hods) with the dashed diagonal lines. (Light solid lines that are not axes are facets of the original LIP.)

affine span) of our LIP, and for which no other candidate bid that is in a fin of our LIP with the same affine span is located at higher coordinates. In this case, there are two such candidates, located at $(4, 3)$ on the LIP fin with normal $(1, -2)$, and at $(5, 4)$ on the LIP fin with normal $(1, -1)$. For each candidate, we choose the multiplicity that matches the weight of its corresponding fin, here 1 and 2 respectively. So we have now created two bids, namely (4,3; 1,2; 1) and (5,4; 1,1; 2), that will be part of the set of bids representing our valuation.

We now subtract the pseudo-LIPs of the bids we have just created from our original LIP. This yields the pseudo-LIP in Figure 6. We see that subtracting the bid located at $(5, 4)$ removed all the facets that emanated from that 0-cell, but created a new pair of contiguous fin facets below $(4, 3)$. (This new pair of fins would be a single facet if they did not cross the vertical facet that starts at $(2, 2)$ and which was also thereby divided into two separate facets.) Meanwhile, subtracting the bid located at $(4, 3)$ removed the fin that originally started at that 0-cell, but created a new fin starting at $(2, 2)$; it also created two new hods starting at $(4, 3)$. Importantly, although the number of facets remaining is no smaller than before (it is actually larger) the remaining fins are "lower down" in their affine spans than previously.

We now repeat the procedure of the previous two paragraphs, except that we apply it to the pseudo-LIP in Figure 6. There are now two candidate bids that did not previously satisfy our conditions for conversion into actual bids, but are now each in a fin of the current pseudo-LIP (that of Figure 6) with no other candidate in this pseudo-LIP above them in the affine span of their respective fin.[30] These candidates are located at $(2, 2)$ on the pseudo-LIP fin with normal $(1, -2)$, and at $(4, 3)$ on the pseudo-LIP fin with normal $(1, -1)$. The multiplicities that match the weights of the corresponding fins, are

[30]Since the affine spans of the fins and horizontal hods of the bids we subtract are always among those we used to create the original grid, we need never concern ourselves with the possibility of new candidate bids.

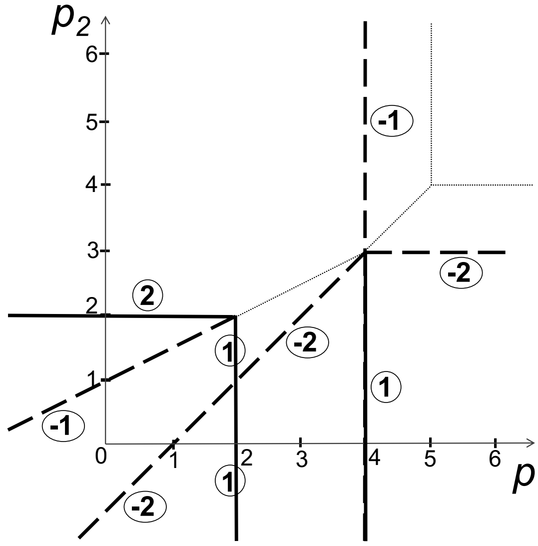


Figure 6: The pseudo-LIP remaining after subtracting the pseudo-LIPs of the bids $(4, 3; 1, 2; 1)$ and $(5, 4; 1, 1; 2)$ from the LIP of Figure 4 is shown as solid and dashed lines; its facet weights are shown in circles. The facets added to the LIP of Figure 4 are the dashed lines; the facets removed from the LIP of Figure 4 are shown as light dotted lines.

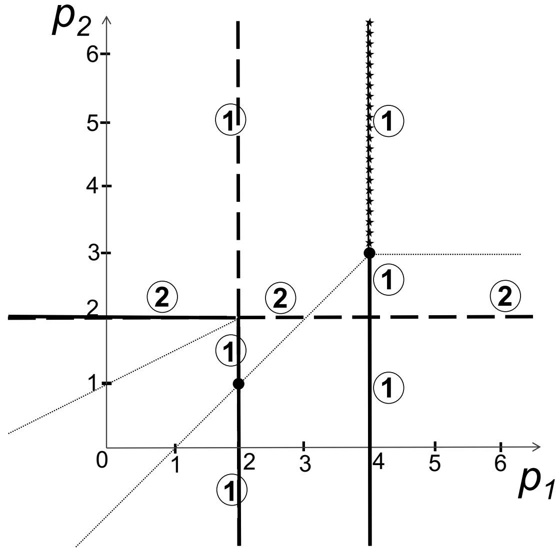


Figure 7: The pseudo-LIP remaining after subtracting the pseudo-LIPs of the bids $(2, 2; 1, 2; -1)$ and $(4, 3; 1, 1; -2)$ from the pseudo-LIP of Figure 6 is shown as solid, dashed, and dotted-solid lines; its facet weights are shown in circles. The facet that is in the pseudo-LIP of Figure 6, but whose weight is different is shown as the dotted-solid line. The facets added to the pseudo-LIP of Figure 6 are the dashed lines; the facets removed from the pseudo-LIP of Figure 6 are shown as light dotted lines.

now negative; they are -1 and -2, respectively. So we have specified two new actual bids, (2,2; 1,2; -1) and (4,3; 1,1; -2), to add to the list of bids that will represent the valuation we started with.

Subtracting the pseudo-LIPs of the bids we have just found from the current pseudo-LIP (that of Figure 6) yields the pseudo-LIP shown in Figure 7. The new subtraction has removed all the remaining facets except the hods to the left of $(2,2)$, below $(2,2)$, and below $(4,3)$ (which latter hod is now divided into two). Moreover, it has added $+2$ to the weight of the hod facet extending vertically upwards from $(4,3)$, thus giving it the same weight as the hods below $(4,3)$, and it has also added new hod facets extending above, and to the right of, $(2,2)$.

There are now no remaining candidate bids satisfying our conditions for conversion into actual bids.

Observe that we considered candidate bids that were "higher" in the affine span of a LIP or pseudo-LIP fin before any that were lower in the same affine span. The reason is that the fins of bids located on higher facets affect the weights of lower facets, but not vice versa. So considering candidates in this order (but not, e.g., in the opposite order) guarantees that, when our consideration of candidate bids terminates, we will have selected a collection of bids for which the sum of their pseudo-LIPs exactly matches the fins of our original LIP. That is, it is a general result that there will then be no fins in the remaining pseudo-LIP that we still need to match.

Observe also that, having eliminated the LIP fins, the remaining hods with the same affine span have the same weights and, moreover, their union "continues indefinitely", that is, coincides with its affine span. The reason for this is that our "editing" of our original LIP has always respected the balancing condition (Definition 2.20), so our remaining (weighted) pseudo-LIP must also obey the balancing condition, so when only horizontal and vertical facets remain, they cannot "stop dead", or change weight, at any point. So it is also a general result that, as here, there can be no isolated hod-segments, and no weight changes along any affine span of hods, in the pseudo-LIP that remains after eliminating all the fins.

It therefore follows that the hods remaining after eliminating the fins all correspond to bids located at extreme prices, i.e., bids that we will place at edges of the bounding box (one bid for each union of hods with the same affine span). In our example, since we chose $[-1,6]^2$ as our bounding box, the three required additional bids are located at $(2,-1)$ and $(4,-1)$, with multiplicity 1, and $(-1,2)$ with multiplicity 2.[31]

In sum, our original (weighted) LIP is generated by the seven bids $(4,3;1,2;1)$, $(5,4;1,1;2), (2,2;1,2;-1), (4,3;1,1;-2), (2,-1;1,1;1), (4,-1;1,1;1), (-1,2;1,1;2)$, recalling that the format for each bid is $\mathbf{b} = (\mathbf{r}(\mathbf{b}); \mathbf{t}(\mathbf{b}); m)$.

3.3 Any Concave Ordinary Substitutes Valuation can be Represented by PMA Bids: the General Case

We show the general result by induction.[32] We need to show that for any LIP of a concave ordinary substitutes valuation in n dimensions, there exists a collection of valid bids for which the sum of their pseudo-LIPs is identical to the given LIP, within any bounding box we define.[33] So we assume the result is true in $n-1$ dimensions:

As in the two-goods case, the harder facets of a n-dimensional valuation's LIP to replicate with pseudo-LIPs of bids are the LIP's fins. We proceed by considering the

[31]By convention we set $\mathbf{t}(\mathbf{b}) = \mathbf{1}$ for bids on lower faces of the bounding box.

[32]As in two dimensions, the proof is constructive, so it implies a recursive algorithm to find the collection of bids that represents the valuation.

[33]If the valuation's domain is an FBD (Definition 2.9), we do not need any bounding box (Theorem 2.11); otherwise the bounding box can be arbitrarily large (Theorem 2.13).

fins that are normal to just one pair of dimensions, say dimensions n and $(n-1)$, and then successively consider fins normal to other pairs of dimensions.

So we first form a grid that is a natural generalisation of the one we formed in two dimensions.[34] Then, exactly as we described for the two-dimensional case, we define a "candidate bid", for each 0-cell of the grid, and separately for every 1-cell in the grid starting at that 0-cell and going down in all coordinates. The candidate bid's root, $\mathbf{r}$, is located at the 0-cell, and its tradeoff, $\mathbf{t}$, is such that the candidate bid's pseudo-LIP's strong diagonal contains the 1-cell.[35] A distinction from the two-dimensional case is that there are also additional candidate bids on the lower faces of the bounding box (with respect to some coordinates other than $(n-1, n)$); for simplicity we ignore these–this simplification affects only this paragraph (and its footnotes). Then, again as in two dimensions, we choose multiplicities for successive sets of candidate bids[36] so that, in each set of candidate bids, part of each candidate bid's pseudo-LIP matches an $(n-1, n)$ fin of our remaining pseudo-LIP.[37] (For the first set of candidate bids, the "remaining pseudo-LIP" is just our original LIP.) For each successive set of candidate bids, we deduct their pseudo-LIPs from our remaining pseudo-LIP.

As in the two-dimensional case, it can be shown that the procedure of the previous paragraph will leave a pseudo-LIP with no $(n-1, n)$ fins.[38] Crucially, it follows from this that any cross-section perpendicular to dimension $n-1$ of any of its (i, n) fins, for $i \neq n-1$, is independent of the value of p_{n-1} at which it is taken. To demonstrate this, we first observe that the fin contains lines parallel to the $n-1$ axis. Second, similarly to our discussion of the two-dimensional case, we observe that because the pseudo-LIP must obey the balancing condition, the fin cannot "stop dead" at any p_{n-1}, except at a particular class of intersections with two other fins, and it is not too hard to check that this cannot happen.[39] That is, if the fin ends at some p_{n-1}, another fin with the same affine span, and same weight, must start at p_{n-1}. So a contiguous series of such fins,

[34] We create an initial grid consisting of (i) the affine spans of the LIP's (i, n) fins, (ii) the affine spans of the $n-$hods, and (iii) the edges of the bounding box. We then add additional $n-$hods that pass through particular $(n-2)$-cells of the initial grid.

[35] As in two dimensions, these 1-cells are all in the affine spans of (i, n) fins for all i.

[36] Choosing an order in which to consider candidates, that ensures that bids chosen later do not upset matches made earlier, is harder than in the two-dimensional case, but Appendix B shows how to create a partial ordering of candidates (given in Definition B.22), which we can use to select the successive sets of bids. (Note that Appendix B uses a nomenclature opposite to that of Section 3.2, describing the bids we want to select earlier as "lower", as is perhaps more natural in multiple dimensions.)

[37] Precisely: the candidate bid has its root in the $(n-1, n)$ fin of our remaining pseudo-LIP; its tradeoff, $\mathbf{t}$, is such that part of its pseudo-LIP's strong diagonal is in the $(n-1, n)$ fin of our remaining pseudo-LIP; and its multiplicity is the weight of the $(n-1, n)$ fin of the remaining pseudo-LIP divided by the greatest common divisor of $t_{(n-1)}$ and t_n (see Lemma 4.9, part (2)(ii), illustrated in Figure 3b).

[38] As in our two-dimensional example, "covering" the $(n-1, n)$ fins in this way will in fact find all the bids we need, except those for which at least one root coordinate is on the bounding box. The reason is that the 1-cells that we use to define the tradeoffs of the candidate bids contain all the information needed to mimic the full structure of the original LIP at the corresponding 0-cells, and we use all that information, so that that structure is indeed respected in all coordinate directions.

[39] If the fin stops at an intersection with a single other facet, the balancing condition can only hold if there is another fin with the same affine span and weight extending in the opposite direction from the intersection. So if there were no such fin extending in the opposite direction from the intersection, the fin stopping at the intersection would have to meet both an $(n-1, n)$ fin and an (i, n) fin. (Corollary A.3 gives details.) But we are now working with a pseudo-LIP which has no $(n-1, n)$ fins, so this is not possible.

continuing in direction $n-1$ to the bounding box, must be contained in the pseudo-LIP.

It follows that projecting the (i,n) fins of our remaining pseudo-LIP into $n-1$ dimensions by deleting (what was) the $n-1$th coordinate loses no information about the (i,n) fins. By the inductive hypothesis, in $n-1$ dimensions we can find bids whose pseudo-LIPs exactly replicate the intersection of our remaining pseudo-LIP with the lower face of the bounding box in coordinate direction $(n-1)$. So, viewing these bids as n-dimensional bids (where r_{n-1} is on the lower face of the bounding box) they *also* exactly replicate the (i,n) fins. Moreover the bids we found in this way can have no $(i,n-1)$ fins within the bounding box, for any i, so will not recreate any of the $(n-1,n)$ fins that we had previously deleted. We therefore delete these bids' pseudo-LIPs from our remaining pseudo-LIP, to yield a pseudo-LIP with no (i,n) fins, for any i.

We now handle the (i,j) fins of our remaining pseudo-LIP, for $i,j \in [1,n-1]$, in a similar way. We project these, by deleting (what was) their nth coordinate, to find bids whose pseudo-LIPs replicate them. For the same reason that we created no new $(n-1,n)$ fins when we eliminated the (i,n) fins in the step of the previous paragraph, no (i,n) fins are created when eliminating the (i,j) fins for $i,j \in [1,n-1]$. So this now gives us a collection of bids whose pseudo-LIPs together exactly replicate all the fins of the original valuation's (weighted) pseudo-LIP.

As in two dimensions, the remaining pseudo-LIP therefore consists of a set of contiguous series of hods, with each series consisting of hods with the same affine spans and weights, and with each series running from one face of the bounding box to the opposite face. And, as we illustrated in the two-dimensional case, it is straightforward to create these hods by adding additional bids located on the faces of the bounding box.[40]

So given that our result is true in $n-1$ dimensions, it is true in n dimensions, and it is easy to see that it holds in one dimension (or indeed, from Section 3.2, in two dimensions).

3.4 Any Valid Collection of PMA Bids Describes a Concave Ordinary Substitutes Valuation

The converse of the preceding result, i.e., that any valid collection of bids describes a concave ordinary substitutes valuation, is relatively straightforward. The sum of the bids' pseudo-LIPs is itself a pseudo-LIP. Moreover, this pseudo-LIP's facets must all have positive weights (because an increase in a good's price across a negatively weighted facet would increase demand for the good, violating validity). So, by the Valuation Complex Equivalence Theorem (Fact 2.21), there exists a (unique) concave valuation corresponding to this (weighted) pseudo-LIP (with demand set at the bounding box's highest price including $\mathbf{0}$, and valuation $v(\mathbf{0}) = 0$). That is, this pseudo-LIP is in fact a LIP. Furthermore, since each individual bid (or its negative) represents an ordinary substitutes valuation, the normals to the bids' pseudo-LIP's facets are all vectors of the ordinary substitutes demand type, from which it follows easily that the normals to the LIP that is the sum of the bids' pseudo-LIPs are also vectors of the ordinary substitutes demand type. So the (weighted) LIP corresponds to a concave ordinary substitutes valuation (Fact 2.22), and the collection of bids therefore describes a concave ordinary

[40] The additional bids will in fact be located on lowest 1-cells of the bounding box, so they create no new fins in the interior of the bounding box.

substitutes valuation.

3.5 Uniqueness of the Representation of a Valuation by PMA Bids

Finally, consider two (weighted) LIPs, each of which is the sum of the pseudo-LIPs of a collection of (valid) bids with no redundancies (including no redundancies relative to the bounding box—see Definition 2.14). If the two collections of bids are different, then either one of the LIPs has a facet that the other LIP does not have (this can be shown to be the case if any of the bids' roots or strong diagonals differ between the collections), or one of the LIPs has a facet with greater weight than the otherwise identical facet in the other LIP (this is the case if any of the bids' multiplicities differ between the collections). In either case, considering a change in price across this facet shows that the valuations corresponding to the LIPs are different on at least one side of this facet. So the two collections of bids cannot both correspond to the same valuation, and it follows that the representation of a valuation by bids is essentially unique.

4 Bids and Weighted Pseudo-LIPs

4.1 Pseudo-LIPs

Recall from Section 2.2 that we introduce both positive bids and negative bids, using both in our representation of valuations. The "multiplicity" of these bids is closely associated with the weights of the facets of the valuation's LIP.

As we explained in Section 2.2, if $m(\mathbf{b}) \notin \mathbb{Z}_{>0}$ then bid $\mathbf{b}$ does not correspond to a meaningful economic valuation. But, considering our definition of their demand sets (Equation (3)) and the properties of weights of facets of LIPs (Definition 2.16 part (7) and Fact 2.19), it is natural to associate this bid with the set $\mathcal{L}_{v_{|\mathbf{b}|}}$, where we recall that we write $|\mathbf{b}| = (\mathbf{r}(\mathbf{b}); \mathbf{t}(\mathbf{b}); |m(\mathbf{b})|)$, but then multiply the weight of each facet by -1. We will indeed do so (see Section 4.3). However, because there is no corresponding economic valuation, and in particular because the law of demand (Definition 2.4) is not satisfied, this is not a true "locus of indifference prices". Thus we must widen our class of objects of study. Moreover, because we will work with collections of both positive and negative bids, we allow combinations of facets of either sign.

The "pseudo-LIPs" $\mathcal{L}$ which we will now work with can differ from true LIPs not only in weighting, but also in shape. In particular, it will not in general be as straightforward to identify polyhedral facets directly from the set $\mathcal{L}$, as it is from a true LIP, as we discuss below. However, we recall that a LIP $\mathcal{L}_v$ is associated with its price complex, a balanced $\mathbb{Z}_{>0}$-weighted rational polyhedral complex whose support is $\mathbb{R}^n$; the facets of the LIP are also the facets of the price complex. We therefore similarly define a pseudo-LIP from such a complex, but now allow a $\mathbb{Z}$-weighting.

Definition 4.1. Fix (Π, w), where Π is a balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$. The *weighted pseudo-LIP* of (Π, w) is the pair $(\mathcal{L}, w)$, where $\mathcal{L}$ is the union of the facets F of Π such that $w(F) \neq 0$, and the weight w on facets $F \subseteq \mathcal{L}$ is inherited from (Π, w).

Here, Π is analogous to the price complex of a standard LIP (Definition 2.16 Part (4)). It is implicitly n-dimensional because its support is $\mathbb{R}^n$. Because we only define a weighted pseudo-LIP of a balanced complex (Π, w), and we only remove 0-weighted facets, it follows that every weighted pseudo-LIP is balanced.

We do not allow 0-weighted facets because we will want to associate a "demand" with the connected components of the complement of the pseudo-LIP; we will be interested in how demand changes across facets (by analogy with Fact 2.19, it would not change across a 0-weighted facet).[41]

By definition, the facets of a pseudo-LIP $\mathcal{L}$ are inherited from Π. The facets of a true LIP $\mathcal{L}_v$ are the faces of the maximal (n-dimensional) price complex cells of v, and so the facets are the maximal $(n-1)$-dimensional subsets of $\mathcal{L}_v$ that only intersect in their boundaries. This is not the case for pseudo-LIPs.[42] We cannot define such sets to be the facets of the pseudo-LIP, because such sets need not be convex, and so need not be polyhedra.[43] However, none the less, such sets do have a consistent weighting in our pseudo-LIPs:

Lemma 4.2. *If F, F' are facets of the weighted pseudo-LIP $(\mathcal{L}, w)$ such that $F \cap F'$ is $(n-2)$-dimensional and no other facets of $(\mathcal{L}, w)$ contain $F \cap F'$ in their boundary, then F and F' share a common affine span and $w(F) = w(F')$.*

Proof. The balancing condition (Definition 2.20) holds around $F \cap F'$. So, if $\mathbf{d}$ and $\mathbf{d}'$ are the respective primitive integer normal vectors to F and F', chosen with respect to a coherent rotational direction, then $w(F)\mathbf{d} + w(F')\mathbf{d}' = \mathbf{0}$. Since F and F' are not coincident, we reject $\mathbf{d}' = \mathbf{d}$, and so $\mathbf{d}' = -\mathbf{d}$, showing their affine spans are the same, and $w(F) = w(F')$. $\square$

If $\mathcal{L} = \mathcal{L}'$ (as sets) and if $w(F) = w(F')$ whenever $F \cap F'$ is $(n-1)$-dimensional for a facet F of $\mathcal{L}$ and a facet F' of $\mathcal{L}'$, then, when there is no ambiguity, we will abuse notation and say that $(\mathcal{L}, w) = (\mathcal{L}', w')$. Similarly, for an open subset $U \subsetneq \mathbb{R}^n$, we will write $(\mathcal{L} \cap U, w) = (\mathcal{L}' \cap U, w')$ to mean that $\mathcal{L} \cap U = \mathcal{L}' \cap U$ and $w(F) = w(F')$ whenever $F \cap F' \cap U$ is $(n-1)$-dimensional for a facet F of $\mathcal{L}$ and a facet F' of $\mathcal{L}'$.

A positive-integer-weighted pseudo-LIP is a "true" LIP of a valuation. Using the abuse of notation just described, we may show:

Proposition 4.3. *$(\mathcal{L}, w)$ is a weighted pseudo-LIP such that $w(F) \in \mathbb{Z}_{>0}$ for all facets F of $\mathcal{L}$ if and only if $(\mathcal{L}, w) = (\mathcal{L}_v, w_v)$ for some concave valuation v.*

Proof. If v is a valuation then, by Fact 2.18 the price complex is a rational polyhedral complex with support $\mathbb{R}^n$, and the induced weighting w_v on the facets is balanced and takes values in $\mathbb{Z}_{>0}$. Such a complex defines a pseudo-LIP (Definition 4.1). Conversely, let (Π, w) be the underlying complex defining $(\mathcal{L}, w)$, and suppose that $w(F) \in \mathbb{Z}_{\geq 0}$ for

[41] See Lemma 4.13 later for the case of pseudo-LIPs induced by bids.

[42] For example, let Π be the set of quadrants of $\mathbb{R}^2$, together with their faces, and define w to be 0 on the vertical facets and 1 on the horizontal facets. Then $\mathcal{L}$ is just given by $\{\mathbf{p} \in \mathbb{R}^2 \mid p_2 = 0\}$. However, its facets, as inherited from Π, are $\{\mathbf{p} \in \mathbb{R}^2 \mid p_1 \leq 0,\, p_2 = 0\}$ and $\{\mathbf{p} \in \mathbb{R}^2 \mid p_1 \geq 0,\, p_2 = 0\}$.

[43] For example, the set $\{\mathbf{p} \in \mathbb{R}^3 \mid p_3 = 0;\, 0 \leq p_1 \leq 1 \textit{ or } 0 \leq p_2 \leq 1\}$ is not convex. However, it is the maximal $(n-1)$-dimensional subset of a pseudo-LIP, as is shown in Example B.11 in Appendix B.3. (Note that this example rests on an understanding of the pseudo-LIPs defined by bid collections, as developed in Section 4 below.)

all facets F. Observe that the set $\widehat{\Pi}$ of all facets F of Π such that $w(F) > 0$, taken together with all faces of these facets, is a rational polyhedral complex of dimension $(n-1)$. Restricting the weight w to the facets of $\widehat{\Pi}$, we see that the balancing condition (Definition 2.20) is still satisfied, as all facets in Π that are not in $\widehat{\Pi}$ have weight 0. We can now invoke the Valuation-Complex Equivalence Theorem (Fact 2.21 above) to see that $\mathcal{L}$, which is the support of $\widehat{\Pi}$, is the LIP of a concave valuation v which induces the same weighting. □

Finally, we extend the definition of demand types to pseudo-LIPs in the natural way:

Definition 4.4. If $\mathcal{D}$ is demand type vector set then *a weighted pseudo-LIP* $(\mathcal{L}, w)$ *is of demand type* $\mathcal{D}$ if all of the facets of the underlying complex (Π, w) have normal vectors in $\mathcal{D}$.

Note that in this way we can refer to a pseudo-LIP as being "of the ordinary substitutes demand type" despite it not corresponding to any valuation (concave ordinary substitutes or not).

4.2 Arithmetic of Pseudo-LIPs

Recall that we seek to express the demand arising from a concave ordinary substitutes valuation as the demand defined by a bid collection. While bids are all positive, the demand set defined by a bid collection is just given by the convex hull of the Minkowski sum of demands from individual bids (Equation (2)). So demand is unique for the collection $\mathcal{B}$ if and only if it is unique for every $\mathbf{b} \in \mathcal{B}$: the LIP $\mathcal{L}_{\mathcal{B}}$ corresponding to $\mathcal{B}$ of positive bids will be the union of the individual LIPs. Similarly, the weight of a facet in $\mathcal{L}_{\mathcal{B}}$ will be the sum of the weights of the facets of the LIPs of individual bids which contain this facet. (We give more details on the LIPs and pseudo-LIPs associated with bids in the subsequent subsections.)

Fundamental to our proofs will be presenting this construction as an "addition", which we notate as $\boxplus$. And, since we allow negative-weightings in our weighted pseudo-LIPs, this naturally extends to subtraction, notated $\boxminus$.[44]

Intuitively, then, we add two weighted pseudo-LIPs by: taking their union; calculating the weight on their facets; and then removing the zero-weighted facets. To construct this formally, we recall that weighted pseudo-LIPs are defined by the non-zero-weighted facets of n-dimensional polyhedral complexes. To combine two such complexes, we consider intersections of their cells: where a facet of one meets an n-cell of the other, we obtain a piece of the union of their facet sets, and if two facets have $(n-1)$-dimensional intersection, we similarly obtain a piece of this union. So we define:

Definition 4.5. Let (Π^1, w^1) and (Π^2, w^2) be the balanced $\mathbb{Z}$-weighted rational polyhedral complexes of dimension n with support $\mathbb{R}^n$, and let $(\mathcal{L}^1, w^1)$ and $(\mathcal{L}^2, w^2)$ be their respective weighted pseudo-LIPs.

(1) $(\Pi^1, w^1) \boxplus (\Pi^2, w^2)$ is the $\mathbb{Z}$-weighted complex (Π, w) where Π has cells $C^1 \cap C^2$ for $C^1 \in \Pi^1$, $C^2 \in \Pi^2$ and, for each facet F of Π, $w(F)$ is the sum of the weight of all facets of Π^1 and Π^2 which contain F.

[44] This construction is conceptually very similar to "tropical intersection theory"; see Allermann and Rau (2010).

(2) $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$ is the weighted pseudo-LIP of $(\Pi^1, w^1) \boxplus (\Pi^2, w^2)$.
(3) $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2) := (\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, -w^2)$

Again, we will slightly abuse notation and, for an open set U, write $(\mathcal{L}^1 \cap U, w^1) \boxplus (\mathcal{L}^2 \cap U, w^2)$ for the set $\mathcal{L} \cap U$, and weighting w restricted to this set, where $(\mathcal{L}, w) = (\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$. We will notate subtraction of subsets of weighted pseudo-LIPs similarly.

The meaning of $\boxplus$ and $\boxminus$ is also clear from:

Lemma 4.6. *Let $(\mathcal{L}^1, w^1)$ and $(\mathcal{L}^2, w^2)$ be weighted pseudo-LIPs. Then $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$ is the weighted pseudo-LIP $(\mathcal{L}, w)$ where*

(1) $\mathcal{L}$ is the closure of the set of points $\mathbf{p}$ such that either $\mathbf{p}$ is in the interior of a facet F^1 of $\mathcal{L}^1$, or $\mathbf{p}$ is in the interior of a facet F^2 of $\mathcal{L}^2$, or both; and if indeed both hold then $w^1(F^1) + w^2(F^2) \neq 0$;
(2) the weight of a facet $F$of $\mathcal{L}$ is the sum of the weights of the facets of $\mathcal{L}^1$ and $\mathcal{L}^2$ containing F.

Once addition "$\boxplus$" of $\mathbb{Z}$-weighted pseudo-LIPs is understood, the subtraction operation "$\boxminus$" is clear, as it is simply addition of the weighted pseudo-LIP whose facets have the opposite sign.

We see that $\boxplus$ and $\boxminus$ have the following standard properties, which allow us to indeed think of them as an "arithmetic" of weighted pseudo-LIPs.

Lemma 4.7. *If $(\mathcal{L}^1, w^1)$, $(\mathcal{L}^2, w^2)$ and $(\mathcal{L}^3, w^3)$ are weighted pseudo-LIPs, then so are $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$ and $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2)$, and the usual rules of addition and subtraction hold, with $(\emptyset, 0)$ playing the role of the identity element. That is:*

(1) $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2) = (\mathcal{L}^2, w^2) \boxplus (\mathcal{L}^1, w^1)$;
(2) $(\mathcal{L}^1, w^1) \boxplus ((\mathcal{L}^2, w^2) \boxplus (\mathcal{L}^3, w^3)) = ((\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)) \boxplus (\mathcal{L}^3, w^3)$;
(3) $(\emptyset, 0) \boxplus (\mathcal{L}^1, w^1) = (\mathcal{L}^1, w^1) \boxplus (\emptyset, 0) = (\mathcal{L}^1, w^1)$;
(4) $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2) = (\emptyset, 0) \boxminus ((\mathcal{L}^2, w^2) \boxminus (\mathcal{L}^1, w^1))$;
(5) $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^1, w^1) = (\emptyset, 0)$.

4.3 The LIP from a Single Positive or Single Negative Bid

Recall, from Section 2.2, that a single bid $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ with root $\mathbf{r}(\mathbf{b}) = \mathbf{r}$, trade-off $\mathbf{t}(\mathbf{b}) = \mathbf{t}$ and positive integer multiplicity $m = m(\mathbf{b}) \in \mathbb{Z}_{>0}$ represents the valuation $v_{\mathbf{b}}$ with $v_{\mathbf{b}}(\mathbf{x}) = \mathbf{r} \cdot \mathbf{x}$ for $\mathbf{x}$ in its domain, $m\mathbf{t} \odot \Delta_{[n]_0}$. To simplify notation, we notate its weighted LIP $(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$. We recall from Equation (1) that the demand set for such a bid is $D_{\mathbf{b}}(\mathbf{p}) = m(\mathbf{b})t(\mathbf{b}) \odot \Delta_{I(\mathbf{b},\mathbf{p})}$, where $I(\mathbf{b}, \mathbf{p}) = \arg\max_{i \in [n]_0} t_i(\mathbf{b})(r_i(\mathbf{b}) - p_i)$. We can now define the corresponding weighted pseudo-LIP for negative-weighted bids:

Definition 4.8. The weighted pseudo-LIP $(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$ of a bid $\mathbf{b}$ where $m(\mathbf{b}) < 0$ is defined by $\mathcal{L}_{\mathbf{b}} := \mathcal{L}_{|\mathbf{b}|}$ and $w_{\mathbf{b}}(F) = -w_{|\mathbf{b}|}(F)$ for all facets F of $\mathcal{L}_{\mathbf{b}}$.

Now:

Lemma 4.9. *If $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ then the weighted pseudo-LIP $(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$ has weighted facets:*

(1) an i-hod for each $i \in [n]$, which we write $F^i_{\mathbf{b}}$, and such that:

(i) $F^i_{\mathbf{b}} = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i;\ p_j \geq r_j \text{ for } j \neq i\}$

(ii) $w_{\mathbf{b}}(F^i_{\mathbf{b}}) = mt_i$;

(iii) *prices* $\mathbf{p}$ *are in* $F^i_{\mathbf{b}}$ *if and only if* $\{0, i\} \subseteq I(\mathbf{b}, \mathbf{p})$, *with equality for prices in the interior of* $F^i_{\mathbf{b}}$.

(2) *an* $\left(i, j; \frac{t_i}{t_j}\right)$*-fin (equivalently, a* $\left(j, i; \frac{t_j}{t_i}\right)$*-fin) for each* $i, j \in [n]$ *with* $i \neq j$*, which we write* $F^{ij}_{\mathbf{b}}$*, and such that:*

(i) $F^{ij}_{\mathbf{b}} = \{\mathbf{p} \in \mathbb{R}^n \mid p_i \leq r_i,\ t_i(p_i - r_i) = t_j(p_j - r_j) \leq t_k(p_k - r_k) \text{ for } k \neq i, j\}$

(ii) $w_{\mathbf{b}}(F^{ij}_{\mathbf{b}}) = m \gcd(t_i, t_j)$;

(iii) *prices* $\mathbf{p}$ *are in* $F^{ij}_{\mathbf{b}}$ *if and only if* $\{i, j\} \subseteq I(\mathbf{b}, \mathbf{p})$, *with equality for prices in the interior of* $F^{ij}_{\mathbf{b}}$.

Because m plays no role in the description of the sets $F^i_{\mathbf{b}}$ and $F^{i,j}_{\mathbf{b}}$ we will also write these as $F^i_{(\mathbf{r};\mathbf{t})}$ and $F^{i,j}_{(\mathbf{r};\mathbf{t})}$ respectively, where it is more convenient to do so.

4.4 The weighted pseudo-LIP of a bid collection

We formally define our collections of bids, and the associated weighted pseudo-LIPs:

Definition 4.10.

(1) A *bid collection* $\mathcal{B}$ is a finite collection of bids $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ where $\mathbf{r} \in \mathbb{R}^n$, $\mathbf{t} \in \mathbb{Z}^n_{>0}$ and $m \in \mathbb{Z} \setminus \{0\}$.

(2) For a bid collection $\mathcal{B}$ define $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}}) := \boxplus_{\mathbf{b} \in \mathcal{B}} (\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$.

We do not need to specify the order of arithmetic $\boxplus$ in Definition 4.10 Part (2) by Lemma 4.7, which also affirms that $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ is a weighted pseudo-LIP. It immediately follows that bid collections may be combined in the following ways:

Lemma 4.11. *Let* $\mathcal{B}^1$ *and* $\mathcal{B}^2$ *be bid collections. Write* $\mathcal{B}^3 = \{(\mathbf{r}(\mathbf{b}); \mathbf{t}(\mathbf{b}); -m(\mathbf{b})) \mid \mathbf{b} \in \mathcal{B}^2\}$.

(1) $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) \boxplus (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2}) = (\mathcal{L}_{\mathcal{B}^1 \cup \mathcal{B}^2}, w_{\mathcal{B}^1 \cup \mathcal{B}^2})$.

(2) $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) \boxminus (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2}) = (\mathcal{L}_{\mathcal{B}^1 \cup \mathcal{B}^3}, w_{\mathcal{B}^1 \cup \mathcal{B}^3})$.

It is immediate from Definitions 4.4, 4.5, and 4.10, and Lemma 4.9 that:

Corollary 4.12. *If* $\mathcal{B}$ *is a bid collection then* $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ *is of the ordinary substitutes demand type.*

Recall that we defined $D_{\mathcal{B}}(\mathbf{p})$ in Section 2.2 (Equation (4)). This corresponds to the weighted pseudo-LIP $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ exactly in the same way as a demand correspondence for a quasilinear valuation corresponds to a LIP (Fact 2.19):

Lemma 4.13. *Let* $\mathcal{B}$ *be a bid collection (not necessarily valid).*

(1) $\mathcal{L}_{\mathcal{B}} = \{\mathbf{p} \in \mathbb{R}^n \mid |D_{\mathcal{B}}(\mathbf{p})| > 1\}$.

(2) $D_{\mathcal{B}}(\mathbf{p})$ *is constant (and a singleton set) for prices in the connected components of the complement of* $\mathcal{L}_{\mathcal{B}}$.

(3) *The change in* $D_{\mathcal{B}}(\mathbf{p})$ *between prices on either side changes of a facet* F *of* $\mathcal{L}_{\mathcal{B}}$ *is* $w_{\mathcal{B}}(F)$ *times the primitive integer vector that is normal to* F*, and points in the the opposite direction to the change in price.*

Recall from Section 2.2 that we defined a bid collection $\mathcal{B}$ as *valid* if the demand correspondence $D_{\mathcal{B}}(\mathbf{p})$ satisfies the law of demand (Definition 2.4). Lemma 4.13 allows us a comprehensive view on validity that subsumes Propositions 2.6 and 2.8:[45]

Proposition 4.14 (Cf. Baldwin et al. 2019, Theorem 1, for the strong substitutes case[46])**.** *Let $\mathcal{B}$ be a finite collection of bids. The following are equivalent:*

(1) $\mathcal{B}$ is valid;
(2) $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ is a (positive-) weighted LIP;
(3) For every $\mathbf{p} \in \mathbb{R}^n$ and every $i \in [n]$, the set $\mathcal{B}'$ of bids $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{p} \in F^i_{\mathbf{b}}$ satisfies $\sum_{\mathbf{b}\in\mathcal{B}'} m(\mathbf{b})t_i(\mathbf{b}) \geq 0$; and for every distinct pair $i, j \in [n]$ the set $\mathcal{B}'$ of bids $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{p} \in F^{ij}_{\mathbf{b}}$ satisfies $\sum_{\mathbf{b}\in\mathcal{B}'} m(\mathbf{b}) \gcd(t_i(\mathbf{b}), t_j(\mathbf{b})) \geq 0$.
(4) there exists a concave ordinary substitutes valuation v such that $(\mathcal{L}_v, w_v) = (\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$;
(5) there exists a concave ordinary substitutes valuation v such that $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.

Moreover, when $\mathcal{B}$ is valid, the valuation $v_{\mathcal{B}}$ of Definition 2.7 is well defined and satisfies $D_{v_{\mathcal{B}}}(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.

There is an analogous version of Proposition 4.14 for the strong substitutes case. If we assume that $\mathcal{B}$ is a finite collection of SSPMA bids, then validity of $\mathcal{B}$ is equivalent to (2), to a version of (3) in which $\mathbf{t} = \mathbf{1}$ in every case, and to (4) and (5) modified to specify that v is a strong substitutes valuation. This may be seen by combining Proposition 4.14 with Baldwin et al. (2019, Theorem 1).

4.5 The Bounding Box

Recall from Lemma 2.10 that if $\mathcal{B}$ is a finite valid collection of bids, then the domain $A_{\mathcal{B}}$ of the corresponding valuation is an FBD. Recall from Section 2.2 that, to handle bids with other domains, we introduced a "bounding box" $\mathbf{H} = [\underline{H}, \overline{H}]^n$ (Definition 2.12), within which we will match the required demand set using our bids.

We will generally assume that $\underline{H}, \overline{H}$ satisfy the following relative to a pseudo-LIP $\mathcal{L}$:

Assumption 4.15. $\underline{H} < \overline{H}$ are respectively sufficiently small and large that $\mathbf{H}^\circ \cap C \neq \emptyset$ for every cell of the polyhedral complex Π defining $\mathcal{L}$.

Such $\underline{H}, \overline{H}$ always exist because there are only finitely many cells of any polyhedral complex.

Recall from Lemma 2.10 that any valuation $v_{\mathcal{B}}$ generated by valid bids must have a finite bids domain, and that, conversely, we show in Theorem 2.11 that concave ordinary substitutes valuations whose domain is an FBD can be generated by a valid set of bids. Crucial to our proof of this result will be the following lemma, demonstrating that in this case bids are strictly inside the bounding box. Recall from Definition 2.14 that a bid set has no redundancies relative to $\mathbf{H}$ if contains at most one bid with any combination of root and trade-off, as well as bids on the boundary of $\mathbf{H}$ satisfying certain criteria.

[45] Proving Proposition 4.14 also enables us to prove Lemma 2.10, whose proof is therefore presented at this point in the appendix.

[46] Baldwin et al. (2019) define validity as concavity of an indirect utility function, which they associate to any set of SSPMA bids. This is more convenient for their purposes, and is equivalent to our definition in the SSPMA case, as they show in their Theorem 1 (their Part 2 is equivalent to our Part (3), and their Part 3 is our Part (5)).

Lemma 4.16. *If v is a concave ordinary substitutes valuation with an FBD, if $\underline{H}, \overline{H}$ satisfy Assumption 4.15 for $\mathcal{L}_v$, and if $\mathcal{B}$ is a bid collection with no redundancies relative to $\mathbf{H}$ and as described in Theorem 2.13, then $\mathbf{r}(\mathbf{b}) \in \mathbf{H}^\circ$ for all $\mathbf{r} \in \mathcal{B}$.*

Moreover, if we are only interested in the pseudo-LIP we generate within the bounding box, then it is sufficient to consider bids in the box and on its boundary:

Proposition 4.17. *If $\mathcal{B}$ is a bid collection and $\underline{H} < \overline{H}$ then there exists a bid collection $\mathcal{B}'$ of bids rooted in $\mathbf{H}$ such that $\{\mathbf{b} \in \mathcal{B} \mid \mathbf{r}(\mathbf{b}) \in \mathbf{H}^\circ\} = \{\mathbf{b}' \in \mathcal{B}' \mid \mathbf{r}(\mathbf{b}') \in \mathbf{H}^\circ\}$, such that there is a 1-1 correspondence between $\{\mathbf{b} \in \mathcal{B} \mid \mathbf{r}(\mathbf{b}) \notin \mathbf{H}^\circ\}$ and $\{\mathbf{b}' \in \mathcal{B}' \mid \mathbf{r}(\mathbf{b}') \notin \mathbf{H}^\circ\}$, and such that $(\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^\circ, w_{\mathcal{B}}) = (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$.*

4.6 Uniqueness for Bid Collections

Recall from Section 2.2 that, in order to avoid ambiguity, we adopt the convention that $t_i(\mathbf{b}) = 1$ whenever $r_i(\mathbf{b}) = \underline{H}$ and that $\sum_{r_i(\mathbf{b}) \neq \underline{H}} \mathbf{e}^i t_i(\mathbf{b})$ is a primitive integer vector. With this understood, different bid collections define the same pseudo-LIP only in the following natural way:

Lemma 4.18. *If $\mathcal{B}^1$ and $\mathcal{B}^2$ are bid collections then the following are equivalent:*

(1) $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) = (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2})$;
(2) the sum of multiplicities of bids at any root with the same trade-off are the same in $\mathcal{B}^1$ and in $\mathcal{B}^2$;
(3) there exist $\underline{H}, \overline{H}$ such that $(\mathcal{L}_{\mathcal{B}^1} \cap \mathbf{H}^\circ, w_{\mathcal{B}^1}) = (\mathcal{L}_{\mathcal{B}^2} \cap \mathbf{H}^\circ, w_{\mathcal{B}^2})$ and such that, for all $\mathbf{b} \in \mathcal{B}^1, \mathcal{B}^2$, we have $\mathbf{r}(\mathbf{b}) \in \mathbf{H}$ and
  (i) if $r_i(\mathbf{b}) = \underline{H}$ then $t_i(\mathbf{b}) = 1$;
  (ii) $\sum_{r_i(\mathbf{b}) \neq \underline{H}} \mathbf{e}^i t_i(\mathbf{b})$ is a primitive integer vector.
  (iii) $\mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^\circ \neq \emptyset$.

Lemma 4.18 shows that there is "essentially" only one bid collection giving rise to any weighted pseudo-LIP, or, by Proposition 4.3, to any LIP; it therefore allows us to infer Theorem 2.15 from Theorems 2.11 and 2.13. As we discuss in Section 2.2, bid collections are unique under either of two possible conventions; for the purposes of this paper we are agnostic between these conventions and so bid collections will only be unique up to the description of Lemma 4.18 Part (2).

5 Proof of Main Theorems

5.1 The Fundamental Result

Our main theorems rest on the following fundamental result:

Theorem 5.1. *For any $n' \geq 1$, if $(\mathcal{L}, w)$ is a weighted pseudo-LIP in $\mathbb{R}^{n'}$, of the ordinary substitutes demand type, and $\mathbf{H} = [\underline{H}, \overline{H}]^{n'}$ for any $\underline{H} < \overline{H} \in \mathbb{R}$, then there exists a bid collection $\mathcal{B}$ rooted in $\mathbf{H}$, with no redundancies relative to $\mathbf{H}$, such that $(\mathcal{L} \cap \mathbf{H}^\circ, w) = (\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^\circ, w_{\mathcal{B}})$.*

If $(\mathcal{L}, w)$ is of the strong substitutes demand type then the bids $\mathcal{B}$ are SSPMA bids.

Note that, in Theorem 5.1, we do not assume that $\underline{H}, \overline{H}$ satisfy Assumption 4.15. It is not necessary, as the statement holds for any $\underline{H} < \overline{H}$, and, because we prove the theorem by induction, it is more convenient not to have to check that this condition holds.

5.2 Proofs of the Main Theorems, Contingent on Theorem 5.1

Theorem 5.1 is very close to giving us the representation result inside the bounding box (Theorem 2.13). We sought a bid collection $\mathcal{B}$ such that, at every price $\mathbf{p} \in \mathbf{H}^\circ$, the demand set $D_\mathcal{B}(\mathbf{p})$ is equal to $D_v(\mathbf{p})$. However, there is a little ambiguity in the bids identified by Theorem 5.1. This is because bids $\mathbf{b}$ rooted at the vertices of $\mathbf{H}$ with the form $\mathbf{r}(\mathbf{b}) = \overline{H}\mathbf{e}^i + \sum_{j \neq i} \underline{H}\mathbf{e}^j$ (for $i \in [n]$) satisfy $\mathcal{L}_\mathbf{b} \cap \mathbf{H}^\circ = \emptyset$. However, the presence of such bids affects $D_\mathcal{B}(\mathbf{p})$ throughout $\mathbf{H}$. So we need to fine-tune the multiplicities of the bids at these points, in order to achieve a perfect match in demand sets.

Proof of Theorem 2.13. By Theorem 5.1, there exists a bid collection $\mathcal{B}'$ rooted in $\mathbf{H}$ such that $(\mathcal{L}_v \cap \mathbf{H}^\circ, w_v) = (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$. By Lemma 4.13 and Definition 2.16, it follows that $|D_v(\mathbf{p}^0)| = 1$ if and only if $|D_{\mathcal{B}'}(\mathbf{p}^0)| = 1$ for prices $\mathbf{p}^0 \in \mathbf{H}^\circ$; fix some such $\mathbf{p}^0$. Write $\{\mathbf{x}\} = D_v(\mathbf{p}^0)$ and $\{\mathbf{x}_{\mathcal{B}'}\} = D_{\mathcal{B}'}(\mathbf{p}^0)$, and let $\mathbf{y} = \mathbf{x} - \mathbf{x}_{\mathcal{B}'}$.

Now, for $i \in [n]$, let $\mathbf{b}^i = (\overline{H}\mathbf{e}^i + \sum_{j \neq i} \underline{H}\mathbf{e}^j, \mathbf{1}, y_i)$. It follows that $D_{\mathbf{b}^i}(\mathbf{p}) = \{y_i \mathbf{e}^i\}$ for all $\mathbf{p} \in \mathbf{H}^\circ$. Thus, if we let $\mathcal{B} := \mathcal{B}' \cup \bigcup_{i \in [n]} \{\mathbf{b}^i\}$ then $D_\mathcal{B}(\mathbf{p}^0) = \{\mathbf{x}_{\mathcal{B}'} + \mathbf{y}\} = \{\mathbf{x}\} = D_v(\mathbf{p}^0)$.

By Fact 2.19 Part (2) and Lemma 4.13 Part (3), it follows that $D_\mathcal{B}(\mathbf{p}) = D_v(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$ such that $|D_v(\mathbf{p})| = |D_{\mathcal{B}'}(\mathbf{p})| = 1$. Because the sets of prices at which any one bundle is demanded are closed, and because all demand sets under $D_\mathcal{B}$ and D_v are discrete-convex, it follows that $D_\mathcal{B}(\mathbf{p}) = D_v(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$.

In particular this implies that $D_\mathcal{B}(\mathbf{p})$ satisfies the law of demand for all suitable $\mathbf{p}, \mathbf{p}' \in \mathbf{H}^\circ$, since this is true of $D_v(\mathbf{p})$. So the bids $\mathcal{B}$ are valid in $\mathbf{H}^\circ$.

If v is a strong substitutes valuation then $\mathcal{B}'$ is a collection of SSPMA bids, by Theorem 5.1. The additional bids $\mathbf{b}^i$ are also SSPMA bid for all $i \in [n]$, so $\mathcal{B}$ is a SSPMA bid collection. $\square$

Recall that Theorem 2.11 handled the special case in which the domain of the valuation is an FBD (or a discrete simplex, if the valuation is strong substitutes). This case is simpler if one does not wish to also prove Theorem 2.13: see Remark 5.13. However, given that we have included a proof of that case, it is more efficient to show that this follows from Theorem 2.13, together with Proposition 4.17 and Lemma 4.16.

Proof of Theorem 2.11. Let $\underline{H}, \overline{H}$ satisfy Assumption 4.15 for $\mathcal{L}_v$. Applying Theorem 5.1 and Lemma 4.16, it follows that there exists a bid collection $\mathcal{B}$ with no redundancies relative to $\mathbf{H}$ and such that $\mathbf{r}(\mathbf{b}) \in \mathbf{H}^\circ$ for all $\mathbf{r} \in \mathcal{B}$ and $(\mathcal{L}_\mathcal{B} \cap \mathbf{H}^\circ, w_\mathcal{B}) = (\mathcal{L} \cap \mathbf{H}^\circ, w)$. Recall that $\mathbf{0} \in A_v$, and so, by Assumption 4.15, $\mathbf{0} \in D_v(\mathbf{p})$ for some $\mathbf{p} \in \mathbf{H}^\circ$. Since $A_v \subseteq \mathbb{Z}^n_{\geq 0}$, it follows that $\{\mathbf{0}\} = D_v(\mathbf{p})$ for prices $\mathbf{p} \in \mathbf{H}^\circ$ close to $(\overline{H}, \ldots, \overline{H})$. But if $\mathbf{r}(\mathbf{b}) \in \mathbf{H}^\circ$ for all $\mathbf{b} \in \mathcal{B}$, then $\{\mathbf{0}\} = D_\mathcal{B}(\mathbf{p})$ for prices $\mathbf{p} \in \mathbf{H}^\circ$ close to $(\overline{H}, \ldots, \overline{H})$. So, by Fact 2.19 Part (2) and Lemma 4.13 Part (3) (and as argued in the proof of Theorem 2.13 above), it follows that $D_\mathcal{B}(\mathbf{p}) = D_v(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$.

We now show that demand also matches for prices not in $\mathbf{H}^\circ$, so suppose that $\mathbf{p}^1 \notin \mathbf{H}^\circ$. Fix $\underline{H}' < \underline{H}$ and $\overline{H}' > \overline{H}$ so that $\mathbf{p}^1 \in \mathbf{H}'^\circ$ where we write $\mathbf{H}'^\circ := (\underline{H}', \overline{H}')^n$. It follows again that there is a bid collection $\mathcal{B}'$, with no redundancies relative to $\mathbf{H}'$, such that $\mathbf{r}(\mathbf{b}') \in \mathbf{H}'^{\circ}$ for all $\mathbf{r} \in \mathcal{B}'$ and $D_v(\mathbf{p}) = D_{\mathcal{B}'}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}'^{\circ}$, and satisfying $(\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}'^{\circ}, w_{\mathcal{B}'}) = (\mathcal{L}_v \cap \mathbf{H}'^{\circ}, w)$.

Now apply Proposition 4.17 to $\mathcal{B}'$: there exists a bid collection $\mathcal{B}''$ of bids rooted in $\mathbf{H}$, such that $\{\mathbf{b} \in \mathcal{B}' \mid \mathbf{r}(\mathbf{b}) \in \mathbf{H}^{\circ}\} = \{\mathbf{b} \in \mathcal{B}'' \mid \mathbf{r}(\mathbf{b}) \in \mathbf{H}^{\circ}\}$, such that there is a 1-1 correspondence between $\{\mathbf{b} \in \mathcal{B}' \mid \mathbf{r}(\mathbf{b}) \notin \mathbf{H}^{\circ}\}$ and $\{\mathbf{b} \in \mathcal{B}'' \mid \mathbf{r}(\mathbf{b}) \notin \mathbf{H}^{\circ}\}$ and such that $(\mathcal{L}_{\mathcal{B}''} \cap \mathbf{H}^{\circ}, w_{\mathcal{B}''}) = (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^{\circ}, w_{\mathcal{B}'})$. But by Lemma 4.16 again, it also follows that $\mathbf{r}(\mathbf{b}'') \in \mathbf{H}^{\circ}$ for all $\mathbf{b}'' \in \mathcal{B}''$. So $\emptyset = \{\mathbf{b} \in \mathcal{B}'' \mid \mathbf{r}(\mathbf{b}) \notin \mathbf{H}^{\circ}\} = \{\mathbf{b} \in \mathcal{B}' \mid \mathbf{r}(\mathbf{b}) \notin \mathbf{H}^{\circ}\}$ and thus $\mathcal{B}' = \{\mathbf{b} \in \mathcal{B}' \mid \mathbf{r}(\mathbf{b}) \in \mathbf{H}^{\circ}\}$. That is, we have shown that $\mathbf{b}' \in \mathbf{H}^{\circ}$ for all $\mathbf{b}' \in \mathcal{B}'$.

Moreover, since $\mathbf{H}^{\circ} \subsetneq \mathbf{H}'^{\circ}$, it follows by definition of $\mathcal{B}'$ that $(\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^{\circ}, w_{\mathcal{B}'}) = (\mathcal{L}_v \cap \mathbf{H}^{\circ}, w) = (\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^{\circ}, w_{\mathcal{B}})$. Since we know both sets of bids are rooted in $\mathbf{H}^{\circ}$, we know by Lemma 4.18 that $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}}) = (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$. We also already know that $D_{\mathcal{B}}(\mathbf{p}) = D_v(\mathbf{p}) = D_{\mathcal{B}'}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^{\circ}$, and so it follows by Fact 2.19 that $D_{\mathcal{B}}(\mathbf{p}) = D_{\mathcal{B}'}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}$. In particular, since $\mathbf{p}^1 \in \mathbf{H}'^{\circ}$ and so $D_{\mathcal{B}'}(\mathbf{p}^1) = D_v(\mathbf{p}^1)$, we can conclude that $D_{\mathcal{B}}(\mathbf{p}^1) = D_v(\mathbf{p}^1)$. But $\mathbf{p}^1 \in \mathbb{R}^n$ was arbitrary, so we can conclude $D_{\mathcal{B}}(\mathbf{p}) = D_v(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$. Now, since D_v satisfies the law of demand (for all prices, see Definition 2.4) so does $D_{\mathcal{B}}$, and so the bids $\mathcal{B}$ are valid.

Proposition 2.8 now tells us that $D_{v_{\mathcal{B}}}(\mathbf{p}) = D_v(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$. Applying Fact 2.21, and the fact that $v(\mathbf{0}) = 0 = v_{\mathcal{B}}(\mathbf{0})$, allows us to conclude that $v_{\mathcal{B}} = v$.

The case of strong substitutes here now follows immediately from the case of strong substitutes in Theorem 2.13. □

Proof of Theorem 2.15. The case of Theorem 2.11 is now straightforward. Given bids $\mathcal{B}$ satisfying Theorem 2.11, simply: replace any subset of bids that have the same root and trade-off with a single bid whose multiplicity is the sum of the multiplicities of the bids it replaces; or remove them all if this sum is zero. This gives existence of a suitable bid collection with no redundancies; uniqueness is immediate from Lemma 4.18.

To prove the case of Theorem 2.13, we first combine bids which have the same root and trade-off, as above. Next, setting $t_i(\mathbf{b}) = 1$ when $r_i(\mathbf{b}) = \underline{H}$ does not affect the demand set at any price in $\mathbf{H}^{\circ}$. And if $\gcd\{t_i \mid r_i \neq \underline{H}\} = g > 1$ then replacing t_i with $\frac{t_i}{g}$ and replacing $m(\mathbf{b})$ with $gm(\mathbf{b})$, will similarly not affect demand at any price in $\mathbf{H}^{\circ}$. Finally, if $\mathbf{r}(\mathbf{b}) = (-\underline{H}, \ldots, -\underline{H})$ then removal of $\mathbf{b}$ does not affect demand at any price in $\mathbf{H}^{\circ}$. So we obtain a bid collection $\mathcal{B}$ with no redundancies (Definition 2.14).

To show uniqueness of this set, first restrict attention to the subset $\mathcal{B}'$ of bids $\mathbf{b}$ such that $\mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^{\circ} \neq \emptyset$. Lemma 4.18 provides uniqueness of this subset $\mathcal{B}'$ of any such set $\mathcal{B}$. Finally, we must consider bids $\mathbf{b}$ such that $\mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^{\circ} = \emptyset$, but such that $\mathbf{b} \neq (-\underline{H}, \ldots, -\underline{H})$. The bids $\mathbf{b}^i$ in the Proof of Theorem 2.13 meet this description: recall that they satisfy $D_{\mathbf{b}^i}(\mathbf{p}) = y_i \mathbf{e}^i$ for all $\mathbf{p} \in \mathbf{H}^{\circ}$. Given uniqueness of $\mathcal{B}'$, the only way to adjust demand $D_{\mathcal{B}}(\mathbf{p}^0)$ at a given price $\mathbf{p}^0 \in \mathbf{H}^{\circ}$ to match $D_v(\mathbf{p}^0)$ is by the inclusion of such bids, and because $\mathbf{b}^i$ uniquely adjusts the ith coordinate of demand, there is a unique such set. This completes the proof. □

5.3 Proving Theorem 5.1

Our proof of Theorem 5.1 proceeds by induction on n', as follows. First, we provide a trivial base case:

Lemma 5.2. *Theorem 5.1 holds when we restrict to the case $n' = 1$.*

5.3.1 Lemmas for the Inductive Step in proving Theorem 5.1

Our use of induction rests on identifying $(n-1)$-dimensional weighted pseudo-LIPs associated with a given n-dimensional weighted pseudo-LIP. Recall, for any set $X \subseteq \mathbb{R}^n$, we denote by $\langle X \rangle$ the *affine span* of X, that is, the set $\langle X \rangle := \{\lambda(\mathbf{x}-\mathbf{x}') \mid \lambda \in \mathbb{R}, \mathbf{x}, \mathbf{x}' \in X\}$.

For any $i \in [n]$ write $\underline{\mathbf{H}}_i := \{\mathbf{p} \in \mathbf{H} \mid p_i = \underline{H}\}$ and $\overline{\mathbf{H}}_i := \{\mathbf{p} \in \mathbf{H} \mid p_i = \overline{H}\}$. Now, for any $i \in [n]$, the hyperplane $\langle \underline{\mathbf{H}}_i \rangle$ can be identified with $\mathbb{R}^{n-1}$ under the standard projection $\pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ which deletes the ith coordinate. Write ρ_i for the restriction of π_i to $\langle \underline{\mathbf{H}}_i \rangle$, so that we can also refer to its inverse $\rho_i^{-1} : \mathbb{R}^{n-1} \to \langle \underline{\mathbf{H}}_i \rangle$. Also write $\langle \underline{\mathbf{H}}_i \rangle^+ := \{\mathbf{p} \in \mathbb{R}^n \mid p_i > \underline{H}\}$.

Definition 5.3. If $(\mathcal{L}, w)$ is a weighted pseudo-LIP, we write $\mathcal{L}^{\rho_i}$ for the union of sets $F' := \rho_i(F \cap \langle \underline{\mathbf{H}}_i \rangle)$, where F is a facet of $\mathcal{L}$ and $F \cap \langle \underline{\mathbf{H}}_i \rangle^+ \neq \emptyset$. If $\dim F' = n-2$ then we set $w_{\mathcal{L}^{\rho_i}}(F') := \sum w(F)$, where the sum is taken over all facets F of $\mathcal{L}$ such that $F \cap \langle \underline{\mathbf{H}}_i \rangle^+ \neq \emptyset$ and $F' = \rho_{n-1}(F \cap \langle \underline{\mathbf{H}}_i \rangle)$.[47]

Lemma 5.4. *If $(\mathcal{L}, w)$ is an ordinary substitutes weighted pseudo-LIP in $\mathbb{R}^n$ then $(\mathcal{L}^{\rho_i}, w_{\mathcal{L}^{\rho_i}})$ is an ordinary substitutes weighted pseudo-LIP in $\mathbb{R}^{n-1}$.*

Under an inductive hypothesis on Theorem 5.1, then, there exist $(n-1)$-dimensional bids $\mathcal{B}$ rooted in $\rho_i(\underline{\mathbf{H}}_i)$ such that $(\mathcal{L}^{\rho_i} \cap \rho_i(\underline{\mathbf{H}}_i)^\circ, w_{\mathcal{L}^{\rho_i}}) = (\mathcal{L}_{\mathcal{B}} \cap \rho_i(\underline{\mathbf{H}}_i)^\circ, w_{\mathcal{B}})$.

Now observe that if $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ is an $(n-1)$-dimensional bid, we can associate an n-dimensional bid $\sigma_i(\mathbf{b}) := (\rho_i^{-1}(\mathbf{r}); \mathbf{t}'; m)$ where $\pi(\mathbf{t}') = \mathbf{t}$ and $t'_i = 1$. Write $\sigma_i(\mathcal{B}) := \{\sigma_i(\mathbf{b}) \mid \mathbf{b} \in \mathcal{B}\}$.

Lemma 5.5. *If $\mathcal{B}$ is an $(n-1)$-dimensional bid collection rooted in $\rho_i(\underline{\mathbf{H}}_i)$ then $\sigma_i(\mathcal{B})$ is an n-dimensional bid collection rooted in $\underline{\mathbf{H}}_i$, and F is a facet of $\mathcal{L}_{\sigma_i(\mathcal{B})}$ if and only if $F \cap \mathbf{H}^\circ = (\rho_i^{-1}(F') + \mathbb{R}\mathbf{e}^i) \cap \mathbf{H}^\circ$ where F' is a facet of $\mathcal{L}_{\mathcal{B}}$, also satisfying $w_{\sigma_i(\mathcal{B})}(F) = w_{\mathcal{B}}(F')$. In particular, $\mathcal{L}_{\sigma_i(\mathcal{B})} \cap \mathbf{H}^\circ$ has no i-hods and no (i,j)-fins for any $j \in [n]$, $j \neq i$.*

For any set $X \subset \mathbb{R}^n$, we will refer to the set $X + \mathbb{R}\mathbf{e}^i$ as the *extrusion* of X in direction $\mathbf{e}^i$.

5.3.2 The Structure of the Inductive Step of the proof of Theorem 5.1

The most substantive technical result in proving the inductive step, comes in finding bids in $\mathbf{H}$ which exactly cover all $(n-1,n)$-fins of $\mathcal{L}$ in $\mathbf{H}^\circ$:

Proposition 5.6. *If $(\mathcal{L}, w)$ is an ordinary substitutes weighed pseudo-LIP in $\mathbb{R}^n$, and $\mathbf{H}$ is any product of intervals in $\mathbb{R}^n$, then there exists a bid collection $\mathcal{B}'$ rooted in $\mathbf{H}$ such that $(\mathcal{L} \cap \mathbf{H}^\circ, w) \boxminus (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$ has no $(n-1,n)$-fins.*

This is significant because if a pseudo-LIP has no $(n-1,n)$-fins in $\mathbf{H}^\circ$ then, for any $i \in [n-2]$, its (i,n)-fins all can be extruded in direction $\mathbf{e}^{n-1}$ and still stay within the pseudo-LIP intersection of the pseudo-LIP and $\mathbf{H}^\circ$.

[47] If $\mathcal{L}$ is a LIP then this is the "stable intersection" of $\mathcal{L}$ and $\langle \underline{\mathbf{H}}_i \rangle$, in the terminology of tropical geometry.

Lemma 5.7. *Suppose $n \geq 2$ and that $(\mathcal{L}, w)$ is an ordinary substitutes weighted pseudo-LIP such that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no $(n-1, n)$-fins. Then, for any $i \in [n-2]$ and any (i, n)-fin F of $\mathcal{L}$ such that $F \cap \mathbf{H}^\circ \neq \emptyset$, it holds that $(F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H} \subseteq \mathcal{L}$. Moreover, for any facet F' of $\mathcal{L}$ such that $\dim(F' \cap (F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ) = n - 1$ it follows that $w(F') = w(F)$.*

It follows that we obtain full information about these (i, n)-fins by taking a cross-section perpendicular to $\mathbf{e}^{n-1}$. We therefore use $\mathcal{L}^{\rho_{n-1}}$, as in Definition 5.3. Now a consequence of Lemmas 5.5 and 5.7 is:

Corollary 5.8. *Suppose $n \geq 2$ and $(\mathcal{L}, w)$ is an ordinary substitutes weighted pseudo-LIP on n goods such that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no $(n-1, n)$-fins. Suppose also the statement of Theorem 5.1 holds for $n' \leq n-1$. Then there exist bids $\mathcal{B}'$ rooted in $\underline{\mathbf{H}}_{n-1}$ such that $(\mathcal{L}, w) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ has no (i, n)-fins meeting $\mathbf{H}^\circ$ for any $i \in [n-1]$.*

The (i, j)-fins for $i, j \in [n-1]$ remain, but we can now handle these in a similar way, by considering direction $\mathbf{e}^n$ and applying the inductive step again:

Lemma 5.9. *Suppose $n \geq 2$ and that $(\mathcal{L}, w)$ is an ordinary substitutes weighted pseudo-LIP such that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no (i, n)-fins for any $i \in [n-1]$. Then, for any $j, k \in [n-1]$ and any (j, k)-fin F of $\mathcal{L}$, it holds that $(F + \mathbb{R}\mathbf{e}^n) \cap \mathbf{H}^\circ \subseteq \mathcal{L}_v$. Moreover, for any facet F' of $\mathcal{L}_v$ such that $\dim(F' \cap (F + \mathbb{R}\mathbf{e}^{n-1})) = n - 1$ it follows that $w(F') = w(F)$.*

Corollary 5.10. *Suppose $n \geq 2$ and $(\mathcal{L}, w)$ is an ordinary substitutes weighted pseudo-LIP on n goods such that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no (i, n)-fins for any $i \in [n-1]$. Suppose also the statement of Theorem 5.1 holds for $n' \leq n-1$. Then there exist bids $\mathcal{B}'$ rooted in $\underline{\mathbf{H}}_n$ such that $(\mathcal{L}, w) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ has no fins meeting $\mathbf{H}^\circ$.*

Thus all fins have been dealt with. The hods remain, but this is simple for a pseudo-LIP with no fins:

Lemma 5.11. *Suppose $n \geq 2$ and that $(\mathcal{L}, w)$ is an ordinary substitutes weighted pseudo-LIP such that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no fins. If F is a i-hod for any $i \in [n]$ then $\langle F \rangle \cap \mathbf{H} \subseteq \mathcal{L}_v$, and all facets F' of $\mathcal{L}$ with $(n-1)$-dimensional intersection with $\langle F \rangle \cap \mathbf{H}$ have the same weight.*

Corollary 5.12. *Suppose $n \geq 2$ and $(\mathcal{L}, w)$ is an ordinary substitutes weighted pseudo-LIP on n goods such that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no fins. Suppose also the statement of Theorem 5.1 holds for $n' \leq 1$. Then there exist bids $\mathcal{B}'$ rooted in $\bigcup_{i \in [n]} \underline{\mathbf{H}}_i$ such that $(\mathcal{L} \cap \mathbf{H}^\circ, w) = (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$.*

Finally, we need to show:

Proof of Theorem 5.1. The existence of $\mathcal{B}$ in the ordinary substitutes case follows from Proposition 5.6 and Corollaries 5.8, 5.10 and 5.12. That is, we first identify a bid set $\mathcal{B}^1$ as in Proposition 5.6 and so a weighted pseudo-LIP $(\mathcal{L}^1, w^1) := (\mathcal{L} \cap \mathbf{H}^\circ, w) \boxminus (\mathcal{L}_{\mathcal{B}^1} \cap \mathbf{H}^\circ, w_{\mathcal{B}^1})$ that has no $(n-1, n)$-fins. We then apply Corollary 5.8 to $(\mathcal{L}^1, w^1)$ to find additional bids $\mathcal{B}^2$ such that $(\mathcal{L}^2, w^2) := (\mathcal{L}^1 \cap \mathbf{H}^\circ, w^1) \boxminus (\mathcal{L}_{\mathcal{B}^2} \cap \mathbf{H}^\circ, w_{\mathcal{B}^2})$ has no (i, n)-fins; apply Corollary 5.10 to $(\mathcal{L}^2, w^2)$ to find additional bids $\mathcal{B}^3$ such that $(\mathcal{L}^3, w^3) := (\mathcal{L}^2 \cap \mathbf{H}^\circ, w^2) \boxminus (\mathcal{L}_{\mathcal{B}^3} \cap \mathbf{H}^\circ, w_{\mathcal{B}^3})$ has no fins; and finally apply Corollary

5.12 to $(\mathcal{L}^3, w^3)$ to find additional bids $\mathcal{B}^4$ such that $(\mathcal{L}^3 \cap \mathbf{H}^\circ, w^3) = (\mathcal{L}_{\mathcal{B}^4} \cap \mathbf{H}^\circ, w_{\mathcal{B}^4})$. So, by Lemmas 4.7 and 4.11, it follows that if we set $\mathcal{B}' = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3 \cup \mathcal{B}^4$ then $(\mathcal{L} \cap \mathbf{H}^\circ, w) = (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$.

We now adjust $\mathcal{B}'$ to remove redundancies. Whenever there exist $\mathbf{b}, \mathbf{b}' \in \mathcal{B}'$ with $(\mathbf{r}(\mathbf{b}), \mathbf{t}(\mathbf{b})) = (\mathbf{r}(\mathbf{b}'), \mathbf{t}(\mathbf{b}'))$ then replace both $\mathbf{b}$ and $\mathbf{b}'$ in $\mathcal{B}$ with $(\mathbf{r}(\mathbf{b}), \mathbf{t}(\mathbf{b}), m(\mathbf{b}) + m(\mathbf{b}'))$, unless $m(\mathbf{b}) + m(\mathbf{b}') = 0$, in which case $\mathbf{b}$ and $\mathbf{b}'$ may simply be removed. Next, if $r_i(\mathbf{b}') = \underline{H}$ for any $i \in [n]$ then $\mathbf{t}(\mathbf{b}')$ may be adjusted to satisfy conditions (1) and (2) of Definition 2.14; we also need to scale up $m(\mathbf{b}')$ by $\gcd\{t_i \mid r_i \neq \underline{H}\}$. Finally, any bid $\mathbf{b}'$ such that $\mathbf{r}(\mathbf{b}) = (\underline{H}, \ldots, \underline{H})$ may simply be removed. Completion of these steps yields a bid collection $\mathcal{B}$ with no redundancies relative to $\mathbf{H}$, and, by Lemma 4.18, such that $(\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^\circ, w_{\mathcal{B}}) = (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'}) = (\mathcal{L} \cap \mathbf{H}^\circ, w)$.

If $\mathcal{B}$ contains any bid with trade-off not equal to $\mathbf{1}$ then it must contain a fin whose normal is not a strong substitutes demand type vector. So if $\mathcal{L}$ is of the strong substitutes demand type then $\mathcal{B}$ is a collection of SSPMA bids. $\square$

Remark 5.13. When $(\mathcal{L}, w)$ is the LIP of a concave ordinary substitutes valuation v with domain an FBD then it is possible to prove a stronger form of Theorem 5.1: that there exists a bid collection $\mathcal{B}$ such that $(\mathcal{L}, w) = (\mathcal{L}_v, w_v) = (\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$. Moreover, $\mathcal{B}$ is the bid collection found in Proposition 5.6.

First, one shows that in this case, the analogue of Proposition 5.6 says that there exist bids $\mathcal{B}'$ such that $(\mathcal{L}_v, w_v) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ (globally) has no $(n-1, n)$-fins.

Second, one shows that for any $\mathcal{L}_v$ where v has an FBD, and for any $\mathcal{L}_{\mathcal{B}}$, every i-hod is bounded below in coordinates $j \neq i$, and every (i,j)-fin is bounded above in coordinates i, j and bounded below in coordinates $k \neq i, j$. As this holds for both $\mathcal{L}_v$ and $\mathcal{L}_{\mathcal{B}'}$, it also holds for $(\mathcal{L}_v, w_v) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$.

Then, a version of Lemma 5.7 shows that for a pseudo-LIP with (globally) no $(n-1, n)$-fins, any (i,n)-fin can be (globally) extruded in coordinate $(n-1)$. But since $(\mathcal{L}_v, w_v) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ can have no (i,n)-finsthat can be globally extruded in coordinate $(n-1)$, it follows that $(\mathcal{L}_v, w_v) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ has no (i,n)-fins at all.

Similarly, a version of Lemma 5.9 shows that for a pseudo-LIP with (globally) no (i,n)-fins, any (j,k)-fin can be (globally) extruded in coordinate n. Again, it follows that $(\mathcal{L}_v, w_v) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ has no fins of any kind.

Finally, a version of Lemma 5.11 shows that for a pseudo-LIP with (globally) no fins, if it contains a hods then it contains the affine span of that hod. Yet again this is impossible for $(\mathcal{L}_v, w_v) \boxminus (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$, which we therefore conclude has no facets at all. We can therefore deduce that $(\mathcal{L}_v, w_v) = (\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$. That is, the bids found in Proposition 5.6 are all that are needed.

We do not take this approach because it either complicate the statements and proofs of lemmas required for the general case, or necessitate a considerable amount of near replication in proofs. Given that we wish to prove Theorem 2.11 in any case, it is more efficient to derive the FBD case Theorem 2.11 from that more general result.

Appendices

A The Geometry of Substitutes Pseudo-LIPs

This appendix develops results which will be used in our proofs in Appendix B. Recall that if C is a polyhedral set, we write $\langle C\rangle$ for the affine span of C.

Definition A.1. For any distinct indices $i, j, k, l \in [n]$, we say that an $(n-2)$-cell C of a pseudo-LIP $\mathcal{L}$ is:

(1) *Type 1 with indices* (i, j), if $\langle C\rangle = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i;\ p_j = r_j\}$ for some $\mathbf{r} \in \mathbb{R}^n$;

(2) *Type 2 with indices* (i, j, k) *and trade-off* (t_i, t_j, t_k), if $\langle C\rangle = \{\mathbf{p} \in \mathbb{R}^n \mid t_i(p_i - r_i) = t_j(p_j - r_j) = t_k(p_k - r_k)\}$ for some $\mathbf{r} \in \mathbb{R}^n$;

(3) *Type 3 with indices* $(i; j, k)$ *and trade-off* (t_j, t_k) if $\langle C\rangle = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i;\ t_j(p_j - r_j) = t_k(p_k - r_k)\}$ for some $\mathbf{r} \in \mathbb{R}^n$;

(4) *Type 4 with indices* $(i, j; k, l)$ *and trade-off* $(t_i, t_j; t_k, t_l)$ if $\langle C\rangle = \{\mathbf{p} \in \mathbb{R}^n : t_i(p_i - r_i) = t_j(p_j - r_j);\ t_k(p_k - r_k) = t_l(p_l - r_l)\}$ for some $\mathbf{r} \in \mathbb{R}^n$.

Note that we do not insist here that the trade-off vectors are primitive integer, and so these vectors are only uniquely defined up to multiplication by a positive scalar: for any $\alpha \in \mathbb{Q}_{>0}$, if C is Type 2 with trade-off (t_i, t_j, t_k) then it is also Type 2 with trade-off $(\alpha t_i, \alpha t_j, \alpha t_k)$, and similar results hold for Types 3 and 4. This will not concern us.

Lemma A.2. *Let $\mathcal{L}$ be a pseudo-LIP of the ordinary substitutes demand type, let F be a facet of $\mathcal{L}$ and let $C \subsetneq F$ be an $(n-2)$-cell. Then C is one of Types 1, 2, 3 and 4 from Definition A.1. The possible form of F depends on the Type of C as follows:*

(1) If C is Type 1 with indices (i, j) then F is an i-hod, a j-hod, or a (i, j)-fin.

(2) If C is Type 2 with indices (i, j, k) and trade-off (t_i, t_j, t_k) then F is an $(i, j; t_i/t_j)$-fin, an $(i, k; t_i/t_k)$-fin or a $(j, k; t_j/t_k)$-fin.

(3) If C is Type 3 with indices $(i; j, k)$ and trade-off (t_j, t_k) then F is an i-hod or a $(j, k; t_j/t_k)$-fin.

(4) If C is Type 4 with indices $(i, j; k, l)$ and trade-off $(t_i, t_j; t_k, t_l)$ then F is a $(i, j; t_i/t_j)$-fin or a $(k, l; t_k/t_l)$-fin.

Proof. An $(n-2)$-cell of $(\mathcal{L}, w)$ is an $(n-2)$-cell of the underlying complex (Π, w) and so is the intersection of (at least) two non-parallel facets F^1, F^2 of (Π, w). As there are limited possible facet normals, we may break this down into 6 cases. Consideration of these cases together proves the Lemma.

Case 1: F^1 has normal $\mathbf{e}^i$ and F^2 has normal $\mathbf{e}^j$ where $i \neq j$. Here C is Type 1 with indices (i, j).

Case 2: F^1 has normal $\mathbf{e}^i$ and F^2 has normal $t_i\mathbf{e}^i - t_j\mathbf{e}^j$, for $j \neq i$. The space of vectors normal to $\langle C\rangle$ is spanned by $\{\mathbf{e}^i, \mathbf{e}^j\}$, so again C is Type 1 with indices (i, j).

Case 3: F^1 has normal $\mathbf{e}^i$ and F^2 has normal $t_j\mathbf{e}^j - t_k\mathbf{e}^k$ for i, j, k distinct. The space of vectors normal to $\langle C\rangle$ is spanned by $\{\mathbf{e}^i, t_j\mathbf{e}^j - t_k\mathbf{e}^k\}$, and so C is Type 3 with indices $(i; j, k)$ and trade-off (t_j, t_k).

Case 4. F^1 has normal $t_i\mathbf{e}^i - t_j\mathbf{e}^j$ and F^2 has normal $t'_i\mathbf{e}^i - t'_j\mathbf{e}^j$. For these to intersect in an $(n-2)$-cell, we must have $\frac{t_i}{t_j} \neq \frac{t'_i}{t'_j}$. The space of vectors normal to $\langle C\rangle$ is spanned by $\{\mathbf{e}^i, \mathbf{e}^j\}$, so that again C is Type 1 with indices (i, j).

Case 5. F^1 has normal $t_i\mathbf{e}^i - t_j\mathbf{e}^j$ and F^2 has normal $t'_j\mathbf{e}^j - t'_k\mathbf{e}^k$ where i, j, k are distinct. The space of vectors normal to $\langle C\rangle$ is spanned by $\{t_it'_j\mathbf{e}^i - t_jt'_j\mathbf{e}^j, t_jt'_j\mathbf{e}^j - t_jt'_k\mathbf{e}^k\}$ and also contains $t_it'_j\mathbf{e}^i - t_jt'_k\mathbf{e}^k$, so C is Type 2 with indices (i, j, k) and corresponding trade-off $(t_it'_j, t_jt'_j, t_jt'_k)$.

Case 6. F^1 has normal $t_i\mathbf{e}^i - t_j\mathbf{e}^j$ and F^2 has normal $t_k\mathbf{e}^k - t_l\mathbf{e}^l$ where i, j, k, l are distinct. The space of vectors normal to $\langle C\rangle$ is spanned by $\{t_i\mathbf{e}^i - t_j\mathbf{e}^j, t_k\mathbf{e}^k - t_l\mathbf{e}^l\}$, so C is Type 4 with indices $(i, j; k, l)$ and trade-off $(t_i, t_j; t_k, t_l)$. ☐

Recall that every LIP is balanced, when paired with the facet weights (Definition 2.20 and Fact 2.21). It follows that, in many cases, when a facet F of an ordinary substitutes weighted pseudo-LIP has an $(n-2)$-cell, there is another facet on the "other side" of that $(n-2)$-cell with the same normal and weight as F. That is, we recall that a rational polyhedron, such as a facet, is the intersection of a finite set of half-spaces (Definition 2.17). An $(n-2)$-cell is then defined by the intersection of the boundary of one of these half-spaces with the facet itself; the intersection between the affine span of the facet and the other half-space associated with that "boundary" provides the "other side" of the $(n-2)$-cell.

Corollary A.3. *Let $\mathcal{L}$ be a weighted pseudo-LIP of the ordinary substitutes demand type, and let C be an $(n-2)$-cell such that either:*

(1) C is of Type 1 or 2, but there are at most two distinct affine spans to the facets containing C; or (2) C is of Type 3 or 4.

Then if $\mathcal{L}$ has a facet F with normal $\mathbf{d}$ containing C in its boundary, it follows that $\mathcal{L}$ also has a facet $F' \neq F$ with weight $w_v(F') = w_v(F)$ and normal $\mathbf{d}$ containing C in its boundary.

Proof. Observe from Lemma A.2 that in all of the cases listed, there are at most two possible normal vectors for facets containing such C in their boundary. These normal vectors are linearly independent. The balancing condition therefore tells us that for each facet on one side of the bounding $(n-2)$-cell there must exist another facet on the "other" side of the cell normal to the same vector and with equal weight. □

Regarding Type 2 $(n-2)$-cells, in fact the balancing condition implies an additional condition relating the weights of the facets to their trade-offs.

Lemma A.4. *Suppose weighted pseudo-LIP $(\mathcal{L}, w)$ has a Type 2 $(n-2)$-cell C with indices (i, j, k) and primitive integer trade-off (t_i, t_j, t_k). Suppose also that C is contained in exactly one $(i, j; t_i/t_j)$-fin F of $\mathcal{L}$. Then $\gcd(t_i, t_j) \mid w(F)$, that is, $\gcd(t_i, t_j)$ divides $w(F)$.*

Proof. The primitive integer normal vector to F is $\frac{t_i\mathbf{e}^i - t_j\mathbf{e}^j}{\gcd(t_i,t_j)}$. By the contrapositive of Corollary A.3 there must be three distinct affine spans to the facets containing C; by Lemma A.2 the other facets are $(i, k; t_i/t_k)$- and $(j, k; t_j/t_k)$-fins, whose primitive integer vectors may be presented similarly to that of F; there are at most two fins with each of these index and trade-off combinations. If all four of these fins are present, label them as F^{ik} and $\widehat{F}^{ik}$, and F^{jk} and $\widehat{F}^{jk}$ respectively, such that the balancing condition around C may be written

$$\frac{w(F)(t_i\mathbf{e}^i - t_j\mathbf{e}^j)}{\gcd(t_i,t_j)} + \frac{\left(w(F^{jk}) - w(\widehat{F}^{jk})\right)(t_j\mathbf{e}^j - t_k\mathbf{e}^k)}{\gcd(t_j,t_k)} - \frac{\left(w(F^{ik}) - w(\widehat{F}^{ik})\right)(t_i\mathbf{e}^i - t_k\mathbf{e}^k)}{\gcd(t_i,t_k)} = 0;$$

if any of these fins are not present we simply set their weight to zero. Taking coefficients of $\mathbf{e}^i$, dividing through by t_i, and multiplying through by the denominators, we obtain:

$$\gcd(t_i,t_k)w(F) = \gcd(t_i,t_j)\left(w(F^{ik}) - w(\widehat{F}^{ik})\right). \tag{5}$$

In particular, both sides of Equation (5) are positive integers. Finally, we show that $\gcd(t_i, t_j) \mid w(F)$.

Consider any prime number q such that $q^\ell \mid \gcd(t_i, t_j)$ for some maximal exponent $\ell \geq 1$. Since (t_i, t_j, t_k) is a primitive integer vector, we know that $q \nmid t_k$ and hence $q \nmid \gcd(t_i, t_k)$. But q^ℓ divides the right-hand side of Equation (5), and hence also its left-hand side. Thus $q^\ell \mid w(F)$. As this is true for all primes q in the prime factorisation of $\gcd(t_i, t_j)$, it follows that $\gcd(t_i, t_j) \mid w(F)$, as required. □

Finally, we will only need the following result for (true) LIPs. Recall that the *Euclidean ordering* $\leq$ for $\mathbb{R}^n$ defines that $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for all $i \in [n]$. Then:

Lemma A.5. *If v is a concave ordinary substitutes valuation then every price complex cell for v is a lattice with respect to the Euclidean ordering.*

Proof. This follows by straightforward consideration of the possible bounds on such sets and the limited range of facet normals permitted in an ordinary substitutes LIP. Alternatively, it also follows from the well-known result that, for any ordinary substitutes valuation, the set of prices for which any given bundle lies in the convex hull of the demand set form a lattice (see, e.g., Milgrom and Strulovici, 2009), together with the fact that the intersection of two lattices in $\mathbb{R}^n$ is another lattice (in $\mathbb{R}^n$). □

B Extra Examples and Proofs of Results in the Text

B.1 General Conventions

Example B.1. Suppose we have two positive bids: $\mathbf{b}^1 = ((4,2),(1,2),1)$ and $\mathbf{b}^2 = ((2,0),(1,1),1)$, and consider demand at $\mathbf{p} = (2,1)$. We calculate that $I(\mathbf{b}^1,(2,1)) = \{1,2\}$ and so $D_{\mathbf{b}^1}(1,2) = \{(1,0),(0,2)\}$, while $I(\mathbf{b}^2,(2,1)) = \{0,1\}$ and so $D_{\mathbf{b}^2}(1,2) = \{(0,0),(1,0)\}$. Thus $\sum_{j=1}^2 D_{\mathbf{b}^j}(1,2) = \{(1,0),(0,2),(2,0),(1,2)\}$. But this set is not discrete-convex: $(1,1) \in \operatorname{conv}\left(\sum_{j=1}^2 D_{\mathbf{b}^j}(1,2)\right)$. This illustrates why, at Equation (2), we must define $D_{\mathcal{B}}(\mathbf{p})$ to be the convex hull of the Minkowski sum of the demands from individual positive bids, to ensure that the corresponding valuation is concave.

B.2 Arithmetic of Pseudo-LIPs

Recall (Definition 4.5) that we defined addition $\boxplus$ for balanced $\mathbb{Z}$-weighted rational polyhedral complexes with support $\mathbb{R}^n$. As with weighted pseudo-LIPs, we define:

Definition B.2. $(\Pi^1,w^1) \boxminus (\Pi^2,w^2) := (\Pi^1,w^1) \boxplus (\Pi^2,-w^2)$.

Lemma B.3. *If (Π^1,w^1) and (Π^2,w^2) are balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$ then so are $(\Pi^1,w^1) \boxplus (\Pi^2,w^2)$ and $(\Pi^1,w^1) \boxminus (\Pi^2,w^2)$.*

Proof. It is well-known that the set of intersections of cells from two polyhedral complexes forms a polyhedral complex Π (see, e.g. Grünbaum, 1967, Chapter 3, Section 3.2, Exercise 7). It clearly inherits support $\mathbb{R}^n$ from Π^1 and Π^2. To show that the balancing condition holds (Definition 2.20), consider an $(n-2)$-cell G of Π. If $G \subseteq G^i$ where G^i is an $(n-2)$-cell of Π^i for $i = 1$ or 2, then the balancing condition is satisfied around G^i by all facets of Π^i containing G^i. On the other hand, if the minimal cell of Π^i containing G is a facet F, then if we isolate F and split it into two facets along the $(n-2)$-cell defined by $\langle G \rangle \cap F$, then the balancing condition is clearly satisfied around G by these two facets. And if the minimal cell of Π^i containing G is an n-cell then there are no facets of Π^i containing G and the balancing condition on Π^i is trivial. Putting these cases together and noting that weights are just added across the two complexes, yields the balancing condition for $(\Pi^1,w^1) \boxplus (\Pi^2,w^2)$.

Finally, if (Π^2,w^2) is a balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$ then so is $(\Pi^2,-w^2)$ and so the result for $(\Pi^1,w^1) \boxminus (\Pi^2,w^2)$ follows from that for $(\Pi^1,w^1) \boxplus (\Pi^2,-w^2)$. $\square$

We now show that we can use the usual rules of addition and subtraction on balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$.

Lemma B.4. *$\boxplus$ and $\boxminus$ satisfy the usual rules of addition and subtraction, with $(\mathbb{R}^n,0)$ playing the role of identity element. That is, for balanced $\mathbb{Z}$-weighted rational polyhedral complexes $(\Pi^1,w^1),(\Pi^2,w^2)$ and (Π^3,w^3) with support $\mathbb{R}^n$ we have:*
(1) $(\Pi^1,w^1) \boxplus (\Pi^2,w^2) = (\Pi^1,w^2) \boxplus (\Pi^1,w^1)$
(2) $(\Pi^1,w^1) \boxplus ((\Pi^1,w^2) \boxplus (\Pi^3,w^3)) = ((\Pi^1,w^1) \boxplus (\Pi^1,w^2)) \boxplus (\Pi^3,w^3)$
(3) $(\mathbb{R}^n,0) \boxplus (\Pi^1,w^1) = (\Pi^1,w^1) \boxplus (\mathbb{R}^n,0) = (\Pi^1,w^1)$
(4) $(\Pi^1,w^1) \boxminus (\Pi^2,w^2) = (\mathbb{R}^n,0) \boxminus ((\Pi^2,w^2) \boxminus (\Pi^1,w^1))$

Additionally, $(\Pi^1,w^1) \boxminus (\Pi^1,w^1) = (\Pi^1,0)$.

Proof. (1) follows immediately from noting that the order of (Π^1,w^1) and (Π^2,w^2) is immaterial in Definition 4.5. (2) is similarly clear when we note that both can be written as the polyhedral complex with cells $C^1 \cap C^2 \cap C^3$ where $C^i \in \Pi^i$, with $w(F)$ similarly adding the weights of all facets from any of these three complexes which contain F. (3) holds because $C^1 \cap \mathbb{R}^n = C^1$ for any cell of Π^1, and $(\mathbb{R}^n,0)$ contains no facets to alter the weighting.

To show (4), re-write the right hand side as $(\mathbb{R}^n,0) \boxminus ((\Pi^2,w^2) \boxplus (\Pi^1,-w^1))$. But this is equal to $(\mathbb{R}^n,0) \boxplus (\Pi^3,w^3)$ where Π^3 is equal to the complex of intersections of cells in Π^2 and Π^1, and w^3 is defined on facets F of this complex by $w^3(F)$ being equal to (-1) times the weight of this facet in $(\Pi^2,w^2) \boxplus (\Pi^1,-w^1)$, that is, $-1 \times \left(\sum_{F' \in \mathcal{F}^2} w^2(F') + \sum_{F' \in \mathcal{F}^1} -w^1(F')\right) = \sum_{F' \in \mathcal{F}^1} w^1(F') - \sum_{F' \in \mathcal{F}^2} w^2(F')$, in which $\mathcal{F}^i$ is the set of all facets of Π^i containing F, for $i = 1,2$. So we have shown that $(\Pi^3,w^3) = (\Pi^1,w^1) \boxminus (\Pi^2,w^2)$, which by application of (3) completes the proof.

Finally, the complex of $(\Pi^1,w^1) \boxminus (\Pi^1,w^1)$ is just Π^1, and it is clear that the weight of every facet is zero. $\square$

The definition of $\boxplus$ for weighted pseudo-LIPs is inherited directly from $\boxplus$ for balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$. For $\boxminus$, we similarly see

Lemma B.5. *If $(\mathcal{L}^i, w^i)$ is the weighted pseudo-LIP of (Π^i, w^i) for $i = 1, 2$ then $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2)$ is the weighted pseudo-LIP of $(\Pi^1, w^1) \boxminus (\Pi^2, w^2)$.*

Proof of Lemmas 4.6 and B.5. Clear from Definitions 4.1 and 4.5. ☐

Proof of Lemma 4.7. First observe that the weighted pseudo-LIP of $(\mathbb{R}^n, 0)$ is $(\emptyset, 0)$. Then the weighted pseudo-LIP property for the $\boxplus$ and $\boxminus$ of two weighted pseudo-LIPs follow from Definitions 4.5 and B.2 and Lemmas B.3 and B.5, and results (1)-(4) of Lemma 4.7 follow from Definition 4.5 and Lemmas B.4 and B.5. It remains to show property (5). But, by Lemma B.5, and the final result of Lemma B.4, we know that $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^1, w^1)$ is the weighted pseudo-LIP of $(\Pi^1, 0)$; since all facets of $(\Pi^1, 0)$ have weight 0 it follows that $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^1, w^1) = (\emptyset, 0)$. ☐

B.3 Bids and Geometry

Fact B.6 (Baldwin and Klemperer (2019) Lemma 2.9(2))**.** The cells of the price complex are the intersections of closures of UDRs.

Lemma B.7. *Write $\mathbf{b} = (\mathbf{r}; \mathbf{t}; 1)$. The valuation $v_\mathbf{b}$ has UDRs as follows:*
(1) $\{\mathbf{0}\} = D_\mathbf{b}(\mathbf{p})$ iff $\mathbf{p} \in \{\mathbf{p} \in \mathbb{R}^n \mid p_j > r_j$ for $j = 1, \ldots, n\}$;
(2) $\{t_i\mathbf{e}^i\} = D_\mathbf{b}(\mathbf{p})$ iff $\mathbf{p} \in \{\mathbf{p} \in \mathbb{R}^n : p_i < r_i,\ t_i(p_i - r_i) < t_j(p_j - r_j)$ for $j = 1, \ldots, n\}$.

Proof. To show (1), observe that $\mathbf{0}$ is uniquely demanded at $\mathbf{p}$ iff for all $\mathbf{x} \in \mathbf{t} \odot \Delta_{[n]_0} \setminus \{\mathbf{0}\}$ we have

$$0 = v_\mathbf{b}(\mathbf{0}) - \mathbf{p} \cdot \mathbf{0} > v_\mathbf{b}(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = (\mathbf{r} - \mathbf{p}) \cdot \mathbf{x} \tag{6}$$

due to quasilinearity of demand. In particular, Equation (6) is required to hold for $\mathbf{x} = \mathbf{e}^j$ for any $j \in [n]$, which reveals that $r_j < p_j$ for all $j \in [n]$. On the other hand, $r_j < p_j$ for $j \in [n]$ is clearly sufficient for (6) to hold for all $\mathbf{x} \in \mathbf{t} \odot \Delta_{[n]_0} \setminus \{\mathbf{0}\}$.

To show (2), observe that $t_i\mathbf{e}^i$ is uniquely demanded at $\mathbf{p}$ iff $t_i(r_i - p_i) = v_\mathbf{b}(t_i\mathbf{e}^i) - \mathbf{p} \cdot (t_i\mathbf{e}^i) > v_\mathbf{b}(\mathbf{0}) = 0$ and, for all $\mathbf{x} \in \mathbf{t} \odot \Delta_{[n]_0}$ with $\mathbf{x} \neq t_i\mathbf{e}^i$, we have

$$t_i(r_i - p_i) = v_\mathbf{b}(t_i\mathbf{e}^i) - \mathbf{p} \cdot (t_i\mathbf{e}^i) > v_\mathbf{b}(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = (\mathbf{r} - \mathbf{p}) \cdot \mathbf{x}. \tag{7}$$

In particular, (7) must hold when $\mathbf{x} = t_j\mathbf{e}^j$ for any $j \neq i$, whence $t_i(p_i - r_i) < t_j(p_j - r_j)$for all $j \neq i$, which is clearly also sufficient for (7) to hold for all $\mathbf{x} \in \mathbf{t} \odot \Delta_{[n]_0}$.

Finally, the UDRs described already are dense in $\mathbb{R}^n$, so no other UDRs are possible. ☐

Proof of Lemma 4.9. Immediate from Facts 2.19 and B.6, and Lemma B.7. ☐

It is useful to identify the "strong diagonal 1-cell" $\bigcap_{i=1}^n F_\mathbf{b}^{ij}$ discussed in Section 3. However, we are only interested in this if it stays within the bounding box; for cases in which a bid lies on the boundary of $\mathbf{H}$, we identify a 1-cell of the union of $\mathcal{L}$ with the faces of this box, which also lies within $\mathcal{L}$. First define:

Definition B.8. If $(\mathbf{r}; \mathbf{t}) \in \mathbf{H} \times \mathbb{Z}^n_{>0}$ then write $\operatorname{inv}(\mathbf{r}; \mathbf{t}) := \sum_{k \in [n], r_k \neq \underline{H}} \frac{\mathbf{e}^k}{t_k}$. If $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ is a bid then write $C_\mathbf{b} := \{\mathbf{r} - \lambda \operatorname{inv}(\mathbf{r}; \mathbf{t}) \mid \lambda \geq 0\}$.

Corollary B.9. *Suppose $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$ and $\mathbf{b}' = (\mathbf{r}; \mathbf{t}'; m')$ are bids such that $\mathbf{r}(\mathbf{b}) = \mathbf{r}(\mathbf{b}') = \mathbf{r}$, and that $r_i, r_j \neq \underline{H}$. Then:*
(1) $C_\mathbf{b} \subseteq F_\mathbf{b}^{ij}$;
(2) $C_\mathbf{b} \subseteq F_{\mathbf{b}'}^{i,j}$ if and only if $\frac{t_j}{t_i} = \frac{t'_j}{t'_i}$ and $\frac{t_k}{t_i} \geq \frac{t'_k}{t'_i}$ for all $k \in [n]$ such that $r_k \neq \underline{H}$; otherwise $C_\mathbf{b} \cap F_{\mathbf{b}'}^{ij} = \{\mathbf{r}\}$

Proof. If $\mathbf{p} \in C_{\mathbf{b}}$ then $\mathbf{p} = \mathbf{r} - \lambda \operatorname{inv}(\mathbf{r}; \mathbf{t})$ for some $\lambda \geq 0$. Observe $p_i \leq r_i$. If $r_k \neq \underline{H}$ then $p_k - r_k = -\frac{\lambda}{t_k}$; if $r_k = \underline{H}$ then $p_k - r_k = 0$.

Part (1) now follows immediately from Lemma 4.9, since $t_i(p_i - r_i) = -\lambda = t_j(p_j - r_j) \leq t_k(p_k - r_k) \in \{-\lambda, 0\}$.

For part Part (2), apply Lemma 4.9 to see that $\mathbf{p} \in F^{ij}_{\mathbf{b}'}$ if and only if $-t'_i \frac{\lambda}{t_i} = -t'_j \frac{\lambda}{t_j} \leq -t'_k \frac{\lambda}{t_k}$ for all $k \neq i, j$ such that $r_k \neq \underline{H}$. This always holds when $\lambda = 0$, but this holding for any, and hence all, $\lambda > 0$ is equivalent to $\frac{t_j}{t_i} = \frac{t'_j}{t'_i}$ and $\frac{t_k}{t_i} \geq \frac{t'_k}{t'_i}$ for all $k \in [n]$ such that $r_k \neq \underline{H}$. □

These results, in particular, allow us to easily show:

Corollary B.10. *Suppose $\mathcal{B}$ is a bid collection with no redundancies relative to $\mathbf{H}$, and fix $i, j \in [n]$ with $i \neq j$. If $r^*_i := \max\{r_i(\mathbf{b}) \mid \mathbf{b} \in \mathcal{B},\ r_j(\mathbf{b}) > \underline{H}\} > \underline{H}$, then there exists an (i,j)-fin F of $\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^\circ$ and a bid $\mathbf{b} \in \mathcal{B}$ such that $r_i(\mathbf{b}) = r^*_i$ and $\mathbf{r}(\mathbf{b}) \in \arg\max_{\mathbf{p} \in F}\{p_i\}$.*

Proof. Identify $\mathbf{r} \in \mathbb{R}^n$ such that $r_i = r^*_i$ and $r_j > \underline{H}$ and $\mathbf{r} = \mathbf{r}(\mathbf{b})$ for some $\mathbf{b} \in \mathcal{B}$, and $\mathbf{r}$ is minimal with respect to the Euclidean ordering with these properties.

Fix $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m) \in \mathcal{B}$. Without loss of generality, assume that there does not exist $\mathbf{b}' \in \mathcal{B}$ such that $\mathbf{r}(\mathbf{b}') = \mathbf{r}$ and $\frac{t_k(\mathbf{b}')}{t_i(\mathbf{b}')} \leq \frac{t_k}{t_i}$ for all $k \in [n]$ such that $r_k \neq \underline{H}$. For if such $\mathbf{b}'$ did exist then, because $\mathcal{B}$ has no redundancies, one such inequality must be strict, and we could then replace $\mathbf{b}$ with such $\mathbf{b}'$.

Consider $\mathbf{b}' \in \mathcal{B}$ such that $\mathbf{r}(\mathbf{b}') \neq \mathbf{r}$. By assumption we know either $r_i(\mathbf{b}') < r^*_i$; or $r_i(\mathbf{b}') = r^*_i$ but $\mathbf{r}(\mathbf{b}') \not\leq \mathbf{r}(\mathbf{b})$; or $r_i(\mathbf{b}') = r^*_i$ but $r_j(\mathbf{b}') = \underline{H} < r_j(\mathbf{b})$. In each of these cases, by Lemma 4.9, it follows that $\mathbf{r} \notin F^{ij}_{\mathbf{b}'}$. It follows that there exists an open neighbourhood $U_{\mathbf{b}'}$ of $\mathbf{r}$ in $F^{ij}_{\mathbf{b}}$ which is not contained in $F^{ij}_{\mathbf{b}'}$.

Now consider $\mathbf{b}' \in \mathcal{B}$ such that $\mathbf{r}(\mathbf{b}') = \mathbf{r}$ but $\mathbf{b}' \neq \mathbf{b}$. By our assumptions on $\mathbf{b}$, we know that $\frac{t_k}{t_i} < \frac{t_k(\mathbf{b}')}{t_i(\mathbf{b}')}$ some $k \in [n]$ such that $r_k \neq \underline{H}$. By Corollary B.9 Part (2) we know that $C_{\mathbf{b}} \cap F^{ij}_{\mathbf{b}'} = \{\mathbf{r}\}$, but by Corollary B.9 Part (1), we know that $C_{\mathbf{b}} \subseteq F^{ij}_{\mathbf{b}}$. Since facets are closed, it follows that there exists an open subset $U_{\mathbf{b}'}$ of $F^{ij}_{\mathbf{b}}$, containing $C_{\mathbf{b}} \setminus \{\mathbf{r}\}$, which is not contained in $F^{ij}_{\mathbf{b}'}$.

As these cases cover all the (finitely many) $\mathbf{b}' \in \mathcal{B}$, we may take the intersection of all such $U_{\mathbf{b}'}$ to obtain an open subset U of $F^{ij}_{\mathbf{b}}$, containing an open neighbourhood of $\{\mathbf{r}\}$ in $C_{\mathbf{b}} \setminus \{\mathbf{r}\}$, which is not contained in $F^{ij}_{\mathbf{b}'}$ for any $\mathbf{b}' \in \mathcal{B}$ for $\mathbf{b}' \neq \mathbf{b}$. It follows from Lemma 4.6 that $U \subseteq \mathcal{L}_{\mathcal{B}}$. And, as $C_{\mathbf{b}} \subseteq \mathbf{H}$, it therefore also follows that $U \cap \mathbf{H}^\circ \neq \emptyset$; as this set is an open subset of an (i,j)-fin of $F^{ij}_{\mathbf{b}}$, and so itself has dimension $(n-1)$, we can conclude that there is an (i,j)-fin F of $\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^\circ$; as $\mathbf{r}$ is in the closure of $U \cap \mathbf{H}^\circ$, we can conclude that $\mathbf{r}$ is contained in such a fin; and as, by assumptions on $\mathbf{b}$, there can be no (i,j)-fin of $\mathcal{L}_{\mathcal{B}}$ containing $\mathbf{p}$ with $p_i > r_i$, we conclude that $\mathbf{r} \in \arg\max_{\mathbf{p} \in F}\{p_i\}$. □

Example B.11. Let $n = 3$ and consider bids $\mathcal{B} = \{(0,0,0;1,1,1;1), (1,1,0;1,1,1;-1), (2,2,0;1,1,1;1)\}$. Then the set $\{\mathbf{p} \in \mathbb{R}^3 \mid p_3 = 0;\ 0 \leq p_1 \leq 1 \text{ or } 0 \leq p_2 \leq 1\}$ is a maximal 1-dimensional subset of $\mathcal{L}_{\mathcal{B}}$, but is not a polyhedron.

Proof of Lemma 4.13. First observe that if $\mathcal{B} = \{\mathbf{b}\}$ then parts (1)–(3) are all immediate from Lemmas B.7 and 4.9 and by definition of $D_{\mathbf{b}}(\mathbf{p})$. Additionally, observe that $(\Pi_{\mathbf{b}}, w_{\mathbf{b}})$ associated with the weighted pseudo-LIP $(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$ is simply given by the closures of the UDRs described in Lemma B.7, together with all their faces.

Now, $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}}) = \boxplus_{\mathbf{b} \in \mathcal{B}} (\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$. By repeated application of Definition 4.5 and Lemma B.4, this is the weighted pseudo-LIP associated with $(\Pi_{\mathcal{B}}, w_{\mathcal{B}}) = \boxplus_{\mathbf{b} \in \mathcal{B}} (\Pi_{\mathbf{b}}, w_{\mathbf{b}})$. We know that $|D_{\mathbf{b}}(\mathbf{p})| = 1$ for $\mathbf{p}$ in the interior of an n-cell of $\Pi_{\mathbf{b}}$, and so $|D_{\mathcal{B}}(\mathbf{p})| = 1$ and is constant for $\mathbf{p}$ in the interiors of the n-cells of $(\Pi_{\mathcal{B}}, w_{\mathcal{B}})$; such $\mathbf{p}$ are simply away from the facets of $\Pi_{\mathcal{B}}$. Observe that such prices are dense in $\mathbb{R}^n$, and at them $D_{\mathcal{B}}(\mathbf{p}) = \sum_{\mathbf{b} \in \mathcal{B}} D_{\mathbf{b}}(\mathbf{p})$. Away from these prices, $D_{\mathcal{B}}(\mathbf{p})$ is defined to be the discrete convex hull of demand at nearby prices (Equation (4)). So $|D_{\mathcal{B}}(\mathbf{p})| = 1$ at a price in the interior of a facet of $(\Pi_{\mathcal{B}}, w_{\mathcal{B}})$ if and only if the same bundle is demanded at prices in the interior of the n-cell of $\Pi_{\mathcal{B}}$ on either side. But we know Part (3) holds for each singleton set $\{\mathbf{b}\}$, where $\mathbf{b} \in \mathcal{B}$. Moreover, $D_{\mathcal{B}}(\mathbf{p}) = \sum_{\mathbf{b} \in \mathcal{B}} D_{\mathbf{b}}(\mathbf{p})$ for prices in the interior of the n-cells of $\Pi_{\mathcal{B}}$, while $w_{\mathcal{B}}$ is defined by adding weights of facets in the individual $\Pi_{\mathbf{b}}$. It follows that the same bundle is demanded on either side of a facet of $\Pi_{\mathcal{B}}$ if and only if the weight of that facet is 0. As $\mathcal{L}_{\mathcal{B}}$ is the union of non-zero weighted facets of $\Pi_{\mathcal{B}}$, this demonstrates both (1) and (2). Part (3) similarly follows by definition of $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ and from the fact that it holds for each individual bid. □

Proof of Propositions 2.6, 2.8 and 4.14. We need simply prove Proposition 4.14, which subsumes Proposition 2.6 and 2.8.

Suppose that a facet F of $(\mathcal{L}_\mathcal{B}, w_\mathcal{B})$ has negative weight. Let i be a coordinate represented in the normal of F, so that either F is an i-hod or an (i,j)-fin for some $j \in [n]$. There exist $\mathbf{p}$ and $\mathbf{p}' = \mathbf{p} + \lambda \mathbf{e}^i$ that lie on either side of F, such that $D_\mathcal{B}(\mathbf{p}) = \{\mathbf{x}\}$ and $D_\mathcal{B}(\mathbf{p}') = \{\mathbf{x}'\}$. The primitive integer vector that is normal to F and points in the opposite direction to the change in price is either $-\mathbf{e}^i$ or $-t_i\mathbf{e}^i + t_j\mathbf{e}^j$ for some $t_i, t_j \in \mathbb{Z}$. Then, by Lemma 4.13 Part (3), we know $\mathbf{x}' - \mathbf{x} = -w_\mathcal{B}(F)\mathbf{e}^i$ or $\mathbf{x}' - \mathbf{x} = w_\mathcal{B}(F)(-t_i\mathbf{e}^i + t_j\mathbf{e}^j)$. Since $w_\mathcal{B}(F) < 0$, in both cases $x'_i > x_i$: $D_\mathcal{B}$ fails to satisfy the law of demand. So (1) $\Longrightarrow$ (2).

Next, we use Lemma 4.9 to convert statement (2) into a statement about bids. If we consider any one i-hod of $\mathcal{L}_\mathcal{B}$ passing through a price $\mathbf{p}$, then this hod having positive weight is equivalent to the set $\mathcal{B}'$ of bids $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{p} \in F^i_\mathbf{b}$ satisfying $\sum_{\mathbf{b}\in\mathcal{B}'} m(\mathbf{b})t_i(\mathbf{b}) \geq 0$; and if we consider any one $(i,j;\alpha)$-fin passing through $\mathbf{p}$, then this having positive weight is equivalent to the set $\mathcal{B}'_\alpha$ of bids $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{p} \in F^{ij}_\mathbf{b}$ and such that $t_i(\mathbf{b})/t_j(\mathbf{b}) = \alpha$, satisfying $\sum_{\mathbf{b}\in\mathcal{B}'_\alpha} m(\mathbf{b}) \gcd(t_i(\mathbf{b}), t_j(\mathbf{b})) \geq 0$. But if this holds for every (i,j)-fin, then it holds for any $\alpha \in \mathbb{Q}$, and so by taking the sum over the union of such $\mathcal{B}'_\alpha$, we see that Part (2) $\Longrightarrow$ (3). Conversely, if (3) holds, then in particular it holds at prices in the interior of exactly one facet, and so (3) $\Longrightarrow$ (2).

That (2)$\Longleftrightarrow$(4) follows from Fact 2.22, Proposition 4.3 and Corollary 4.12. But if (4) holds, then by Fact 2.21 we can moreover choose v such that $D_v(\mathbf{p}) = \{\mathbf{0}\}$, where $\mathbf{p}$ is a sufficiently high price that $D_\mathcal{B}(\mathbf{p}) = \{\mathbf{0}\}$. By Lemma 4.13 and Fact 2.19, it follows that $D_v(\mathbf{p}) = D_\mathcal{B}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$ such that $|D_\mathcal{B}(\mathbf{p})| = 1$. Finally, since such prices are dense, and since both $D_v(\mathbf{p})$ and $D_\mathcal{B}(\mathbf{p})$ are given by the convex hull of their values at such prices in a sufficiently small open neighbourhood (the former because v is concave, the latter by definition at Equation (4)), it follows that $D_v(\mathbf{p}) = D_\mathcal{B}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$; that is, we know (4) $\Longrightarrow$ (5). But if (5) holds then clearly $\mathcal{B}$ satisfies the law of demand, that is, (5) $\Longrightarrow$ (1).

The final statement is Proposition 2.8. We work with $\Pi_\mathcal{B}$ as in the proof of Lemma 4.13. Let v satisfy $D_v(\mathbf{p}) = D_\mathcal{B}(\mathbf{p})$, whose existence has been established above, since Part (1) $\Longrightarrow$ Part (5). For prices $\mathbf{p}', \mathbf{p}''$ in the interior of an n-cell of $\Pi_\mathcal{B}$, we know that $I(\mathbf{b}, \mathbf{p}') = I(\mathbf{b}, \mathbf{p}'')$, and so the choice between $\mathbf{p}'$ and $\mathbf{p}''$ does not affect $i_{\mathbf{b},\mathbf{x}}$.

Let $\mathbf{p}$ be in a facet of $\Pi_\mathcal{B}$, and let $\mathbf{p}'$ and $\mathbf{p}''$ be prices in the interiors of the n-cells on either side, so that $|I(\mathbf{b}, \mathbf{p}')| = |I(\mathbf{b}, \mathbf{p}'')| = 1$ for all $\mathbf{b} \in \mathcal{B}$; write these goods as $i(\mathbf{b}, \mathbf{p}')$ and $i(\mathbf{b}, \mathbf{p}'')$, respectively. Let $\mathbf{x}', \mathbf{x}''$ satisfy $\{\mathbf{x}'\} = D_\mathcal{B}(\mathbf{p}')$ and $\{\mathbf{x}''\} = D_\mathcal{B}(\mathbf{p}'')$. By continuity we know that $I(\mathbf{b}, \mathbf{p}'), I(\mathbf{b}, \mathbf{p}'') \subset \arg\max_{i\in[n]_0} t_i(\mathbf{b})(r_i(\mathbf{b}) - p_i)$ for all $\mathbf{b} \in \mathcal{B}$, from which it follows that

$$\begin{aligned}
&\sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b}) t_{i(\mathbf{b},\mathbf{p}')} \left(r_{i(\mathbf{b},\mathbf{p}')}(\mathbf{b}) - p_{i(\mathbf{b},\mathbf{p}')}\right) = \sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b}) t_{i(\mathbf{b},\mathbf{p}'')} \left(r_{i(\mathbf{b},\mathbf{p}'')}(\mathbf{b}) - p_{i(\mathbf{b},\mathbf{p}'')}\right) \\
\Longleftrightarrow &\sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b}) t_{i(\mathbf{b},\mathbf{p}')} r_{i(\mathbf{b},\mathbf{p}')}(\mathbf{b}) - \mathbf{p}\cdot\mathbf{x}' = \sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b}) t_{i(\mathbf{b},\mathbf{p}'')} r_{i(\mathbf{b},\mathbf{p}'')}(\mathbf{b}) - \mathbf{p}\cdot\mathbf{x}''
\end{aligned} \tag{8}$$

Consider first the case that $\mathbf{x}' = \mathbf{x}''$. Then Equation (8) demonstrates that

$$\sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b}) t_{i(\mathbf{b},\mathbf{p}')} r_{i(\mathbf{b},\mathbf{p}')}(\mathbf{b}) = \sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b}) t_{i(\mathbf{b},\mathbf{p}'')} r_{i(\mathbf{b},\mathbf{p}'')}(\mathbf{b}).$$

For quasilinear valuations, the set of all prices at which $\mathbf{x}$ is demanded is convex; this holds here since the existence of v has been established. So we can apply this equality repeatedly between pairs of n-cells of $\Pi_\mathcal{B}$ at which $\mathbf{x}'$ is demanded, allows us to demonstrate that $\widehat{v}_\mathcal{B}$ is well-defined, as it is independent of the choice of $\mathbf{p}^\mathbf{x}$.

Now suppose $\mathbf{x}' \neq \mathbf{x}''$. Equation (8) demonstrates that in this case $\widehat{v}_\mathcal{B}(\mathbf{x}') - \mathbf{p}\cdot\mathbf{x}' = \widehat{v}_\mathcal{B}(\mathbf{x}'') - \mathbf{p}\cdot\mathbf{x}''$. But, since $\mathbf{x}', \mathbf{x}'' \in D_\mathcal{B}(\mathbf{p}) = D_v(\mathbf{p})$, we know that $v(\mathbf{x}') - \mathbf{p}\cdot\mathbf{x}' = v(\mathbf{x}'') - \mathbf{p}\cdot\mathbf{x}''$. Subtracting the second of these equations from the first and applying repeatedly across price space allows us to conclude that $\widehat{v}_\mathcal{B}(\mathbf{x}) - v(\mathbf{x})$ is constant for all $\mathbf{x}$ which are uniquely demanded for any price. But then $\widehat{v}_\mathcal{B}(\mathbf{x}) = v(\mathbf{x}) + k$ for all bundles $\mathbf{x}$ uniquely demanded under v, where k is some constant. As v is concave, we can conclude that $v_\mathcal{B} = v + k$. As addition of a constant does not affect the demand set of a valuation, this demonstrates that $D_{v_\mathcal{B}}(\mathbf{p}) = D_v(\mathbf{p}) = D_\mathcal{B}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}$. Finally, concavity of $v_\mathcal{B}$ is immediate by definition. $\square$

To prove Lemma 2.10, we require a slight strengthening of the law of demand:

Definition B.12. For any demand correspondence D, say that D *satisfies the strict law of demand* if, given $\mathbf{p}, \mathbf{p}' = \mathbf{p} + \lambda \mathbf{e}^i \in \mathbb{R}^n$, where $i \in [n]$ and $\lambda > 0$, such that $\mathbf{x} \in D(\mathbf{p})$ and $\mathbf{x}' \in D(\mathbf{p}')$, it holds that $x'_i \leq x_i$, with equality iff $\mathbf{x}' \in D(\mathbf{p})$ and $\mathbf{x} \in D(\mathbf{p}')$.

Lemma B.13 (See also Baldwin et al. 2021b, Lemma A.1)**.** *If $v : A_v \to \mathbb{R}$ is a valuation then D_v satisfies the strict law of demand.*

Proof. Let $\mathbf{p}, \mathbf{p}'$ and $\mathbf{x}, \mathbf{x}'$ be as in Definition B.12. $\mathbf{x}' \in D_v(\mathbf{p}')$ implies that $v(\mathbf{x}') - \mathbf{p}' \cdot \mathbf{x}' \geq v(\mathbf{x}) - \mathbf{p}' \cdot \mathbf{x}$, that is, $v(\mathbf{x}') - (\mathbf{p} + \lambda \mathbf{e}^i) \cdot \mathbf{x}' \geq v(\mathbf{x}) - (\mathbf{p} + \lambda \mathbf{e}^i) \cdot \mathbf{x}$, and so $v(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}' - \lambda x'_i \geq v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} - \lambda x_i$. Meanwhile, $\mathbf{x} \in D_v(\mathbf{p})$ implies that $v(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}' \leq v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$. Subtracting the previously derived inequality and recalling that $\lambda > 0$ yields that $x'_i \leq x_i$. Moreover, equality holds here if and only if it holds in both of the original equalities, which in turn holds if and only if $\mathbf{x} \in D_v(\mathbf{p}')$ and $\mathbf{x}' \in D_v(\mathbf{p})$. $\square$

Proof of Lemma 2.10. To show Part (1) of Definition 2.9, observe that, since the bids $\mathcal{B}$ are valid, by Proposition 4.14 Part (5), there exists a concave ordinary substitutes valuation v such that, in particular, $A_{\mathcal{B}} = \bigcup \{D_v(\mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^n\}$. But, since v is concave, the latter object is precisely the domain of v and is therefore discrete-convex (see, e.g., Baldwin and Klemperer 2019 Definition 2.10 and Lemma 2.11) . To show that $\mathbf{0} \in A_{\mathcal{B}}$, observe that, if we set $p_i \gg 0$ for all $i \in [n]$ then $I(\mathbf{b}, \mathbf{p}) = \{0\}$ for all $\mathbf{b} \in \mathcal{B}$ and so $D_{\mathcal{B}}(\mathbf{p}) = \{\mathbf{0}\}$. To show that $A_{\mathcal{B}} \subseteq \mathbb{Z}^n_{\geq 0}$, suppose not, and that $\mathbf{x} \in A_{\mathcal{B}}$ with $x_i < 0$ for some $i \in [n]$. By definition of $A_{\mathcal{B}}$ there exists $\mathbf{p}$ such that $\mathbf{x} \in D_{\mathcal{B}}(\mathbf{p})$. Let $R \gg 0$ and set $\mathbf{p}' = \mathbf{p} + R\mathbf{e}^i$. Then, as long as R has been chosen to be sufficiently large, it follows that $i \notin I(\mathbf{b}, \mathbf{p}')$ for all $\mathbf{b} \in \mathcal{B}$, and so that if $\mathbf{x}' \in D_{\mathcal{B}}(\mathbf{p}')$ then $x'_i = 0$. But then $x'_i > x_i$, which contradicts the law of demand, hence validity of $\mathcal{B}$, showing that indeed $A_{\mathcal{B}} \subsetneq \mathbb{Z}^n_{\geq 0}$ as required.

To show Part (2), fix $\mathbf{x} \in A_{\mathcal{B}}$ and write $\widehat{\mathbf{x}} = \mathbf{x} - x_i \mathbf{e}^i$. We wish to show that $\widehat{\mathbf{x}} \in A_{\mathcal{B}}$. As in the preceding paragraph, we know there exists $\mathbf{p}$ such that $\mathbf{x} \in D_{\mathcal{B}}(\mathbf{p})$ and we can choose $\mathbf{p}' = \mathbf{p} + R\mathbf{e}^i$ for $R \gg 0$ such that if $\mathbf{x}' \in D_{\mathcal{B}}(\mathbf{p}')$ then $x'_i = 0$. However, $\mathbf{x}' \neq \widehat{\mathbf{x}}$ in general.

Assume at first that there exists $\mathbf{p}$ such that $D_{\mathcal{B}}(\mathbf{p}) = \{\mathbf{x}\}$. Recalling that the set of prices meeting this condition is dense in $\mathbb{R}^n$, we can choose $\mathbf{p}$ so that $|D_{\mathcal{B}}(\mathbf{p}')| = 1$, and so $\{\mathbf{x}'\} = D_{\mathcal{B}}(\mathbf{p}')$ is uniquely defined. Again applying Proposition 4.14 Part (5), and Definition 2.1 for ordinary substitutes, we observe that $x'_j \geq x_j = \hat{x}_j$ for all $j \in [n]$ with $j \neq i$. So, since $\hat{x}_i = 0 = x'_i$, we know that $\mathbf{x}' \geq \hat{\mathbf{x}}$.

Now, $\hat{x} \in A_{\mathcal{B}}$ will hold iff, for all $\mathbf{d} \in \mathbb{R}^n$ there exist $\mathbf{y}^{\mathbf{d}} \in A_{\mathcal{B}}$ such that $\mathbf{d} \cdot \widehat{\mathbf{x}} \leq \mathbf{d} \cdot \mathbf{y}^{\mathbf{d}}$. Given $\mathbf{d} \in \mathbb{R}^n$, write $\mathbf{d}^+ := \sum_{j=1}^n \max(d_i, 0)\mathbf{e}^i$ and $\mathbf{d}^- := \sum_{j=1}^n \max(-d_i, 0)\mathbf{e}^i$. Then $\mathbf{d} = \mathbf{d}^+ - \mathbf{d}^-$ and so

$$\mathbf{d} \cdot \widehat{\mathbf{x}} = \mathbf{d}^+ \cdot \widehat{\mathbf{x}} - \mathbf{d}^- \cdot \widehat{\mathbf{x}} \leq \mathbf{d}^+ \cdot \widehat{\mathbf{x}} \leq \mathbf{d}^+ \cdot \mathbf{x}', \tag{9}$$

the first inequality holding since $\mathbf{d}^- \geq 0$ and $\widehat{\mathbf{x}} \geq 0$, and the second holding since $\mathbf{x}' \geq \hat{\mathbf{x}}$ and $\mathbf{d}^+ \geq 0$. If $\mathbf{d}^+ = \mathbf{d}$ then we are done. Otherwise, let $\mathbf{p}_{\mathbf{d}} = \mathbf{p}' + R^{\mathbf{d}} \sum_{d_j < 0} \mathbf{e}^j$ where $R^{\mathbf{d}} \gg 0$, adjusting $\mathbf{p}$ and $\mathbf{p}'$ if necessary so that $|D_{\mathcal{B}}(\mathbf{p}_{\mathbf{d}})| = 1$, and write $\{\mathbf{y}^{\mathbf{d}}\} = D_{\mathcal{B}}(\mathbf{p}_{\mathbf{d}}) \subseteq A_{\mathcal{B}}$, noting that if $R^{\mathbf{d}}$ is sufficiently large then, for all j such that $d_j < 0$, we know $j \neq I(\mathbf{b}, \mathbf{p}^{\mathbf{d}})$ for all $\mathbf{b} \in \mathcal{B}$ and so $y^{\mathbf{d}}_j = 0$; thus $\mathbf{d}^- \cdot \mathbf{y}^{\mathbf{d}} = 0$. However, again applying the definition of ordinary substitutes (Definition 2.1), we know that $y^{\mathbf{d}}_j \geq x'_j$ for all j such that $d_j \geq 0$. It follows that

$$\mathbf{d} \cdot \mathbf{y}^{\mathbf{d}} = \mathbf{d}^+ \cdot \mathbf{y}^{\mathbf{d}} - \mathbf{d}^- \cdot \mathbf{y}^{\mathbf{d}} = \mathbf{d}^+ \cdot \mathbf{y}^{\mathbf{d}} \geq \mathbf{d}^+ \cdot \mathbf{x}'. \tag{10}$$

Combining Equations (9) and (10) demonstrates that $\mathbf{d} \cdot \widehat{\mathbf{x}} \leq \mathbf{d} \cdot \mathbf{y}^{\mathbf{d}}$.

Finally, if there does not exist $\mathbf{p}$ such that $D_{\mathcal{B}}(\mathbf{p}) = \{\mathbf{x}\}$, then $D_{\mathcal{B}}(\mathbf{p})$ is the discrete-convex hull of a finite set $\{\mathbf{x}^k \mid k \in K\}$ of bundles in $A_{\mathcal{B}}$, for each of which there exists a price $\mathbf{p}^k$ such that $D_{\mathcal{B}}(\mathbf{p}^k) = \{\mathbf{x}^k\}$. So $\mathbf{x} = \sum_{k \in K} \lambda_k \mathbf{x}^k$ for some weights $\lambda_k \in [0, 1]$ such that $\sum_{k \in K} \lambda_k = 1$. But, by the argument above, we know that $\mathbf{x}^k - x^k_i \mathbf{e}^i \in A_{\mathcal{B}}$ for all $k \in K$. So $\sum_{k \in K} \lambda_k (\mathbf{x}^k - x^k_i \mathbf{e}^i) = \mathbf{x} - x_i \mathbf{e}^i \in A_{\mathcal{B}}$ also, as required.

Finally, to show part (3), let $i \in [n]$ and suppose that $\mathbf{x}' \in \arg\max_{\mathbf{x} \in A_{\mathcal{B}}} \{x_i\}$. By definition of $A_{\mathcal{B}}$ there exists $\mathbf{p}'$ such that $\mathbf{x}' \in D_{\mathcal{B}}(\mathbf{p}')$. Let $R \gg 0$ and set $\mathbf{p} = \mathbf{p}' - R\mathbf{e}^i$, so that $\mathbf{p}' = \mathbf{p} + R\mathbf{e}^i$. Then, as long as R has been chosen to be sufficiently large, it follows that $I(\mathbf{b}, \mathbf{p}) = \{i\}$ and so $D_{\mathbf{b}}(\mathbf{p}) =$

$\{m(\mathbf{b})t_i(\mathbf{b})\mathbf{e}^i\}$ for all $\mathbf{b} \in \mathcal{B}$. So, letting $W_i = \sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b})t_i(\mathbf{b})$, we have that $\{W_i\mathbf{e}^i\} = D_{\mathcal{B}}(\mathbf{p})$. By definition of $\mathbf{x}'$ we know that $x'_i \geq W_i$. However, the strict law of demand also holds (Lemma B.13), so $x'_i \leq W_i$, so $x'_i = W_i$. Now the strict law of demand dictates that $\mathbf{x}' \in D_{\mathcal{B}}(\mathbf{p}) = \{W_i\mathbf{e}^i\}$. That is, if $\mathbf{x}' \in \arg\max_{\mathbf{x}\in A_{\mathcal{B}}}\{x_i\}$ then $\mathbf{x}' = W_i\mathbf{e}^i$, so $\arg\max_{\mathbf{x}\in A_{\mathcal{B}}}\{x_i\} = \{W_i\mathbf{e}^i\}$.

The case of SSPMA bids is given by Baldwin et al. (2021a) Proposition 3. □

We write $\underline{\mathbf{H}}_i := \{\mathbf{p} \in \mathbf{H} \mid p_i = \underline{H}\}$ and $\overline{\mathbf{H}}_i := \{\mathbf{p} \in \mathbf{H} \mid p_i = \overline{H}\}$.

Proof of Lemma 4.16. Let $i \in [n]$. As A_v is an FBD, if $\mathbf{x} \in A_v$ then $\mathbf{x}' := \mathbf{x} - x_i\mathbf{e}^i \in A_v$ (Definition 2.9); by construction $x'_i = 0$. Then if $p_i x_i > v(\mathbf{x}) - v(\mathbf{x}')$ it follows that $v(\mathbf{x}') - \mathbf{p}\cdot\mathbf{x}' > v(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}$, that is, $\mathbf{x}'$ is preferred to $\mathbf{x}$. So, for any $\mathbf{p}$ such that p_i is large enough to satisfy this condition for all (finitely many) $\mathbf{x} \in A_v$, it follows that $y_i = 0$ for all $\mathbf{y} \in D_v(\mathbf{p})$. In particular, note that the ith coordinate is the same for all such bundles. Thus, by Lemma 4.9, there exists $H'_i > 0$ such that for all $\mathbf{p} \in \mathbb{R}^n$ with $p_i > H'_i$, and for all $j \in [n]$ there does not exist an (i,j)-fin F satisfying $\mathbf{p} \in F$.

So, if F is an (i,j)-fin of $\mathcal{L}_v$, then it is bounded above in coordinate i: there exists a set $C = \arg\max\{p_i \mid \mathbf{p} \in F\}$. But C is a face of F, and so is a price complex cell for v (Definition 2.17 Part (4)(i) and Fact 2.18). By definition of $\mathbf{H}$ it follows that $C \cap \mathbf{H}^\circ \neq \emptyset$; by definition of C and $\mathbf{H}^\circ$ it follows that $p_i < \overline{H}$ for all $\mathbf{p} \in F$, that is, $F \cap \overline{\mathbf{H}}_i = \emptyset$. Note that this holds for any (i,j)-fin of $\mathcal{L}_v$, for any $i, j \in [n]$.

Now, if there exists any bid $\mathbf{b} \in \mathcal{B}$ such that $r_i(\mathbf{b}) = \overline{H}$ and $r_j(\mathbf{b}) > \underline{H}$ for any $i, j \in [n]$, then, by Corollary B.10, there exists an (i,j)-fin F of $\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^\circ$ such that $\overline{H} = \max\{p_i \mid \mathbf{p} \in F\}$. So if $\mathbf{b} \in \mathcal{B}$ and $r_i(\mathbf{b}) = \overline{H}$ for any $i \in [n]$ then $r_j(\mathbf{b}) = \underline{H}$ for all $j \in [n]$, $j \neq i$.

Now we turn to the lower faces of $\mathbf{H}$. Consider a (j,k)-fin F of $\mathcal{L}_v$, and for any $\mathbf{p}^0 \in F^\circ$, the slice of this fin, $F_0 := \{\mathbf{p} \in F \mid p_j = p^0_j\}$. We show that F_0 is bounded below in coordinate $i \neq j, k$. Let $\mathbf{y} \in D_v(\mathbf{p}^0)$, so that also (without loss of generality) $\mathbf{y} + t_j\mathbf{e}^j - t_k\mathbf{e}^k \in D_v(\mathbf{p}^0) \subseteq A_v$, for some trade-offs t_j, t_k.

Since A_v is an FBD, there exists $W_i \in \mathbb{Z}_{>0}$ such that $\{W_i\mathbf{e}^i\} = \arg\max\{x_i \mid \mathbf{x} \in A_v\}$. In particular, since there is only one bundle maximising x_i, we know that $\mathbf{y} \neq W_i\mathbf{e}^i$ and $y_i < W_i$. Now, if $R \in \mathbb{R}$ satisfies $R(W - y_i) > v(\mathbf{y}) - v(W\mathbf{e}^i) - \mathbf{p}\cdot\mathbf{y} + Wp_i$ then $v(\mathbf{y}) - (\mathbf{p} - R\mathbf{e}^i)\cdot\mathbf{y} < v(W\mathbf{e}^i) - (\mathbf{p} - R\mathbf{e}^i)\cdot W\mathbf{e}^i$, that is, the bundle $W\mathbf{e}^i$ is preferred to $\mathbf{y}$ at $\mathbf{p} - R\mathbf{e}^i$, and so in particular $\mathbf{p} - R\mathbf{e}^i \notin F_0$.

So there must exist an $(n-2)$-cell C' of F between $\mathbf{p}$ and $\mathbf{p} - R\mathbf{e}^i$. Now by Definition A.1 and Corollary A.3, if C' is not Type 2 with indices (i,j,k), there is another j-hod F' for $\mathcal{L}_v$ on the opposite side of C' with respect to coordinate i. Moreover, then, if F' is bounded below in coordinate i, then so is F. Since $\mathcal{L}_v$ has finitely many facets, we can apply this process repeatedly. So assume that $C' = C$ which is indeed of Type 2 with indices (i,j,k). But, by Assumption 4.15, it follows that C has non-empty intersection with $\mathbf{H}^\circ$. It follows that, if $\mathbf{p}^j \in \arg\max_{\mathbf{p}\in F} p_j$, then $p^j_i > \underline{H}$.

Now let $\mathcal{B}' := \{\mathbf{b} \in \mathcal{B} \mid r_i(\mathbf{b}) = \underline{H}\}$, and let $\mathcal{B}'' := \mathcal{B} \setminus \mathcal{B}'$. By Corollary B.10, if $r^*_j := \max\{r_j(\mathbf{b}') \mid \mathbf{b}' \in \mathcal{B}', r_k(\mathbf{b}') > \underline{H}\} > \underline{H}$, then there exists a (j,k)-fin F of $\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ$ such that $\mathbf{r} \in \arg\max\{p_j \mid \mathbf{p} \in F\}$, where $\mathbf{r} = \mathbf{r}(\mathbf{b}')$ for some $\mathbf{b}' \in \mathcal{B}'$ and $r_j = r^*_j$. So this provides the required contradiction, unless the part of F closest to $\underline{\mathbf{H}}_i$ is contained in a facet of $\mathcal{L}_{\mathcal{B}''}$ of equal and opposite weight to that of F, in which case it is "deleted" in $\mathcal{L}_{\mathcal{B}}$ itself. However, in that case, because $\mathcal{L}_{\mathcal{B}}$ is itself balanced, and positive-weighted within $\mathbf{H}^\circ$, there must exist another (j,k)-finF', meeting F along an $(n-2)$-cell of Type 1, and providing the required contradiction.

Now we rule out bids $\mathbf{b}$ for which $\underline{H} < r_j(\mathbf{b}) < \overline{H}$ for some $j \in [n]$ and $r_i(\mathbf{b}) = \underline{H}$ for all $i \in [n]$ with $i \neq j$. But in such cases, the fact that $\mathcal{B}$ has no redundancies relative to $\mathbf{H}$ implies that $\mathbf{t}(\mathbf{b}) = \mathbf{1}$ and hence that $\mathbf{b}$ is the only bid in $\mathcal{B}$ rooted at $\mathbf{r}(\mathbf{b})$. Then, Lemma 4.9 implies that $\mathcal{L}_{\mathcal{B}}$ has a non-zero weighted j-hod F passing through $\mathbf{r}$ and such that $F \cap \mathbf{H}^\circ \neq \emptyset$. In exactly the same way as we did for the (j,k)-fin, we can see that this provides a contradiction with the domain of v being an FBD.

So, for all $\mathbf{r} \in \mathcal{B}$, if $\mathbf{r}(\mathbf{b}) \in \mathbf{H} \setminus \mathbf{H}^\circ$ then $\mathbf{r} = \overline{H}\mathbf{e}^i + \sum_{j\in[n], j\neq i} \underline{H}\mathbf{e}^j$ for some $i \in [n]$. But if such a bid $\mathbf{b}$ exists, then, if we set $\mathbf{p} = \sum_{i\in[n]}(\overline{H} - \epsilon)\mathbf{e}^i$ for any small $\epsilon > 0$, we find that $D_{\mathbf{b}}(\mathbf{p}) = \{t_i(\mathbf{b})m(\mathbf{b})\mathbf{e}^i\}$. If $\mathbf{r}(\mathbf{b}') \in \mathbf{H}^\circ$ and we choose ϵ small enough then $D_{\mathbf{b}'}(\mathbf{p}) = \{\mathbf{0}\}$, and so we conclude that the ith coordinate of $D_{\mathcal{B}}(\mathbf{p})$ is given by $t_i(\mathbf{b})m(\mathbf{b})$ for the bid as described. But v is a concave ordinary substitutes valuation with domain an FBD, and so $\mathbf{0} \in A_v$; it follows that $D_v(\mathbf{p}') = \{\mathbf{0}\}$ for sufficiently large prices, and in particular, by Assumption 4.15, that $D_v(\mathbf{p}) = \{\mathbf{0}\} = D_{\mathcal{B}}(\mathbf{p})$. So the bid $\mathbf{b}$ described cannot exist, and $\mathbf{r}(\mathbf{b}) \in \mathbf{H}^\circ$ for all $\mathbf{b} \in \mathcal{B}$. □

Proof of Proposition 4.17. Let $\mathbf{b} \in \mathcal{B}$ and suppose $r_i(\mathbf{b}) \leq \underline{H}$. Define $\mathbf{r}'$ by setting $r'_j = r_j(\mathbf{b})$ for $j \neq i$ and $r'_i = \underline{H}$. Write $\mathbf{b}' := (\mathbf{r}'; \mathbf{t}(\mathbf{b}); m(\mathbf{b}))$. Then one may check that $\mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^\circ = \mathcal{L}_{\mathbf{b}'} \cap \mathbf{H}^\circ$ by verifying that this holds for all facets of these LIPs, using the description of Lemma 4.9.

Now suppose $r_i(\mathbf{b}) \geq \overline{H}$. Define $\mathbf{r}'$ by setting $r'_j = r_j - \frac{t_i}{t_j}(r_i \underline{H})$ for all $j \in [n]$. Write $\mathbf{b}' := (\mathbf{r}'; \mathbf{t}(\mathbf{b}); m(\mathbf{b}))$. Again one may check that $\mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^\circ = \mathcal{L}_{\mathbf{b}'} \cap \mathbf{H}^\circ$ by verifying that this holds for all facets of these LIPs, using the description of Lemma 4.9 (in this case all the hods have empty intersection with $\mathbf{H}^\circ$).

Thus, by adjusting each coordinate in turn as necessary, we obtain $\mathbf{b}''$ such that $\mathbf{r}(\mathbf{b}'') \in \mathbf{H}$ and such that $\mathcal{L}_{\mathbf{b}''} \cap \mathbf{H}^\circ = \mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^\circ$. Repeating this procedure for every $\mathbf{b} \in \mathcal{B}$, and noting that there is nothing to do for bids $\mathbf{b} \in \mathcal{B}$ for which $\mathbf{r}(\mathbf{b}) \in \mathbf{H}^\circ$, while every bid for which $\mathbf{r}(\mathbf{b}) \notin \mathbf{H}^\circ$ is mapped to exactly one new bid rooted on $\mathbf{H} \setminus \mathbf{H}^\circ$, completes the proof. □

Proof of Lemma 4.18. First we show that (2) $\Longrightarrow$ (1). Suppose bids $\mathbf{b}^1, \mathbf{b}^2$ satisfy $\mathbf{r}(\mathbf{b}^1) = \mathbf{r}(\mathbf{b}^2)$ and $\mathbf{t}(\mathbf{b}^1) = \mathbf{t}(\mathbf{b}^2)$. Define $\mathbf{b}$ by $\mathbf{r}(\mathbf{b}) = \mathbf{r}(\mathbf{b}^1)$, $\mathbf{t}(\mathbf{b}) = \mathbf{t}(\mathbf{b}^1)$ and $m(\mathbf{b}) = m(\mathbf{b}^1) + m(\mathbf{b}^2)$. Now $\mathcal{L}_{\mathbf{b}^1} \boxplus \mathcal{L}_{\mathbf{b}^2} = \mathcal{L}_{\mathbf{b}}$. By extension over the whole bid collection, if the sum of multiplicities of bids with any root and trade-off are the same in $\mathcal{B}^1$ as in $\mathcal{B}^2$ then both $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1})$ and $(\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2})$ are equal to the weighted LIP of a bid collection with only one bid at each root, whose multiplicity is this sum of multiplicities. So $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) = (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2})$ in this case.

That (1) $\Longrightarrow$ (3) is evident: we may simply choose suitable $\underline{H}, \overline{H}$ so that $\mathbf{r}(\mathbf{b}) \in \mathbf{H}^\circ$ for all $\mathbf{b} \in \mathbf{H}^\circ$. The additional conditions are then automatically satisfied.

Finally, we assume that (3) holds for some $\underline{H}, \overline{H}$, and show that this implies (2). By Lemmas 4.6, 4.7 and 4.11, we know that $\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ = \emptyset$ where $\mathcal{B}' := \mathcal{B}^1 \cup \{(\mathbf{r}(\mathbf{b}); \mathbf{t}(\mathbf{b}); -m(\mathbf{b})) \mid \mathbf{b} \in \mathcal{B}^2\}$. Write $\mathcal{B}$ for the bid collection consisting of nonzero sums of multiplicities of bids in $\mathcal{B}'$ with the same root and trade-off, that is, $\mathcal{B} = \{(\mathbf{r}; \mathbf{t}; m) \mid (\mathbf{r}; \mathbf{t}) = (\mathbf{r}(\mathbf{b}); \mathbf{t}(\mathbf{b}))$ for some $\mathbf{b} \in \mathcal{B}'$, $m = \sum_{\mathbf{b}' \in \mathcal{B}', \mathbf{r}(\mathbf{b}') = \mathbf{r}(\mathbf{b}); \mathbf{t}(\mathbf{b}') = \mathbf{t}(\mathbf{b})} m(\mathbf{b}')$, $m \neq 0\}$. By construction, and by the conditions given on $\mathcal{B}^1$ and $\mathcal{B}^2$, we know $\mathcal{B}$ has no redundancies relative to $\mathbf{H}$. Observe that $\mathcal{L}_{\mathcal{B}} = \mathcal{L}_{\mathcal{B}'}$, because (2) $\Longrightarrow$ (1), and that if $\mathcal{L}_{\mathcal{B}'} = \emptyset$ then (2) holds, by Lemmas 4.7 and 4.11. We will show this is so by showing that $\mathcal{B} = \emptyset$. Suppose, for a contradiction, that $\mathcal{B} \neq \emptyset$.

First suppose that there exists some $\mathbf{b} \in \mathcal{B}$ such that $r_i(\mathbf{b}) > \underline{H}, r_j(\mathbf{b}) > \underline{H}$ for some $i, j \in [n]$ with $i \neq j$. Then Corollary B.10 applies, and shows that $\mathcal{L}_{\mathcal{B}} \cap \mathbf{H}^\circ$ has a (i,j)-fin. This is a contradiction.

Next suppose that there exists $i \in [n]$ and $\mathbf{b} \in \mathcal{B}$ such that $\underline{H} < r_i(\mathbf{b}) < \overline{H}$ and such that $r_j(\mathbf{b}) = \underline{H}$ for all $j \in [n]$ with $j \neq i$. Observe that our conventions enforce that $\mathbf{t}(\mathbf{b}) = \mathbf{1}$ and so that $\mathbf{b}$ is the only bid in $\mathcal{B}$ rooted at $\mathbf{r}(\mathbf{b})$. Then, considering Lemma 4.9 we see that $\mathcal{L}_{\mathcal{B}}$ has a non-zero weighted i-hod F passing through $\mathbf{r}$ such that $F \cap \mathbf{H}^\circ \neq \emptyset$, providing another contradiction.

Finally, observe that in either the case $r_i(\mathbf{b}) = \underline{H}$ for all $i \in [n]$, or the case $r_i(\mathbf{b}) = \overline{H}$ for some $i \in [n]$ and $r_j(\mathbf{b}) = \underline{H}$ for all $j \neq i$, result in $\mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^\circ = \emptyset$. But by assumption there are no such bids in $\mathcal{B}^1$ or $\mathcal{B}^2$, and hence no such bids in $\mathcal{B}$. So all possible cases lead to contradiction. This completes the proof. □

B.4 Proof of the main result

This section contains the proofs of results in Section 5 except for Proposition 5.6, the key technical result, which is proved in Section B.5 below.

Proof of Lemma 5.2. When $n' = 1$ then a weighted pseudo-LIP is just a finite collection of points, each assigned a weight in $\mathbb{Z}$. Setting every trade-off equal to 1, these points and their weights define the bids directly. □

Proof of Lemma 5.4. Since $(\mathcal{L}, w)$ is a weighted pseudo-LIP, there exists an associated balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$ (Π, w) of dimension n. Let Π^{ρ_i} be the image under ρ_i of the set of intersections between cells C of Π and $\langle \underline{\mathbf{H}}_i \rangle$. This is also a rational polyhedral complex, of dimension $(n-1)$ with support $\mathbb{R}^{n-1}$; the defining properties are inherited from Π.

If F' is a facet of Π^{ρ_i} (so it has dimension $n-2$), then $F' = \rho_i(C \cap \langle \underline{\mathbf{H}}_i \rangle)$, where C is a cell of Π, which must be: either a facet of Π, meeting $\langle \underline{\mathbf{H}}_i \rangle$ in its interior; or C is an $(n-2)$-cell of Π contained in $\langle \underline{\mathbf{H}}_i \rangle$. But in the latter case, C is contained in facets F of Π, and since Π is balanced around C, at least one such facet F must satisfy $F \cap \langle \underline{\mathbf{H}}_i \rangle^+ \neq \emptyset$. So we can define a weighting on Π^{ρ_i} exactly as in Definition 5.3.

To show that Π^{ρ_i} is balanced, we first define a similar complex, for which the intersection that we take is "transverse".[48] Write $K_i := \langle \underline{\mathbf{H}}_i \rangle + \epsilon \mathbf{e}^i$, where $\epsilon > 0$ is small and chosen such that no cells of Π are contained in K_i; this is possible as Π has finitely many cells. Let Σ be the image under ρ_i of the set of intersections $C \cap K_i$, where C is a cell of Π. Again, this is a rational polyhedral complex, of dimension $(n-1)$, with support $\mathbb{R}^{n-1}$. By choice of ϵ, all facets of Π with non-empty intersection with $\langle \underline{\mathbf{H}}_i \rangle$ and with $\langle \underline{\mathbf{H}}_i \rangle^+$ will have an $(n-2)$-dimensional intersection with K_i and these intersections meet only on their boundaries; and these are the only facets of Σ. For each such facet F', set $w'(F')$ to be the weight $w(F)$ of the corresponding facet F of Π. Similarly, the $(n-3)$-cells of Σ are exactly the intersections with K_i of $(n-2)$-cells of Π which have non-empty intersection with $\langle \underline{\mathbf{H}}_i \rangle$ and with $\langle \underline{\mathbf{H}}_i \rangle^+$. So the weighting w' on Σ inherits the balancing property from w, and (Σ, w') is a balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$ in $\mathbb{R}^{n-1}$.

Now observe that if C is an $(n-3)$-cell of Π^{ρ_i}, then it has one of two forms. It may be the projection under ρ_i of the intersection of prices in the relative interior of an $(n-2)$-cell of Π with $\langle \underline{\mathbf{H}}_i \rangle$ – with that cell having non-zero intersection with $\langle \underline{\mathbf{H}}_i \rangle^+$. In this case, all facets F of Π containing C correspond to facets of Π^{ρ_i}, and are in each case the unique facet to do so. Then the weighting $w_{\mathcal{L}^{\rho_i}}$ is clearly balanced around C. Alternatively, C may be the projection under ρ_i of an $(n-3)$-cell $\hat{C}$ of Π that lies within $\langle \underline{\mathbf{H}}_i \rangle$. In this case, $\hat{C}$ must be the intersection of a set of $(n-2)$-cells of Π that meet $\langle \underline{\mathbf{H}}_i \rangle^+$. Each of these gives rise to an $(n-2)$-cell in Σ, around which the weighting w' is balanced. As $\epsilon \to 0$ these $(n-2)$-cells converge, meaning that any facets containing more than one of these $(n-2)$-cells will disappear, but it is not hard to check that the sum of the balancing conditions around these $(n-2)$-cells in Σ imply the balancing condition is satisfied around C.

Thus $(\Pi^{\rho_i}, w_{\mathcal{L}^{\rho_i}})$ is a balanced $\mathbb{Z}$-weighted rational polyhedral complex with support $\mathbb{R}^n$. By definition, $(\mathcal{L}^{\rho_i}, w_{\mathcal{L}^{\rho_i}})$ is the corresponding weighted pseudo-LIP.

Finally we observe that $(\mathcal{L}^{\rho_i}, w_{\mathcal{L}^{\rho_i}})$ is an ordinary substitutes weighted pseudo-LIP by noting from Lemma A.2 that if F is a facet of $\mathcal{L}^{\rho_i}$ then it has the form of: either a Type 1 $(n-2)$-cell of $\mathcal{L} \boxplus \langle \underline{\mathbf{H}}_i \rangle$ with indices (i, j) for some $j \in [n]$, $j \neq i$; or a Type 3 $(n-2)$-cell of $\mathcal{L} \boxplus \langle \underline{\mathbf{H}}_i \rangle$ with indices $(i; j, k)$ and trade-off (t_j, t_k) for some $j, k \in [n]$ distinct from each other and from i. In the former case F is a j-hod of $\mathcal{L}^{\rho_i}$ and in the latter it is a (j,k)-fin of $\mathcal{L}^{\rho_i}$. It follows that $\mathcal{L}^{\rho_i}$ is an ordinary substitutes pseudo-LIP. □

Proof of Lemma 5.5. It is clear by definition that $\sigma_i(\mathcal{B})$ is an n-dimensional bid collection rooted in $\underline{\mathbf{H}}_i$.

First suppose that $\mathcal{B} = \{\mathbf{b}\}$ for $\mathbf{b}$ rooted in $\rho_i(\underline{\mathbf{H}}_i)$. The result follows straightforwardly in this simple case, by considering the descriptions of Lemma 4.9.

Now, recall that $\mathcal{L}_{\mathcal{B}} = \boxplus_{\mathbf{b} \in \mathcal{B}} \mathcal{L}_{\mathbf{b}}$ and $\mathcal{L}_{\sigma_i(\mathcal{B})} = \boxplus_{\mathbf{b} \in \mathcal{B}} \mathcal{L}_{\sigma(\mathbf{b})}$. So the result follows from the simple case above, by repeatedly applying the definition of $\boxplus$. □

Note that Proposition 5.6 is proved in Appendix B.5 below. We proceed with the remaining proofs for this section.

Proof of Lemma 5.7. Let $i \in [n-2]$, let F be an (i,n)-fin of $\mathcal{L}$, and let $\mathcal{F}$ be the set of (i,n)-fins F' of $\mathcal{L}$ such that $\dim(F' \cap (F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ) = n-1$ and such that $w(F') = w(F)$. We will show that $(F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ \subseteq \bigcup \mathcal{F}$. Since $\bigcup \mathcal{F} \subseteq \mathcal{L}$ by definition, it follows that $(F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ \subseteq \mathcal{L}$, and since $F + \mathbb{R}\mathbf{e}^{n-1}$ is a closed and convex set, and $\mathcal{L}$ is a closed set, it follows that $(F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H} \subseteq \mathcal{L}$. The condition on weights of facets follows because $(F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ \subseteq \bigcup \mathcal{F}$, and we defined $\mathcal{F}$ to be a set of fins whose weight matches that of F.

Fix $\mathbf{p} \in F \cap \mathbf{H}^\circ$. Assume for a contradiction that $\{\lambda' > 0 \mid \mathbf{p} + \lambda' \mathbf{e}^{n-1} \in \mathbf{H}^\circ \setminus \bigcup \mathcal{F}\} \neq \emptyset$ and let λ be the infimum of this set. Observe that $\mathbf{r} := \mathbf{p} + \lambda \mathbf{e}^{n-1} \in \mathbf{H}^\circ$.

Now $\mathbf{r} \in \bigcup \mathcal{F} \cap \mathbf{H}^\circ$, since facets are topologically closed, so $\mathbf{r} \in F'$ for some $F' \in \mathcal{F}$; but $\mathbf{r} + \epsilon \mathbf{e}^{n-1} \notin \bigcup \mathcal{F} \cap \mathbf{H}^\circ$ for all $\epsilon > 0$, so $\mathbf{r} + \epsilon \mathbf{e}^{n-1} \notin F'$. It follows that $\mathbf{r}$ is in the boundary of F', which we recall to be a union of intersections of hyperplanes with F', each intersection giving an $(n-2)$-cell. So $\mathbf{r}$ must be in an $(n-2)$-face C of F' such that $\mathbf{r} + \mathbf{e}^{n-1} \notin \langle C \rangle$. That is, a vector in direction $\mathbf{e}^{n-1}$ does not lie in the affine span of C. Since F' is an (i,n)-fin, C is contained in such a facet. We now consider the possible cases for C.

[48] See Maclagan and Sturmfels (2015), Definition 3.4.9, or Baldwin and Klemperer (2019), Definition 4.10.

If C is Type 1 then by Lemma A.2 it must have indices (i, n), and so $\langle C \rangle = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i, p_n = r_n\}$. This contradicts $\mathbf{r} + \mathbf{e}^{n-1} \notin \langle C \rangle$.

Suppose C is Type 2 with indices (i, j, n) for some $j \neq i, n$ and tradeoffs (t_i, t_j, t_k), and additionally suppose $j \neq n-1$. Then $\mathbf{r} + \mathbf{e}^{n-1} \in \langle C \rangle = \{\mathbf{p} \in \mathbb{R}^n \mid t_i(p_i - r_i) = t_j(p_j - r_j) = t_n(p_n - r_n)\}$, providing a contradiction.

Now suppose C is Type 2 with indices $(i, n-1, n)$; it must have trade-off (t_i, t_{n-1}, t_n) where t_i/t_n is the trade-off of F'. Recall that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no $(n-1, n)$-fins, and so since $\mathbf{r} \in \mathbf{H}^\circ$, there is no such fin containing C. By Lemma A.2 it follows that there are at most two different affine spans to the facets of $\mathcal{L}$ containing C. Then by Corollary A.3 it follows that there is a facet F'' with the same affine span and weight as F', on the other side of C from F', contradicting the definition of λ.

If C is Type 3 or Type 4 then obtain a contradiction by applying Corollary A.3 in the same way as in the case of Type 2 with indices $(i, n-1, n)$.

So $\{\lambda' > 0 \mid \mathbf{p} + \lambda' \mathbf{e}^{n-1} \in \mathbf{H} \setminus \bigcup \mathcal{F}\} = \emptyset$ for all $\mathbf{p} \in F \cap \mathbf{H}^\circ$, and hence $(F + \mathbb{R}_{\geq 0}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ \subseteq \bigcup \mathcal{F}$. We show the same for $(F + \mathbb{R}_{\leq 0}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ$ by repeating the argument above but subtracting, instead of adding, copies of $\mathbf{e}^{n-1}$, and considering the supremum instead of the infimum. Together these results show that $(F + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ \subseteq \bigcup \mathcal{F}$. $\square$

Proof of Corollary 5.8. First consider the weighted pseudo-LIP $(\mathcal{L}, w) \boxplus (\langle \underline{\mathbf{H}}_{n-1} \rangle, 1)$. Observe by Lemma A.2 that, if $i \in [n-2]$, when an (i, n)-fin F of this pseudo-LIP meets $\underline{\mathbf{H}}_{n-1}$ in an $(n-2)$-cell C, then this must be a Type 3 $(n-2)$-cell with indices $(n-1; i, n)$ with trade-off equal to the trade-off of the (i, n)-fin itself. It follows that there can be at most one such facet meeting both $\mathbf{H}^\circ$ and C, and so, if $F' = \rho_{n-1}(F \cap \langle \underline{\mathbf{H}}_{n-1} \rangle)$ then $w_{\mathcal{L}^{\rho_{n-1}}}(F') = w(F)$ by Definition 5.3.

Now, applying Lemma 5.7, it follows that, if $\mathcal{F}$ is the set of all (i, n)-fins of $\mathcal{L}$, and $\mathcal{F}'$ is the set of all (i, n)-fins of $\mathcal{L}^{\rho_{n-1}}$, then $\bigcup \mathcal{F} \cap \mathbf{H}^\circ = (\bigcup \mathcal{F}' + \mathbb{R}\mathbf{e}^{n-1}) \cap \mathbf{H}^\circ$ and that if F' is an (i, n)-fin of $\mathcal{L}^{\rho_{n-1}}$ then $w_{\mathcal{L}^{\rho_{n-1}}}(F') = w(F)$ for all (i, n)-fins of $\mathcal{L}$ such that $\dim((F' + \mathbb{R}\mathbf{e}^{n-1}) \cap F \cap \mathbf{H}^\circ) = n-1$.

But by Lemma 5.4 we know that $(\mathcal{L}^{\rho_{n-1}}, w_{\mathcal{L}^{\rho_{n-1}}})$ is an ordinary substitutes weighted pseudo-LIP in $\mathbb{R}^{n-1}$. By the inductive hypothesis we have made on Theorem 5.1, there exists a bid collection $\mathcal{B}$, rooted in $\rho_{n-1}(\underline{\mathbf{H}}_{n-1})$, such that $(\mathcal{L}^{\rho_{n-1}} \cap \rho_{n-1}(\underline{\mathbf{H}}_{n-1})^\circ, w_{\mathcal{L}^{\rho_{n-1}}}) = (\mathcal{L}_\mathcal{B} \cap \rho_{n-1}(\underline{\mathbf{H}}_{n-1})^\circ, w_\mathcal{B})$. Set $\mathcal{B}' := \sigma_{n-1}(\mathcal{B})$, which is (Lemma 5.5) an n-dimensional bid collection, rooted in $\underline{\mathbf{H}}_i$.

Now by Lemma 5.5 and by our previous observation, the union of all (i, n)-fins of $(\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ in $\mathbf{H}^\circ$ is identical to the union of all (i, n)-fins of $(\mathcal{L}, w)$ in $\mathbf{H}^\circ$, with the weights also corresponding.

We conclude that $(\mathcal{L} \cap \mathbf{H}^\circ, w) \boxminus (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$ has no (i, n)-fins for all $i \in [n-2]$. But we originally assumed that $(\mathcal{L} \cap \mathbf{H}^\circ, w)$ had no $(n-1, n)$-fins. The same is true of $(\mathcal{L}_{\mathcal{B}'}, w_{\mathcal{B}'})$ by Lemma 5.5. So $(\mathcal{L} \cap \mathbf{H}^\circ, w) \boxminus (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$ has no (i, n)-fins, for any $i \in [n-1]$. $\square$

Proof of Lemma 5.9. The proof of this lemma is similar to that of Lemma 5.7. Fix $j, k \in [n-1]$ such that $j \neq k$, let F be a (j, k)-fin of $\mathcal{L}$, and let $\mathcal{F}$ be the set of (j, k)-fins F' of $\mathcal{L}$ such that $\dim(F' \cap (F + \mathbb{R}\mathbf{e}^n) \cap \mathbf{H}^\circ) = n-1$ and such that $w(F') = w(F)$. Fix $\mathbf{p} \in F \cap \mathbf{H}^\circ$ and assume for a contradiction that $\{\lambda' > 0 \mid \mathbf{p} + \lambda' \mathbf{e}^n \in \mathbf{H}^\circ \setminus \bigcup \mathcal{F}\} \neq \emptyset$; let λ be the infimum of this set and observe that $\mathbf{r} := \mathbf{p} + \lambda \mathbf{e}^n \in \mathbf{H}^\circ$.

As in the proof of Lemma 5.7, $\mathbf{r}$ is on an $(n-2)$-cell C of a facet $F' \in \mathcal{F}$ such that $\mathbf{r} + \mathbf{e}^n \notin \langle C \rangle$. Now consider the possible cases for C, recalling that F' is a (j, k)-fin.

If C is Type 1 then by Lemma A.2 it must have indices j, k, in which case $\mathbf{r} + \mathbf{e}^n \in \langle C \rangle$: a contradiction.

If C is Type 2, then by Lemma A.2 it must have indices (i, j, k) for some $i \neq j, k$. If $i \neq n$ then $\mathbf{r} + \mathbf{e}^n \in \langle C \rangle$: again, a contradiction.

Now suppose C is Type 2 with indices (j, k, n) and trade-off (t_j, t_k, t_n). By Lemma A.2, and by assumption that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no (i, n)-fins for any $i \in [n-1]$, the only facets of $\mathcal{L}$ containing C are $(j, k; t_j/t_k)$-fins, that is, they all have the same affine span. But by Corollary A.3 it follows that there is a facet F'' with the same affine span as F' on the other side of C from F', contradicting the definition of λ.

If C is Type 3 or Type 4 then we obtain a contradiction by applying Corollary A.3 in the same way as the case of Type 2 with indices (j, k, n).

Thus $\{\lambda' > 0 \mid \mathbf{p} + \lambda' \mathbf{e}^n \in \mathbf{H}^\circ \setminus \bigcup \mathcal{F}\} = \emptyset$ for all $\mathbf{p} \in F \cap \mathbf{H}^\circ$, and thus $(F + \mathbb{R}_{\geq 0}\mathbf{e}^n) \cap \mathbf{H}^\circ \subseteq \bigcup \mathcal{F}$. Again we may show that $(F + \mathbb{R}_{\leq 0}\mathbf{e}^n) \cap \mathbf{H}^\circ \subseteq \bigcup \mathcal{F}$ in the same way. So, as in the proof of Lemma 5.7, the result follows. $\square$

Proof of Corollary 5.10. This follows from Lemmas 5.5 and 5.9 in exactly the same way as Corollary 5.8 follows from Lemmas 5.5 and 5.7. $\square$

Proof of Lemma 5.11. The proof of this lemma is again similar to that of Lemma 5.7. Fix $i \in [n]$, let F be a i-hod of $\mathcal{L}$, and write $\mathcal{F}$ for the set of i-hods F' of $\mathcal{L}$ such that $F' \cap \langle F \rangle \cap \mathbf{H}^\circ \neq \emptyset$ and such that $w(F') = w(F)$. Suppose for a contradiction that $\bigcup \mathcal{F} \cap \mathbf{H}^\circ \neq \langle F \rangle \cap \mathbf{H}^\circ$. Then there must exist a point $\mathbf{r} \in \mathbf{H}^\circ$ such that $\mathbf{r} \in C$, where C is an $(n-2)$-cell contained in a facet $F' \in \mathcal{F}$, and such that $\mathbf{r} + \epsilon \mathbf{e}^j \notin \bigcup \mathcal{F}$ for some $j \in [n]$ with $j \neq i$ and some small $\epsilon > 0$.

However, there can only be at most two distinct affine spans to the facets containing C: this is clear by considering the possibilities of Lemma A.2 and recalling that we assumed that $\mathcal{L} \cap \mathbf{H}^\circ$ contains no fins, so C is of Type 1. So, by Corollary A.3, it follows that there is a facet $F'' \in \mathcal{F}$ with the same affine span as F' on the other side of C from F' containing $r + \epsilon \mathbf{e}^j$, contradicting what we had inferred about C.

The contradiction implies that $\langle F \rangle \cap \mathbf{H}^\circ = \bigcup \mathcal{F} \cap \mathbf{H}^\circ \subseteq \mathcal{L}$, whence $\langle F \rangle \cap \mathbf{H} \subseteq \mathcal{L}$ since $\langle F \rangle$ is closed and convex, and $\mathcal{L}$ is closed. The required property on weights of facets follows immediately from the definition of $\mathcal{F}$. $\square$

Proof of Corollary 5.12. This follows from Lemmas 5.5 and 5.11 in exactly the same way as Corollary 5.8 follows from Lemmas 5.5 and 5.7. $\square$

B.5 The Proof of Proposition 5.6

Definition B.14. Suppose $\mathcal{L}$ is an ordinary substitutes pseudo-LIP. Write $\widehat{\mathrm{grid}}(\mathcal{L})$ for the union of sets $\langle F \rangle$, where either

(1) F is an (i,n)-fin $\mathcal{L} \cap \mathbf{H}^\circ$ for $i \in [n-1]$; or
(2) F is an n-hod of $\mathcal{L} \cap \mathbf{H}^\circ$; or
(3) $F = \underline{\mathbf{H}}_i$ or $\overline{\mathbf{H}}_i$ for some $i \in [n]$.

The *grid of* $\mathcal{L}$, denoted $\mathrm{grid}(\mathcal{L})$, is then the union of $\widehat{\mathrm{grid}}(\mathcal{L})$ with additionally the affine spans $\langle F \rangle$ of the sets F, where

(4) $F = C + \mathbb{R}\mathbf{e}^i$ for any Type 1 $(n-2)$-cell C of $\widehat{\mathrm{grid}}(\mathcal{L})$ with indices (i,n) for some $i \in [n-1]$.

Write $\mathcal{F}_{n-1,n}$ for the set of $(n-1,n)$-fins of $\mathrm{grid}(\mathcal{L})$ contained in $\mathbf{H}$.

Part (4) of Definition B.14 introduces additional n-hods into our structure, including potentially some which are not affine spans of n-hods in $\mathcal{L}$, but instead are defined by intersections of affine spans of (i,n)-fins.

Observe that, because any pseudo-LIP $\mathcal{L}$ has only finitely many facets and $(n-2)$-cells, $\mathrm{grid}(\mathcal{L})$ is indeed also a finite rational polyhedral complex. If we endow every facet of $\mathrm{grid}(\mathcal{L})$ with weight 1, then this has the structure of a balanced weighted rational polyhedral complex, and is thus a (true) LIP. Its $(n-1,n)$-fins, $\mathcal{F}_{n-1,n}$, will be be our building blocks; the addition and subtraction of these will provide the $(n-1,n)$-fins of $\mathcal{L}$.

However, we must create the fins of $\mathcal{F}_{n-1,n}$ by using bids. The geometry of $\mathrm{grid}(\mathcal{L})$ gives us a natural set of roots and trade-offs for bids:

Definition B.15.

(1) If $(\mathbf{r};\mathbf{t}) \in \mathbf{H} \times \mathbb{Z}^n_{>0}$ then define $\mathrm{inv}(\mathbf{r};\mathbf{t}) := \sum_{j \in [n],\, r_j \neq \underline{H}} \frac{\mathbf{e}^j}{t_j}$.
(2) The set of *candidates*, $\mathcal{C}$, is the set of pairs $(\mathbf{r};\mathbf{t}) \in \mathbf{H} \times \mathbb{Z}^n_{>0}$ where:
  (i) $\mathbf{r}$ is a 0-cell of $\mathrm{grid}(\mathcal{L})$;
  (ii) $r_{n-1} \neq \underline{H}$ and $r_n \neq \underline{H}$;
  (iii) for $i \in [n-2]$, if $r_i = \underline{H}$ then $t_i = 1$;
  (iv) there is a 1-cell of $\mathrm{grid}(\mathcal{L})$ of the form $\{\mathbf{r} - \lambda\, \mathrm{inv}(\mathbf{r},\mathbf{t}) \mid 0 \leq \lambda \leq \overline{\lambda}\}$ for some $\overline{\lambda} \in \mathbb{R}_{>0} \cup \{\infty\}$
  (v) $\sum_{j \in [n]; r_j \neq \underline{H}} t_j \mathbf{e}^j$ is a primitive integer vector.

Write $C_{(\mathbf{r};\mathbf{t})}$ for the 1-cell of $\mathrm{grid}(\mathcal{L})$ of Part (2)(iv).

There is a close relationship between $\mathcal{C}$ and $\mathcal{F}_{n-1,n}$.

Definition B.16. If $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$ and $F \in \mathcal{F}_{n-1,n}$ then F and $(\mathbf{r};\mathbf{t})$ are *associated* if $C_{(\mathbf{r};\mathbf{t})} \subsetneq F \subseteq F^{n-1,n}_{(\mathbf{r};\mathbf{t})}$.

Proposition B.17. *For every $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$ there is an unique associated $F \in \mathcal{F}_{n-1,n}$. Conversely, for every $F \in \mathcal{F}_{n-1,n}$ there is a unique associated $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$.*

To prove this, first we show:

Lemma B.18. *If $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$ then there is an n-hod of* $\mathrm{grid}(\mathcal{L})$ *containing $\mathbf{r}$ and, for all $i \in [n-1]$, either $r_i = \underline{H}$, and there is an i-hod of* $\mathrm{grid}(\mathcal{L})$ *containing $C_{(\mathbf{r};\mathbf{t})}$, or $r_i > \underline{H}$, and there is an $(i, n; t_i/t_n)$-fin of* $\mathrm{grid}(\mathcal{L})$ *containing $C_{(\mathbf{r};\mathbf{t})}$.*

Proof. Since $C_{(\mathbf{r};\mathbf{t})}$ is one-dimensional, there must be $n-1$ linearly independent normal vectors represented in the facets of $\mathrm{grid}(\mathcal{L})$ containing $C_{(\mathbf{r};\mathbf{t})}$. Fix a set V of such vectors. Normalise the vectors in V so that they are either $\mathbf{e}^i$ for some $i \in [n]$, or $\alpha_i \mathbf{e}^i - \mathbf{e}^n$, where $\alpha_i \neq 0$ is a rational number.

If $\mathbf{e}^i \in V$ for $i \in [n]$ then $\mathbf{e}^i \cdot \left(\sum_{r_j \neq \underline{H}} \frac{\mathbf{e}^j}{t_j}\right) = 0$, which is only consistent with $r_i = \underline{H}$. In particular, as $r_n \neq \underline{H}$, we have $\mathbf{e}^n \notin V$.

On the other hand, if $\alpha_i \mathbf{e}^i - \mathbf{e}^n \in V$ then $(\alpha_i \mathbf{e}^i - \mathbf{e}^n) \cdot \left(\sum_{r_j \neq \underline{H}} \frac{\mathbf{e}^j}{t_j}\right) = 0$, that is, $\frac{\alpha_i}{t_i} = \frac{1}{t_n}$ if $r_i \neq \underline{H}$, or $\frac{1}{t_n} = 0$ if $r_i = \underline{H}$. The latter case is obviously a contradiction, so we conclude that $r_i \neq \underline{H}$ and $\alpha_i = \frac{t_i}{t_n}$.

Observing that therefore there is only one vector in V containing nonzero ith coordinate for all $i \in [n-1]$, and that there are $n-1$ vectors in V, it follows by the pigeonhole principle that each coordinate $i \in [n-1]$ must be represented exactly once. In summary, then, for all $i \in [n-1]$, either $r_i = \underline{H}$ and $\mathbf{e}^i \in V$, or $r_i \neq \underline{H}$ and $\frac{t_i}{t_n}\mathbf{e}^i - \mathbf{e}^n \in V$. But in the former case, there is an i-hod of $\mathrm{grid}(\mathcal{L})$ containing $C_{(\mathbf{r};\mathbf{t})}$, and in the latter case, there is an $(i, n; t_i/t_n)$-fin of $\mathrm{grid}(\mathcal{L})$ containing $C_{(\mathbf{r};\mathbf{t})}$, as required.

It remains to show that there exists an n-hod containing $\mathbf{r}$. Since $\mathrm{grid}(\mathcal{L})$ has a 0-cell at $\mathbf{r}$, it follows that there is an additional facet whose normal $\mathbf{v}'$ is not in V, but which contains $\mathbf{r}$. So either $\mathbf{v}' = \mathbf{e}^n$; or $V \cup \{\mathbf{v}'\}$ contains both $\mathbf{e}^i$ and $\alpha_i \mathbf{e}^i - \mathbf{e}^n$ for some $i \in [n-1]$ and $\alpha_i \in \mathbb{Q}_{>0}$; or $V \cup \{\mathbf{v}'\}$ contains both $\alpha_i \mathbf{e}^i - \mathbf{e}^n$ and $\beta_i \mathbf{e}^i - \mathbf{e}^n$ for some $i \in [n-1]$ and some $\beta_i \in \mathbb{Q}_{>0}$ with $\beta_i \neq \alpha_i$. In the first of these cases, $\mathrm{grid}(\mathcal{L})$ has an n-hod containing $\mathbf{r}$. In the second and third, the affine spans of these facets are in $\widehat{\mathrm{grid}}(\mathcal{L})$, and the intersection of these is an $(n-2)$-cell C' of Type 1 with indices (i, n), in which case $\langle C' + \mathbf{e}^i \rangle$ is contained in $\mathrm{grid}(\mathcal{L})$, and so (by Definition B.14 Part (4)) $\mathrm{grid}(\mathcal{L})$ has an n-hod containing $\mathbf{r}$. ☐

It immediately follows from this that if $(\mathbf{r}(\mathbf{b}), \mathbf{t}(\mathbf{b})) \in \mathcal{C}$ then many of the facets of $\mathcal{L}_\mathbf{b}$ are contained in $\mathrm{grid}(\mathcal{L})$:

Corollary B.19. *If $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$ then $F^n_{(\mathbf{r};\mathbf{t})} \subsetneq \mathrm{grid}(\mathcal{L})$ and $F^{i,n}_{(\mathbf{r};\mathbf{t})} \subsetneq \mathrm{grid}(\mathcal{L})$ for all $i \in [n-1]$ such that $r_i \neq \underline{H}$.*

Proof. By definition $\langle F \rangle \subseteq \mathrm{grid}(\mathcal{L})$ for every facet F of $\mathrm{grid}(\mathcal{L})$, so the result follows immediately from Lemma B.18. ☐

Recall that by Definition B.15, $r_{n-1} \neq \underline{H}$ for $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$, so that by Corollary B.19 we always have $F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \subsetneq \mathrm{grid}(\mathcal{L})$. Now:

Corollary B.20. *If $F \in \mathcal{F}_{n-1,n}$ and $\dim(F \cap F^{n-1,n}_{(\mathbf{r};\mathbf{t})}) = n-1$ then $F \subseteq F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}$.*

Proof. Recall that $F \subseteq \mathbf{H}$ for every $F \in \mathcal{F}_{n-1,n}$. The result follows because facets do not meet in their interiors, and, by Lemma B.18 and the fact that $\langle F \rangle \subseteq \mathrm{grid}(\mathcal{L})$ for every facet F of $\mathrm{grid}(\mathcal{L})$, we know that every $(n-2)$-cell in the boundary of $F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}$ is the intersection of $F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}$ with another $(n-1)$-dimensional set also contained in $\mathrm{grid}(\mathcal{L})$. ☐

As a final consequence of Lemma B.18:

Corollary B.21. *If $F \in \mathcal{F}_{n-1,n}$ and $(\mathbf{r};\mathbf{t})$ is associated with F, then for all $i \in [n-2]$, either $r_i = \underline{H}$ or there is an $(i, n; t_i/t_n)$-fin of* $\mathrm{grid}(\mathcal{L})$ *meeting F in an $(n-2)$-cell containing $\mathbf{r}$.*

Proof. If $r_i \neq \underline{H}$ then, by Lemma B.18, there is an $(i, n; t_i/t_n)$-fin of grid($\mathcal{L}$) containing $C_{(\mathbf{r};\mathbf{t})}$. By definition of grid($\mathcal{L}$), the affine span of this fin also lies within grid($\mathcal{L}$), and meets F along $C_{(\mathbf{r};\mathbf{t})}$. We need to show that this intersection defines an $(n-2)$-cell of F. But, the only affine spans of facets of grid($\mathcal{L}$) which *can* contain $C_{(\mathbf{r};\mathbf{t})}$ are the affine spans of those described by Lemma B.18, due to the limited range of facets in grid($\mathcal{L}$) and the definition of $C_{(\mathbf{r};\mathbf{t})}$ itself. And Lemma B.18 describes facets with a linearly independent set of normal vectors. So the affine span of each facet there described must meet F in an $(n-2)$-cell of F, and in particular there exists an $(i, n; t_i/t_n)$-fin of grid($\mathcal{L}$) meeting F in an $(n-2)$-cell containing $C_{(\mathbf{r};\mathbf{t})}$, and hence containing $\mathbf{r}$. □

Next, we develop a partial order on $\mathcal{C}$, which will be useful first in our proof of Proposition B.17, and second in our development of bids to demonstrate Proposition 5.6. Write $\leq$ for the Euclidean partial ordering on $\mathbb{R}^n$. Now define a partial ordering $\leq_{\mathcal{C}}$ on $\mathcal{C}$ as follows.

Definition B.22. If $(\mathbf{r};\mathbf{t}), (\mathbf{r}';\mathbf{t}') \in \mathcal{C}$ then $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$ when the following all hold:
(1) $\langle F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \rangle = \langle F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')} \rangle$;
(2) $r'_n \geq r_n$;
(3) if $r'_n = r_n$ then $\mathbf{r}' \leq \mathbf{r}$;
(4) if $\mathbf{r}' = \mathbf{r}$ then $\frac{t'_i}{t'_n} \leq \frac{t_i}{t_n}$ for all $i \in [n-2]$ such that $r_i \neq \underline{H}$.

We must check:

Lemma B.23. *$\leq_{\mathcal{C}}$ is a partial order.*

Proof. Reflexivity is clear. To show anti-symmetry, assume that $(\mathbf{r};\mathbf{t}) \leq_{\mathcal{C}} (\mathbf{r}';\mathbf{t}')$ and $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$. Then $r_n = r'_n$ by (2), and so $\mathbf{r} = \mathbf{r}'$ by (3) and because $\leq$ is a partial order itself. Then, by Part (4), $\frac{t'_i}{t'_n} = \frac{t_i}{t_n}$ for all $i \in [n-2]$ such that $r_i \neq \underline{H}$ (which holds iff $r'_i \neq \underline{H}$). So $t'_i = \frac{t'_n}{t_n} t_i$ for all i such that $r_i \neq \underline{H}$. Since $\sum_{r_j \neq \underline{H}} t_j \mathbf{e}^j$ is a primitive integer vector, it follows that $t'_i = t_i$ for these values of i. Finally, $t_i = 1 = t'_i$ for all i such that $r_i = \underline{H}$.

To show transitivity, suppose $(\mathbf{r};\mathbf{t}) \leq_{\mathcal{C}} (\mathbf{r}';\mathbf{t}')$ and $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r}'';\mathbf{t}'')$. Condition (1) evidently then holds for $(\mathbf{r};\mathbf{t})$ and $(\mathbf{r}'';\mathbf{t}'')$. It is immediate that $r_n \geq r''_n$. If $r_n = r''_n$ then necessarily $r_n = r'_n = r''_n$ and so $\mathbf{r} \leq \mathbf{r}' \leq \mathbf{r}''$. But now, if $\mathbf{r} = \mathbf{r}''$ then necessarily $\mathbf{r} = \mathbf{r}' = \mathbf{r}''$, and so $\frac{t''_i}{t''_n} \leq \frac{t'_i}{t'_n} \leq \frac{t_i}{t_n}$ for all $i \in [n-2]$ such that $r_i \neq \underline{H}$. □

It is useful to note how to write Condition (1) of Definition B.22 in terms of the roots and trade-offs:

Lemma B.24. $\langle F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \rangle = \langle F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')} \rangle$ *if and only if* $\frac{t'_{n-1}}{t'_n} = \frac{t_{n-1}}{t_n}$ *and* $t_{n-1}(r'_{n-1} - r_{n-1}) = t_n(r'_n - r_n)$.

Proof. The normal to $F^{n-1,n}_{(\mathbf{r};\mathbf{t})}$ is $t_{n-1}\mathbf{e}^{n-1} - t_n\mathbf{e}^n$; the normal to $F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')}$ is $t'_{n-1}\mathbf{e}^{n-1} - t'_n\mathbf{e}^n$; clearly these are parallel if and only if $\frac{t'_{n-1}}{t'_n} = \frac{t'_{n-1}}{t'_n}$. If this holds, then $\langle F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \rangle = \langle F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')} \rangle$ if and only if $\mathbf{r}' \in \langle F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \rangle$, which holds if and only if $t_{n-1}(r'_{n-1} - r_{n-1}) = t_n(r'_n - r_n)$ by Lemma 4.9. □

Lemma B.25. *If* $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$ *and* $F \in \mathcal{F}_{n-1,n}$ *are associated, and* $\dim(F \cap F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')}) = n-1$ *then* $(\mathbf{r}', \mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$.

Proof. Since $(\mathbf{r};\mathbf{t})$ and F are associated, we know $C_{(\mathbf{r};\mathbf{t})} \subsetneq F \subseteq F^{n-1,n}_{(\mathbf{r};\mathbf{t})}$, and by Corollary B.20, we know that $F \subseteq F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')}$.

Since F itself is $(n-1)$-dimensional, it must now hold that both $\langle F \rangle = \langle F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \rangle$ and $\langle F \rangle = \langle F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')} \rangle$, that is, Part (1) of Definition B.22 holds.

Next, $\mathbf{r} \in C_{(\mathbf{r};\mathbf{t})}$ so $\mathbf{r} \in F \subseteq F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')}$. It follows by Lemma 4.9 that $r_n \leq r'_n$. And if $r_n = r'_n$ then, for all $i \in [n-2]$ we see by Lemma 4.9 again that $0 = t'_n(r_n - r'_n) \leq t'_i(r_i - r'_i)$, and so it follows that $\mathbf{r}' \leq \mathbf{r}$. So Parts (2) and (3) of Definition B.22 hold.

Finally, suppose that $\mathbf{r}' = \mathbf{r}$, and pick $\epsilon > 0$ sufficiently small that $\mathbf{r} - \epsilon \operatorname{inv}(\mathbf{r};\mathbf{t}) \in C_{(\mathbf{r};\mathbf{t})} \subsetneq F$. Applying Lemma 4.9 to $\mathbf{r} - \epsilon \operatorname{inv}(\mathbf{r};\mathbf{t})$, if $r_i \neq \underline{H}$ then $t'_n(r_n - \frac{\epsilon}{t_n} - r'_n) \leq t'_i(r_i - \frac{\epsilon}{t_i} - r'_n)$, which after re-arrangement shows that $\frac{t_i}{t_n} \geq \frac{t'_i}{t'_n}$. So Part (4) of Definition B.22 holds, and we have established that $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$. □

Proof of Proposition B.17. Since $r_{n-1} > \underline{H}$ and $r_n > \underline{H}$, the description of Lemma 4.9 implies that $\dim(F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}) = n-1$. We know $F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H} \subsetneq \mathrm{grid}(\mathcal{L})$ (by Definition B.15 and Corollary B.19), so $F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}$ is contained in the union of a finite set of facets of grid($\mathcal{L}$); by Corollary B.20 it is in fact equal to this finite set of facets of grid($\mathcal{L}$). Meanwhile $C_{(\mathbf{r};\mathbf{t})}$ is by definition a 1-cell of grid($\mathcal{L}$), and $C_{(\mathbf{r};\mathbf{t})} \subsetneq F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}$ by inspection of Lemma 4.9. Thus there exists a facet F of grid such that $C_{(\mathbf{r};\mathbf{t})} \subsetneq F \subseteq F^{n-1,n}_{(\mathbf{r};\mathbf{t})}$. Uniqueness of F follows because facets containing $C_{(\mathbf{r};\mathbf{t})}$ with normals as described in Lemma B.18 have affine spans meeting the boundary of $F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}$, and so cannot bisect F, and it is straightforward to check that no other facets of grid($\mathcal{L}$) have normals consistent with them containing $C_{(\mathbf{r};\mathbf{t})}$. So $F^{n-1,n}_{(\mathbf{r};\mathbf{t})}$ is not bisected as a facet of grid($\mathcal{L}$) along $C_{(\mathbf{r};\mathbf{t})}$.

To show the converse, suppose $F \in \mathcal{F}_{n-1,n}$ and let $r_n = \max\{p_n \in \mathbb{R} \mid \mathbf{p} \in F\}$ and let $\mathbf{r} = \inf_{\leq}\{\mathbf{p} \in F \mid p_n = r_n\}$, where we take the infimum with respect to the Euclidean order $\leq$. Recall, as we discussed after Definition B.14, that grid($\mathcal{L}$) can be endowed with the structure of a (true) LIP, and so we may apply Lemma A.5 to it. Therefore, because $\{\mathbf{p} \in F \mid p_n = r_n\}$ is a face of F, it is a cell of grid($\mathcal{L}$) and so its infimum is well-defined and is a 0-cell of F, and thus of grid($\mathcal{L}$). Observe here that $r_n > \underline{H}$ and $r_{n-1} > \underline{H}$ since F is contained in $\mathbf{H}$ and is $(n-1)$-dimensional.

Since F has a 0-cell at $\mathbf{r}$, and since $\{\mathbf{p} \in F \mid p_n = r_n\}$ is bounded below at $\mathbf{r}$, there must be facets F' of grid($\mathcal{L}$), whose intersection with F is a $(n-2)$-cell of F containing $\mathbf{r}$, whose normal vectors, together with the normal vector of F, span $\mathbb{R}^n$; and such that F is on the side of $F' \cap F$ on which p_i takes weakly higher values. Given the limited normals of facets in grid($\mathcal{L}$), and the fact that the 0-cell at $\mathbf{r}$ is bounded from below in coordinate i for $i \in [n-2]$, it follows that there exists such an F' which is either an i-hod or an (i,n)-fin of grid($\mathcal{L}$); but that F' can only be i-hod of grid($\mathcal{L}$) if $r_i = \underline{H}$ (as otherwise $r_i = \overline{H}$, which contradicts $F \subseteq \mathbf{H}$ being $(n-1)$-dimensional and lying on the side of F' on which p_i takes weakly higher values). So either $r_i = \underline{H}$ or there is an (i,n)-fin $F^{i,n}$ of grid($\mathcal{L}$) containing $\mathbf{r}$ and meeting F in an $(n-2)$-cell. For each i with $r_i \neq \underline{H}$, let the normal to $F^{i,n}$, expressed as a primitive integer vector, be $\hat{s}_i\mathbf{e}^i - \hat{s}^i_n\mathbf{e}^n$. Moreover, let $t_n = \mathrm{lcm}\{\hat{s}^i_n \mid i \in [n],\, r_i \neq \underline{H}\}$ and let $t_i = \hat{s}_i \frac{t_n}{\hat{s}^i_n}$ for $r_i \neq \underline{H}$, and $t_i = 1$ for $r_i = \underline{H}$. Now, for $r_i \neq \underline{H}$ we also have $t_i\mathbf{e}^i - t_n\mathbf{e}^n$ normal to $F^{i,n}$, with t_n consistent across these facets. So the 1-cell given by the intersection of the facets $F^{i,n}$ for $r_i \neq \underline{H}$ and $\underline{\mathbf{H}}_i$ for $r_i = \underline{H}$ is in direction $\mathrm{inv}(\mathbf{r};\mathbf{t})$. That is, $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$. But by construction now $C_{(\mathbf{r};\mathbf{t})} \subsetneq F \subseteq F^{n-1,n}_{(\mathbf{r};\mathbf{t})} \cap \mathbf{H}$, so F and $(\mathbf{r};\mathbf{t})$ are associated.

Finally, if also $(\mathbf{r}';\mathbf{t}')$ is associated with F then $C_{(\mathbf{r}';\mathbf{t}')} \subsetneq F \subseteq F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')} \cap \mathbf{H}$. Lemma B.25 now shows that both $(\mathbf{r};\mathbf{t}) \leq_{\mathcal{C}} (\mathbf{r}';\mathbf{t}')$, and that $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$. But, by Lemma B.23, it therefore follows that $(\mathbf{r}';\mathbf{t}') = (\mathbf{r};\mathbf{t})$. ☐

In order to prove Proposition 5.6, we will construct $\mathcal{B}'$ from bids $(\mathbf{r};\mathbf{t};m)$ where $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$. We do this by incrementally "deleting" all the $(n-1,n)$-fins of $\mathcal{L}$. These are not necessarily the same as the fins in $\mathcal{F}_{n-1,n}$. However, there is a close relationship between $\mathcal{F}_{n-1,n}$ and the $(n-1,n)$-fins of $\mathcal{L}$, and of the pseudo-LIP we obtain by subtracting any collection of bids associated with fins in $\mathcal{F}_{n-1,n}$, as follows:

Corollary B.26. *Suppose that $\mathcal{B}$ is a (possibly empty) set of bids $\mathbf{b} = (\mathbf{r};\mathbf{t};m)$ where $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$. Write $(\mathcal{L}', w') = (\mathcal{L}, w) \boxminus (\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ Then:*

(1) If F' is a $(n-1,n)$-fin of $\mathcal{L}'$ then $F' \subseteq \mathrm{grid}(\mathcal{L})$.
(2) If $F \in \mathcal{F}_{n-1,n}$ and $F \cap \mathcal{L}'$ is $(n-1)$-dimensional then $F \subseteq \mathcal{L}'$. Moreover, if $F \cap F'$, $F \cap F''$ is $(n-1)$-dimensional for $(n-1,n)$-fins F', F'' of $\mathcal{L}'$ then $w'(F') = w'(F'')$

Proof. We show a more general form of Part (1). Suppose F' is an (i,n)-fin or an n-hod of $(\mathcal{L}', w') = (\mathcal{L}, w) \boxminus (\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ for some $i \in [n-1]$. We show that $F' \subseteq \mathrm{grid}(\mathcal{L})$ if $r_i \neq \underline{H}$. First suppose that $F' \cap \mathcal{L}$ is $(n-1)$-dimensional. Then $F' \subseteq \mathrm{grid}(\mathcal{L})$ by construction of grid($\mathcal{L}$). Next, if $F' \cap \mathcal{L}$ is not $(n-1)$-dimensional, then $F' \subseteq \mathcal{L}_{\mathcal{B}}$. But then $F' \cap F^{i,n}_{\mathbf{b}}$ or $F' \cap F^{n}_{\mathbf{b}}$ is $(n-1)$-dimensional for some $\mathbf{b} \in \mathcal{B}$. But $\mathbf{b} \in \mathcal{B}$ implies that $F^{i,n}_{\mathbf{b}}, F^{n}_{\mathbf{b}} \subseteq \mathrm{grid}(\mathcal{L})$ if $r_i \neq \underline{H}$ by Corollary B.19; by construction of grid($\mathcal{L}$) this implies that again $F' \subseteq \mathrm{grid}(\mathcal{L})$. In particular, if $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$ then $r_{n-1} \neq \underline{H}$ by definition of $\mathcal{C}$, and so $F' \subseteq \mathrm{grid}(\mathcal{L})$ if F' is an $(n-1,n)$-fin of $\mathcal{L}$.

For Part (2), suppose $F \in \mathcal{F}_{n-1,n}$ and $F \cap \mathcal{L}'$ is $(n-1)$-dimensional. Let F' be a facet of $\mathcal{L}'$ such that $F \cap F'$ is $(n-1)$-dimensional, and suppose that $F \not\subseteq F'$. It follows that there is an $(n-2)$-cell C in the boundary of F', such that $C \cap F \neq \emptyset$ but C is not in the boundary of F. Such C is an intersection of facets of $\mathcal{L}'$. But if F'' is another facet of $\mathcal{L}'$ containing C and F'' is an (i,n)-fin or an n-hod of $\mathcal{L}'$ then $F'' \subseteq \operatorname{grid}(\mathcal{L})$ by the proof of Part (1) given above. But then C contains an intersection of facets of $\operatorname{grid}(\mathcal{L})$, which is a contradiction. So any facet F'' of $\mathcal{L}'$ containing C must not be an (i,n)-fin or an n-hod of $\mathcal{L}'$. Now Corollary A.3 implies that $\mathcal{L}'$ has a facet F''' with the same weight and affine span as F', on the other side of C. As $\mathcal{L}'$ has only finitely many facets, we may continue in this way until we have a collection of facets of $\mathcal{L}'$ whose union contains F, all having the same weight. $\square$

Part (2) of Corollary B.26 allows us to abuse notation as follows: if $F \in \mathcal{F}_{n-1,n}$ and $F \subseteq \mathcal{L}'$ then write $w'(F)$ to denote the weight of any facet F' of $\mathcal{L}'$ such that $F \cap F'$ is $(n-1)$-dimensional.

We will incrementally define sets of bids to match more and more of the $(n-1,n)$-fins of $\mathcal{L}$. The following lemma ensures that we can define such bids with integer multiplicities.

Lemma B.27. *Suppose that $\mathcal{B}$ is a (possibly empty) set of bids $\mathbf{b} = (\mathbf{r};\mathbf{t};m)$ where $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$. Write $(\mathcal{L}',w') = (\mathcal{L},w) \boxminus (\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$. Suppose that $(\mathbf{r};\mathbf{t})$ is minimal with respect to $\leq_{\mathcal{C}}$ such that their associated facet F satisfies $F \subseteq \mathcal{L}'$. Then $\gcd(t_{n-1},t_n) \mid w'(F)$.*

Proof. By Corollary B.21, for every $i \in [n-2]$ such that $r_i \neq \underline{H}$, we have F meeting an $(i,n;t_i/t_n)$-fin of $\operatorname{grid}(\mathcal{L})$ in an $(n-2)$-cell C^i containing $\mathbf{r}$, which by Lemma A.2 must be an Type 2 $(n-2)$-cell with indices $(i,n-1,n)$ and trade-off (t_i,t_{n-1},t_n). By assumed minimality of $(\mathbf{r};\mathbf{t})$ such that the associated facet is contained in $\mathcal{L}'$, and by Corollary B.26, it follows that there is only one $(n,n-1)$-fin F' of $\mathcal{L}'$ containing C^i, and that $w'(F') = w'(F)$. So we can apply Lemma A.4. However, (t_i,t_{n-1},t_n) is not necessarily a primitive integer. Write $(t_i,t_{n-1},t_n) = K_i(t'_i,t'_{n-1},t'_n)$ where $K_i \in \mathbb{Z}$ and (t'_i,t'_{n-1},t'_n) is indeed a primitive integer vector. Now, by Lemma A.4, we know that $\gcd(t'_{n-1},t'_n) \mid w'(F)$.

Consider any prime number q such that $q^\ell \mid \gcd(t_{n-1},t_n)$ for some maximal exponent ℓ. Since $\sum_{j\in[n];r_j\neq\underline{H}} t_j\mathbf{e}^j$ is a primitive integer vector, there must exist $i \in [n-2]$ with $r_i \neq \underline{H}$ such that $q \nmid t_i$. Since q is prime, it follows that $\gcd(q,t_i) = 1$ and so, since $K_i t'_i = t_i$, that $\gcd(q,K_i) = 1$. Since $K_i t'_{n-1} = t_{n-1}$ and $K_i t'_n = t_n$, we can conclude that $q^\ell \mid \gcd(t'_{n-1},t'_n)$. Thus, by the conclusion above, we know that $q^\ell \mid w'(F)$. As this follows for all primes in the prime factorisation of $\gcd(t_{n-1},t_n)$ we can conclude that $\gcd(t_{n-1},t_n) \mid w'(F)$. $\square$

We will now see that we can generate $\mathcal{B}'$ by defining (integer) bid sets as follows:

Definition B.28. Write $\mathcal{L}^0 = \mathcal{L}$. For $s \geq 0$ inductively define:

(1) $\mathcal{B}^s$ for bids $\mathbf{b} = (\mathbf{r};\mathbf{t};m)$, where $(\mathbf{r};\mathbf{t})$ is minimal with respect to $\leq_{\mathcal{C}}$ among the subset of those $(\mathbf{r};\mathbf{t}) \in \mathcal{C}$ such that their associated facet F satisfies $F \subseteq \mathcal{L}^s$, and where $m = \frac{w^s(F)}{\gcd(t_{n-1},t_n)}$;

(2) $(\mathcal{L}^{s+1}, w^{s+1}) := (\mathcal{L}^s, w^s) \boxminus (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s})$.

Corollary B.29. *The bid $\mathcal{B}^s$ in Definition B.28 are well defined.*

Proof. It is sufficient to check the definition of the multiplicity $m(\mathbf{b})$ for a bid $\mathbf{b} \in \mathcal{B}^s$, as the remainder of the definition is without ambiguity. We proceed by induction; both the base case and inductive step are proved in the same way. For the base case, observe that $(\mathcal{L}^0,w^0) = (\mathcal{L},w) \boxminus (\mathcal{L}_\emptyset, w_\emptyset)$. Our inductive hypothesis is that bids $\mathcal{B}^s$ are well defined for $s \geq 0$; we can then express $(\mathcal{L}^{s+1},w^{s+1}) := (\mathcal{L},w) \boxminus (\mathcal{L}_{\widehat{\mathcal{B}}^s}, w_{\widehat{\mathcal{B}}^s})$, where $\widehat{\mathcal{B}}^s := \bigcup_{s'=0}^{s} \mathcal{B}^{s'}$. Thus Corollary B.26 and Lemma B.27 can be applied to $\mathcal{L}^s$ for all $s \geq 0$. In particular $w^s(F)$ is well-defined, by Corollary B.26, and $\gcd(t_{n-1},t_n) \mid w'(F)$ so that $m(\mathbf{b}) \in \mathbb{Z}_{>0}$ for all $\mathbf{b} \in \mathcal{B}^s$, by Lemma B.27. This completes both the base and inductive steps. $\square$

Lemma B.30. *For $s \geq 1$, if $\mathbf{b} = (\mathbf{r};\mathbf{t};m) \in \mathcal{B}^{s+1}$ then there exists $\mathbf{b}' = (\mathbf{r}';\mathbf{t}';m') \in \mathcal{B}^s$ such that $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$ and $(\mathbf{r}';\mathbf{t}') \neq (\mathbf{r};\mathbf{t})$.*

Proof. By Definition B.28 we know that the $(n-1,n)$-fin F of $\operatorname{grid}(\mathcal{L})$ associated with $(\mathbf{r};\mathbf{t})$ satisfies $F \subseteq \mathcal{L}^{s+1} = (\mathcal{L}^s,w^s) \boxminus (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s})$.

Suppose first that $F \subseteq \mathcal{L}_{\mathcal{B}^s}$. Then $\dim(F \cap F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')}) = n-1$ for some $(\mathbf{r}';\mathbf{t}') \in \mathcal{C}$, whence by Lemma B.25 we know $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$.

The alternative case is that $F \not\subseteq \mathcal{L}_{\mathcal{B}^s}$. Then $F \cap \mathcal{L}^s$ is $(n-1)$-dimensional, whence $F \subseteq \mathcal{L}^s$ by Corollary B.26. So, by Definition B.28, there exists $(\mathbf{r}';\mathbf{t}';m') \in \mathcal{B}^s$ such that $(\mathbf{r}';\mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r};\mathbf{t})$.

Finally we show that if $\mathbf{b}' \in \mathcal{B}^s$ has associated facet F' then $F' \not\subseteq \mathcal{L}^{s+1}$. We know by Definition B.16 that $F' \subseteq F^{n-1,n}_{(\mathbf{r}';\mathbf{t}')} \subsetneq \mathcal{L}_{\mathbf{b}'}$, and by Definition B.28 that $F' \subseteq \mathcal{L}^s$. Moreover we know by Lemma

4.9 that $w_{\mathbf{b}'}(F_{\mathbf{b}'}^{n-1,n}) = \frac{w^s(F')}{\gcd(t_{n-1},t_n)}\gcd(t_{n-1},t_n) = w^s(F')$. So F' has at most $(n-2)$-dimensional intersection with $(\mathcal{L}^s, w^s) \boxminus (\mathcal{L}_{\mathbf{b}'}, w_{\mathbf{b}'})$.

Now suppose that $\dim\left(F' \cap F_{\mathbf{b}''}^{n-1,n}\right) = n-1$ for some other bid $\mathbf{b}'' \in \mathcal{B}^s$. By Lemma B.25 it follows that $(\mathbf{r}''; \mathbf{t}'') \leq_{\mathcal{C}} (\mathbf{r}'; \mathbf{t}')$. By minimality of $(\mathbf{r}'; \mathbf{t}')$ it follows that $(\mathbf{r}''; \mathbf{t}'') = (\mathbf{r}'; \mathbf{t}')$ and so that $\mathbf{b}'' = \mathbf{b}'$.

We can therefore conclude that F' has at most $(n-2)$-dimensional intersection with $(\mathcal{L}^s, w^s) \boxminus (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s})$, and in particular that $F' \not\subseteq \mathcal{L}^{s+1}$, as claimed. It follows by Definition B.28 that $(\mathbf{r}'; \mathbf{t}') \notin \mathcal{B}^{s+1}$, which proves that $(\mathbf{r}'; \mathbf{t}') \neq (\mathbf{r}; \mathbf{t})$. □

Corollary B.31. *There exists $S \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{B}^S = \emptyset$, and such that $\mathcal{L}^S$ has no $(n-1,n)$-fins meeting $\mathbf{H}^\circ$.*

Proof. If $\mathcal{B}^{s'} \neq \emptyset$ for $s' = 1, \ldots, s$ then, by Lemma B.30, there exists a chain of bids $\mathbf{b}_{s'} = (\mathbf{r}_{s'}; \mathbf{t}_{s'}, m_{s'}) \in \mathcal{B}^{s'}$ such that $(\mathbf{r}_1; \mathbf{t}_1) \lneqq_{\mathcal{C}} \cdots \lneqq_{\mathcal{C}} (\mathbf{r}_t; \mathbf{t}_t)$. Since $\leq_{\mathcal{C}}$ is a partial order (Lemma B.23), it follows that these bids are all distinct. But $\mathcal{C}$ is a finite set, so we conclude that $\mathcal{B}^S = \emptyset$ some $S \in \mathbb{Z}_{\geq 0}$.

Now suppose for a contradiction that $\mathcal{L}^S$ does have an $(n-1,n)$-fin F' meeting $\mathbf{H}^\circ$. Then, by Corollary B.26 Part (1) we know $F' \subseteq \operatorname{grid}(\mathcal{L})$. Hence there exists $F \in \mathcal{F}_{n-1,n}$ with $\dim(F \cap \mathcal{L}^S) = n-1$, which implies $F \subseteq \mathcal{L}^S$ by Corollary B.26 Part (2). But then F is, by Proposition B.17, the associated facet of some $(\mathbf{r}; \mathbf{t}) \in \mathcal{C}$, so that either $(\mathbf{r}; \mathbf{t}; m) \in \mathcal{B}^S$ for some $m \in \mathbb{Z}_{>0}$, or there exists $(\mathbf{r}'; \mathbf{t}'; m') \in \mathcal{B}^S$ with $(\mathbf{r}'; \mathbf{t}') \leq_{\mathcal{C}} (\mathbf{r}; \mathbf{t})$. As $\mathcal{B}^S = \emptyset$, this provides the required contradiction, and so completes the proof. □

Proof of Proposition 5.6. Let $\mathcal{B} := \bigcup_{s'=0}^{S} \mathcal{B}^{s'}$, with S as in Corollary B.31. These bids are rooted in $\mathbf{H}$ as $\mathbf{r} \in \mathbf{H}$ for any $(\mathbf{r}; \mathbf{t}) \in \mathcal{C}$ (Definition B.15). Moreover $(\mathcal{L}, w) \boxminus (\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}}) = (\mathcal{L}^S, w^S)$ and we know that $\mathcal{L}^S$ has no $(n-1,n)$-fins meeting $\mathbf{H}^\circ$ by Corollary B.31. □

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