Incorporation of a Leverage Effect in a Stochastic Volatility Model

Ole E. Barndorff-Nielsen
The Centre for Mathematical Physics and Stochastics (MaPhySto), University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark. *
oebn@imf.au.dk

AND Neil Shephard
Nuffield College, Oxford OX1 1NF, UK.
neil.shephard@nuf.ox.ac.uk

Abstract

In this note we show how the stochastic volatility model of Barndorff-Nielsen and Shephard (1998a) can be generalised to allow for the leverage effect. That is where a negative return sequence is associated with increases in volatility. This is important in empirical work on stock returns. This form of model allows a great deal of analytic tractability — inheriting from our original model formulation many attractive features.

*MaPhySto is funded by the Danish National Research Council

Contents

1 Introduction

A class of stochastic volatility models, discussed in Barndorff-Nielsen and Shephard (1998a), incorporates a number of the well-established common features of observational series of financial assets, in particular series of stock prices and of exchange rates. One fairly important such stylized feature, the so called leverage effect, was however not covered. This is where negative return sequences are associated with increases in the volatility of stock returns. Such asymmetries are not usually observed for exchange rates.

The leverage effect was studied in some early work by Black (1976), while it motivated the introduction of the EGARCH model of Nelson (1991) and the threshold ARCH model of Glosten, Jagannathan, and Runkle (1993). An economic theory behind such effects is discussed by Campbell and Kyle (1993).

In the present note we indicate a way of extending the type of models referred to so that they reflect the leverage effect, and we calculate a few of the consequences. Only the simplest, one-dimensional, version of the models will be considered here. A fuller treatment will be given elsewhere.

2 Incorporating leverage

2.1 Model construction

As a model for the log price process of, for instance, a stock we consider a stochastic process $x^*(t)$, $0 \leq t < \infty$, defined by a stochastic differential equation

$$\mathrm{d}x^*(t) = \left\{\mu + \beta\sigma^2(t)\right\}\mathrm{d}t + \sigma(t)\mathrm{d}w(t) + \rho\mathrm{d}\bar{z}(\lambda t) \tag{1}$$

where w is the Wiener process, $\bar{z}$ is the centered version of a Lévy process $\grave{z}$ (that is $\bar{z}(t) = \grave{z}(t) - \mathrm{E}\grave{z}(t)$), and σ is a stationary process defined, also in terms of $\grave{z}$, by the stochastic differential equation

$$\mathrm{d}\sigma^2(t) = -\lambda\sigma^2(t)\mathrm{d}t + \mathrm{d}\grave{z}(\lambda t) \tag{2}$$

with $0 < \lambda < \infty$. This volatility process is thus of Ornstein-Uhlenbeck (OU) type although it will not have Gaussian increments. The process $\grave{z}$ is a homogeneous Lévy process (so it has independent and stationary increments) with positive increments (also termed a subordinator). As it is the driving process for the OU process we call it a background driving Lévy process (BDLP). The filtration determined by w and $\grave{z}$ jointly will be denoted by $\mathcal{F} = \{\mathcal{F}_t : 0 \leq t < \infty\}$. Throughout we will assume $\mu = \beta = 0$ for simplicity of exposition as these terms raise no new issues.

Although we have focused on the simplest OU volatility process, our model extends to where volatility follows a weighted sum of independent Ornstein-Uhlenbeck processes with different persistence rates. That is

$$\sigma^2(t) = \sum_{j=1}^{m} w_j^* \sigma_j^2(t), \qquad \text{where} \qquad \sum_{j=1}^{m} w_j^* = 1,$$

with

$$\mathrm{d}\sigma_j^2(t) = -\lambda_j \sigma_j^2(t)\mathrm{d}t + \mathrm{d}\grave{z}_j(\lambda_j t),$$

where the $\{\grave{z}_j(t)\}$ are independent (not necessarily identically distributed) BDLPs. Hence some of the components of the volatility may represent short term variation in the process while others represent long term movements. In such a case we would have a process for the price of the type

$$\mathrm{d}x^*(t) = \left\{\mu + \beta\sigma^2(t)\right\}\mathrm{d}t + \sigma(t)\mathrm{d}w(t) + \sum_{j=1}^{m} \rho_j \mathrm{d}\bar{z}_j(\lambda_j t),$$

where the leverage effect could be different for the various components of volatility. As this type of extension will raise no new technical issues we will not deal with it in this note and instead focus on the case where $m = 1$ and so the model is made up of (1) and (2).

For $\rho = 0$ this setup reduces to the elemental version of the models studied in the earlier paper (Barndorff-Nielsen and Shephard (1998a)). If ρ is negative a positive (infinitesimal) increment $\mathrm{d}\grave{z}(\lambda t)$ in the volatility process $\sigma^2(t)$ will have a negative effect on the stock price. This expresses, at least qualitatively, the stylized leverage effect — that negative returns are associated with increases in observed volatility. Notice however that there is no feedback from the x^* process to the volatility process σ^2 in our model - the innovations from $\grave{z}$ affects x^* and σ^2 simultaneously. Hence this model differs from the EGARCH and threshold ARCH models of leverage referenced above. However, it is in keeping with the non-symmetrical stochastic volatility models of leverage previously discussed in the literature (see, for example, the review in Ghysels, Harvey, and Renault (1996)). A typical example of that style of models is where

$$\mathrm{d}\log\sigma^2(t) = -\lambda\left\{\log\sigma^2(t) - \mu\right\}\mathrm{d}t + \varkappa\mathrm{d}s(t),$$

a geometric Gaussian Ornstein-Uhlenbeck process, whose increments of the standard Brownian motion $s(t)$ are correlated with those of $w(t)$. Our construction is mathematically more tractable.

2.2 Volatility and the BDLP

Denoting the cumulants and moments of $\grave{z}(1)$ (when they exist) respectively by $\grave{\kappa}_m$ and $\grave{\mu}_m$ $(m = 1, 2, ...)$ we have

$$\bar{z}(t) = \grave{z}(t) - t\grave{\kappa}_1 = \grave{z}(t) - t\xi \tag{3}$$

where for brevity we have written $\xi = \grave{\kappa}_1$. For some purposes it maybe helpful to note that $\bar{z}(t)$ is bounded from below and that the (marginal) cumulants of $\sigma^2(t)$ follow directly from the cumulants of $\grave{z}(1)$. In particular, if we write the cumulants of σ^2 as $\acute{\kappa}_m$ $(m = 1, 2, ...)$ then it follows (see the proof in Barndorff-Nielsen (1998)) that

$$\grave{\kappa}_m = m\acute{\kappa}_m, \qquad m = 1, 2, ...$$

It will be helpful later to establish the notation that the corresponding cumulant generating functions will be written as $\acute{k}(\theta) = \log\left[\mathrm{E}\exp\left\{\theta\sigma^2(t)\right\}\right]$ and $\grave{k}(\theta) = \log\left\{\mathrm{E}\exp\left(\grave{z}(1)\right)\right\}$ for $\sigma^2(t)$ and $\grave{z}(1)$ respectively. Indeed they are related by the fundamental equality (Barndorff-Nielsen (1998))

$$\acute{k}(\theta) = \int_0^\infty \grave{k}(\theta e^{-s})\mathrm{d}s, \tag{4}$$

which can be reexpressed as

$$\grave{k}(\theta) = \theta\acute{k}'(\theta) \tag{5}$$

(where $\acute{k}'(\theta) = \theta\mathrm{d}\acute{k}(\theta)/\mathrm{d}\theta$). The common feature of this notation is the BDLP objects, $\grave{z}(1)$, have graves over them while the volatility itself , $\sigma^2(t)$, have acutes. This style of notation will be maintained throughout this note. Important special cases of this are

$$\begin{aligned}
\grave{\kappa}_1 &= \mathrm{E}\{\sigma^2(t)\} = \acute{\kappa}_1 = \xi \quad \text{and} \\
\grave{\kappa}_2 &= 2\mathrm{Var}\{\sigma^2(t)\} = 2\acute{\kappa}_2 = 2\omega^2.
\end{aligned}$$

Now, let

$$x_0^*(t) = \int_0^t \sigma(s)\mathrm{d}w(s), \tag{6}$$

which is the log-price process minus the leverage effect. The solution of the equation (1) is then

$$x^*(t) = x_0^*(t) + \rho\bar{z}(\lambda t) \tag{7}$$

2.3 Moments of returns

As in the previous paper, we shall determine some of the properties of the increments over time spans of length Δ of the model process x^*. Thus, let

$$\begin{aligned}
y_n &= x^*(\Delta n) - x^*\left\{\Delta(n-1)\right\} \\
&= y_{0n} + \rho\bar{z}_n
\end{aligned}$$

where

$$\bar{z}_n = \grave{z}(\lambda\Delta n) - \grave{z}\left\{\lambda\Delta(n-1)\right\} - \lambda\Delta\xi$$

and

$$y_{0n} = x_0^*(\Delta n) - x_0^*\left\{\Delta(n-1)\right\}. \tag{8}$$

The implication is that

$$y_n|\sigma_n^2, \bar{z}_n \sim N(\rho\bar{z}_n, \sigma_n^2), \tag{9}$$

where integrated volatility influences the distribution of returns through

$$\sigma_n^2 = \sigma^{2*}(\Delta n) - \sigma^{2*}\left\{\Delta(n-1)\right\} \tag{10}$$

and

$$\sigma^{2*}(t) = \int_0^t \sigma^2(u)\mathrm{d}u. \tag{11}$$

We will write

$$\grave{z}_n = \grave{z}(\lambda\Delta n) - \grave{z}\left\{\lambda\Delta(n-1)\right\}, \tag{12}$$

and

$$\acute{z}_n = \sigma^2\left(\Delta n\right) - \sigma^2\left\{\Delta\left(n-1\right)\right\}.$$

Then an important implication (Barndorff-Nielsen and Shephard (1998a)) of these constructions is that

$$\sigma_n^2 = \lambda^{-1}\left(\grave{z}_n - \acute{z}_n\right). \tag{13}$$

It is this linear structure which will allow us to perform a number of analytic calculations which are not possible for other models.

The formula for the returns (9) is informative for it shows that the effect of the leverage is to shift the distribution. If the volatility innovations are unexpected large and ρ is negative, then the mean return will be negative. Hence negative returns are associated with increases in volatility. Likewise small innovations in the volatility process will happen at the same time as positive returns.

We now define

$$\bar{z}_n = \grave{z}_n - \lambda\Delta\xi \tag{14}$$

$$\bar{\sigma}_n^2 = \sigma_n^2 - \mathrm{E}\left(\sigma_n^2\right) = \sigma_n^2 - \Delta\xi. \tag{15}$$

Further, let

$$\bar{\sigma}^2(t) = \sigma^2(t) - \xi.$$

After these preparations we have that

$$\mathrm{E}\{y_n^2\} = \lambda\Delta(\xi + \rho^2\grave{\kappa}_2) \tag{16}$$

and

$$\mathrm{Var}\{y_n^2\} = \mathrm{Var}\{y_{0n}^2\} + 2\rho^2\{2\lambda^2\Delta^2\xi\grave{\kappa}_2 + 3(e^{-\lambda\Delta} - 1 + \lambda\Delta)\grave{\mu}_3\} + \rho^4\lambda\Delta(\grave{\kappa}_4 + 2\lambda\Delta\grave{\kappa}_2^2), \tag{17}$$

where the expression for

$$\begin{aligned}\operatorname{Var}\{y_{0n}^2\} &= 6\operatorname{Var}\left\{\sigma^2(t)\right\}\lambda^{-2}\left(e^{-\lambda\Delta}-1+\lambda\Delta\right)+2\Delta^2\xi^2\\ &= 3\grave{\kappa}_2\lambda^{-2}\left(e^{-\lambda\Delta}-1+\lambda\Delta\right)+2\Delta^2\xi^2\end{aligned}$$

was derived in Barndorff-Nielsen and Shephard (1998a).

Likewise, for $s=1,2,...$ we derive that

$$\mathrm{E}\{y_n y_{n+s}\}=0, \tag{18}$$

$$\operatorname{Cov}\left(y_n,y_{n+s}^2\right)=\mathrm{E}\{y_n y_{n+s}^2\}=\rho\grave{\kappa}_2(1-e^{-\lambda\Delta})^2\exp\left\{-\lambda\Delta(s-1)\right\} \tag{19}$$

$$\operatorname{Cov}(y_n^2,y_{n+s}^2)=\operatorname{Cov}(y_{0n}^2,y_{0n+s}^2)+\rho^2(1-e^{-\lambda\Delta})^2\exp\left\{-\lambda\Delta(s-1)\right\}\grave{\mu}_3 \tag{20}$$

where the expression

$$\begin{aligned}\operatorname{Cov}(y_{0n}^2,y_{0n+s}^2) &= \operatorname{Var}\left\{\sigma^2(t)\right\}\lambda^{-2}(1-e^{-\lambda\Delta})^2\exp\left\{-\lambda\Delta(s-1)\right\}\\ &= \frac{\grave{\kappa}_2}{2\lambda^2}(1-e^{-\lambda\Delta})^2\exp\left\{-\lambda\Delta(s-1)\right\}\end{aligned}$$

is given in Barndorff-Nielsen and Shephard (1998a). We note then that there is some simplification as

$$\operatorname{Cov}(y_n^2,y_{n+s}^2)=\left(\frac{\grave{\kappa}_2}{2\lambda^2}+\rho^2\grave{\mu}_3\right)(1-e^{-\lambda\Delta})^2\exp\left\{-\lambda\Delta(s-1)\right\}.$$

Only the formulae (17), (19) and (20) require some steps of calculation, which we place in the appendix.

The effect of the leverage on the dynamic properties of discrete time returns is made quite clear by these formulae. First both $\mathrm{E}\left(y_n y_{n+s}^2\right)$ and $\operatorname{Cov}(y_n^2,y_{n+s}^2)$ damp down exponentially with the lag length. In other words, the leverage effect diminishes exponentially with s. The effect of the leverage on the covariance between the squares is to increase (decrease) volatility clustering if the BDLP is positively (negatively) skewed. In practice we would expect the BDLP to be highly positively skewed.

The simplicity of the effect of the leverage term means that we can still compute analytically the spectrum of squared returns. We may write this as

$$\begin{aligned}f(\psi) &= \sum_{s=-\infty}^{\infty}\operatorname{cor}\{y_n^2 y_{n+s}^2\}\cos(s\psi)\\ &= 1-c\phi^{-1}+c\phi^{-1}a(\psi;\phi)\end{aligned}$$

where $\phi=\exp(-\lambda\Delta)$,

$$a(\psi;\phi)=\frac{1-\phi^2}{1-2\phi\cos\psi+\phi^2}$$

and

$$c=\frac{(1-e^{-\lambda\Delta})^2(\lambda^{-2}\omega^2+\rho^2\grave{\mu}_3)}{\operatorname{Var}\{y_n^2\}},$$

which generalises the previous result of Barndorff-Nielsen and Shephard (1998a).

3 Simulation

3.1 Simulating return sequences

A major advantage of the way we construct the leverage effect in our models is that we can simulate returns without any form of discretisation being introduced. This will work off the result that

$$y_n|\sigma_n^2, \bar{z}_n \sim N(\rho\bar{z}_n, \sigma_n^2),$$

which means we just simply have to draw sequences of $\{\bar{z}_n, \sigma_n^2\}$. In particular we can use the following setup.

Suppose f is a positive and integrable function on $[0, \lambda]$ then

$$\int_0^\lambda f(s)\mathrm{d}\grave{z}(s) \stackrel{L}{=} \sum_{i=1}^\infty \bar{Q}^{-1}(a_i^*/\lambda) f(\lambda r_i) \tag{21}$$

where $\{a_i^*\}$ and $\{r_i\}$ are two independent sequences of random variables with the r_i independent copies of a uniform random variable r on $[0, 1]$ and $a_1^* < ... < a_i^* < ...$ as the arrival times of a Poisson process with intensity 1. Further,

$$\bar{Q}^{-1}(x) = \inf\{y > 0 : \bar{Q}(y) \leq x\},$$

where the Lévy density of $\grave{z}(1)$ is written as w and we define

$$\overline{Q}(x) = \int_x^\infty w(y)\mathrm{d}y \tag{22}$$

This result follows from work of Marcus (1987) and Rosinski (1991). A thorough exposition with self-contained proofs is given in Barndorff-Nielsen and Shephard (1998b).

The point of this development is as follows. We first note that the integrated volatility over the period of the $n-th$ return is given in (13), so if we note that

$$\sigma^2(\Delta n) = \exp(-\lambda\Delta)\sigma^2\{\Delta(n-1)\} + w_n,$$

then we can simulate from a sequence $\{\bar{z}_n, \sigma_n^2\}$ by working with the innovations

$$\begin{pmatrix} w_n \\ \grave{z}_n \end{pmatrix} \stackrel{L}{=} \left\{ \begin{array}{l} \exp(-\lambda\Delta)\int_0^{\lambda\Delta} e^s \mathrm{d}\grave{z}(s) \\ \int_0^{\lambda\Delta} \mathrm{d}\grave{z}(s) \end{array} \right\}. \tag{23}$$

The simulation is carried out using the infinite series representations of the integrals given in (21).

3.2 Simulating forecast volatilities and leverage effects

3.2.1 Basics

The above argument easily generalises to the case where we wish to simulate from

$$\sigma^{2*}(t) - \sigma^{2*}(\Delta n), \grave{z}(t) - \grave{z}(\Delta n)|\sigma^2(\Delta n), \qquad t \geq \Delta n,$$

which determines the forecast distribution of log-returns as

$$\begin{aligned} & x^*(t) - x^*(\Delta n) \left| \sigma^{2*}(t) - \sigma^{2*}(\Delta n), \grave{z}(t) - \grave{z}(\Delta n) \right. \\ \sim\ & N\left[\left\{\mu(t-\Delta n) + \beta\left\{\sigma^{2*}(t) - \sigma^{2*}(\Delta n)\right\}\right\} + \rho\left\{\grave{z}(t) - \grave{z}(\Delta n)\right\}, \left\{\sigma^{2*}(t) - \sigma^{2*}(\Delta n)\right\}\right]. \end{aligned}$$

Hence the distribution of log-returns is a mixture of Gaussians, which will be typically skewed due to the leverage effect.

One of the implications of this type of argument is that we can estimate via simulation, for an arbitrary function g and for $t > \Delta n$,

$$E\left[g\left\{x^*(t)\right\} | x^*(\Delta n), \sigma^2(\Delta n)\right],$$

by simply drawing from the mixed Gaussian conditional distribution of log-prices. In the special case where it is possible to analytically evaluate

$$E\left[g\left\{x^*(t)\right\} | x^*(\Delta n), \sigma^{2*}(t) - \sigma^{2*}(\Delta n), \grave{z}(t) - \grave{z}(\Delta n)\right],$$

it would be possible to potentially dramatically improve the efficiency of the simulation based estimator by applying a Rao-Blackwellisation of the simulation estimator. This will take on the form of an estimator

$$\widehat{g} = \frac{1}{M}\sum_{j=1}^{M} g\left[\left\{x^*(t)\right\} | x^*(\Delta n), \sigma^{2*}(t)^j - \sigma^{2*}(\Delta n), \grave{z}(t)^j - \grave{z}(\Delta n)\right],$$

where the $\{\sigma^{2*}(t)^j, \grave{z}(t)^j\}$ are simulated as

$$\sigma^{2*}(t)^j, \grave{z}(t)^j \overset{iid}{\sim} \sigma^{2*}(t), \grave{z}(t) - \grave{z}(\Delta n) | \sigma^2(\Delta n).$$

Such arguments may lead to very fast ways of computing the fair price of some general European options under leverage effects — although such arguments will not be based on the usual mathematical finance arguments of unique equivalent martingale measures as the existence of stochastic volatility means the market is not complete. Previously option prices have been computed by simulating the whole path of price over the duration of the option (Hull and White (1987)) and so is enormously less efficient than the scheme given above.

3.2.2 Control variables

For some problems it may be helpful to employ control variables to improve the efficiency of this simulation estimator. Let us write

$$\nu_j = \sigma^{2*}(t)^j - \sigma^{2*}(\Delta n), \grave{z}(t)^j - \grave{z}(\Delta n).$$

We can compute the mean of $\nu_j|\sigma^2(\Delta n)$ which we will write as μ_1 and restate $\widehat{g}$ as

$$\widehat{g} = \frac{1}{M}\sum_{j=1}^{M} g\left[\{x^*(t)\}|x^*(\Delta n), v_j\right].$$

This can be potentially improved by constructing a control variable (see, for example, Ripley (1987, pp. 123-128)) off a term from the first order Taylor's expansion:

$$\widetilde{g} = \frac{1}{M}\sum_{j=1}^{M} (v_j - \mu_1)' \left.\frac{\partial g\left[\{x^*(t)\}|x^*(\Delta n), v\right]}{\partial v}\right|_{v=\mu_1}.$$

Then we will estimate the mean of g by the unbiased (over the simulations) estimator $\widehat{g} - \widetilde{g}$. This argument can be generalised to higher order expansions in a straightforward way. When quite a few terms in the expansion are used it may become pointless to use any form of simulation at all as the analytic approximation will become, in effect, exact.

4 Integrals with respect to the price

The distributional properties of x^* are embodied in the class of integrals of the form

$$f \bullet x^* = \int_0^\infty f(t)\mathrm{d}x^*(t) \tag{24}$$

where f is a deterministic real function. We therefore proceed to determine the cumulant function of such integrals (when they exist). In particular, by suitable choice of f (see below) one obtains the cumulant functions of the multivariate marginal distributions of x^*. Noting that x^* is a local martingale, we interpret $f \bullet x^*$ as a stochastic integral (as defined, for instance, in Protter (1992)).

Since

$$\mathrm{d}x^*(t) = \sigma(t)\mathrm{d}w(t) + \rho\mathrm{d}\overline{z}(\lambda t) = \sigma(t)\mathrm{d}w(t) + \rho\mathrm{d}\grave{z}(\lambda t) - \rho\xi\lambda\mathrm{d}t$$

we have

$$f \bullet x^* = (f\sigma) \bullet w + \rho f(\lambda^{-1}\cdot) \bullet \grave{z} - \rho\xi\lambda \int_0^\infty f(t)\mathrm{d}t. \tag{25}$$

Now,

$$\mathrm{E}\{\exp(i\zeta(f\sigma) \bullet w)|\grave{z}(\cdot)\} = \exp\left\{-\frac{1}{2}\zeta^2 \int_0^\infty f^2(t)\sigma^2(t)\mathrm{d}t\right\}$$

and hence

$$\begin{aligned}\mathrm{E}\{\exp(i\zeta f \bullet x^*)\} &= \mathrm{E}\left[\exp\left\{-\frac{1}{2}\zeta^2 \int_0^\infty f^2(t)\sigma^2(t)\mathrm{d}t + i\zeta\rho \int_0^\infty f(\lambda^{-1}t)\mathrm{d}\grave{z}(t)\right\}\right] \\ &\quad \cdot \exp\left\{-i\zeta\rho\xi\lambda \int_0^\infty f(t)\mathrm{d}t\right\}.\end{aligned}$$

Furthermore, using the representation

$$\sigma^2(t) = e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} \mathrm{d}\grave{z}(\lambda s)$$

we find

$$\int_0^\infty f^2(t)\sigma^2(t)\mathrm{d}t = I_0 + I_1$$

where

$$\begin{aligned} I_0 &= \int_0^\infty f^2(t) e^{-\lambda t} \int_{-\infty}^{0} e^{\lambda s} \mathrm{d}\grave{z}(\lambda s)\mathrm{d}t \\ &= \left\{ \int_0^\infty e^{-\lambda t} f^2(t)\mathrm{d}t \right\} \int_{-\infty}^{0} e^{s} \mathrm{d}\grave{z}(s) \end{aligned}$$

and

$$\begin{aligned} I_1 &= \int_0^\infty \int_s^\infty e^{-\lambda(t-s)} f^2(t)\mathrm{d}t\mathrm{d}\grave{z}(\lambda s) \\ &= \int_0^\infty \int_0^\infty e^{-\lambda u} f^2(s+u)\mathrm{d}u\mathrm{d}\grave{z}(\lambda s) \\ &= \int_0^\infty \int_0^\infty e^{-\lambda u} f^2(\lambda^{-1}s+u)\mathrm{d}u\mathrm{d}\grave{z}(s) \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{E}\{\exp(i\zeta f \bullet x^*)\} &= \mathrm{E}\left\langle \exp\left[-\frac{1}{2}\zeta^2 \left\{ \int_0^\infty e^{-\lambda t} f^2(t)\mathrm{d}t \right\} \int_{-\infty}^{0} e^{s}\mathrm{d}\grave{z}(s) \right] \right\rangle \\ &\cdot \mathrm{E}\left\langle \exp\left[\int_0^\infty \left\{ -\frac{1}{2}\zeta^2 \int_0^\infty e^{-\lambda u} f^2(\lambda^{-1}t+u)\mathrm{d}u + i\zeta\rho \int_0^\infty f(\lambda^{-1}t) \right\} \mathrm{d}\grave{z}(t) \right] \right\rangle \\ &\cdot \exp\left\{ -i\zeta\rho\xi\lambda \int_0^\infty f(t)\mathrm{d}t \right\} \end{aligned}$$

Consequently, writing $\mathrm{C}\{\zeta \ddagger x\} = \log \mathrm{E}\{e^{i\zeta x}\}$ for the cumulant function of any random variable x, and recalling $\grave{k}(\theta) = \log \mathrm{E}\{\exp \theta \grave{z}(1)\}$, and using the known result that for an arbitrary Lévy process z

$$\mathrm{C}\{\zeta \ddagger f \bullet z)\} = \int_0^\infty \mathrm{C}\{\zeta f(t) \ddagger z(1)\}\mathrm{d}t$$

(a proof may be found, for instance, in Barndorff-Nielsen (1998)), we obtain

$$\mathrm{C}\{\zeta \ddagger f \bullet x^*\} = \int_0^\infty M(t)\mathrm{d}t$$

where

$$\begin{aligned} M(t) &= \grave{k}\left[-\frac{1}{2}\zeta^2 \left\{ \int_0^\infty e^{-\lambda u} f^2(u)\mathrm{d}u \right\} e^{-t} \right] \\ &+ \lambda\grave{k}\left\{ -\frac{1}{2}\zeta^2 \int_0^\infty e^{-\lambda u} f^2(t+u)\mathrm{d}u + i\zeta\rho \int_0^\infty f(t) \right\} \\ &- i\zeta\lambda\rho\xi f(t) \end{aligned}$$

In particular, letting $\zeta = 1$ and

$$f(t) = \zeta_1 \mathbf{1}_{[0,t_1]} + ... + \zeta_m \mathbf{1}_{[0,t_m]}$$

(where $0 < t_1 < ... < t_m$) we obtain the joint cumulant function of $x^*(t_1), ..., x^*(t_m)$.

The special case of $m = 1$ and $t_1 = t$, when $\rho = 0$, deserves explicit consideration as it yields the cumulant function of the (unconditional) log-prices at time t. In particular

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger x^*(t)\} &= \lambda \int_0^\infty \grave{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})e^{-\lambda s}\right\} \mathrm{d}s \\ &\quad +\lambda \int_0^t \grave{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-\lambda s}-e^{-\lambda t})\right\} \mathrm{d}s. \end{aligned}$$

This can be reduced somewhat further. In fact, recalling $\acute{k}(\theta) = \log \mathrm{E}\{\exp \theta\sigma^2\}$, the first integral is simply, following from (4),

$$\acute{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right\}.$$

Using the same result (4) we may rewrite the second integral as

$$\begin{aligned} \lambda \int_0^t \grave{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-\lambda s}-e^{-\lambda t})\right\} \mathrm{d}s &= \int_0^{\lambda t} \grave{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-s}-e^{-\lambda t})\right\} \mathrm{d}s \\ &= -\frac{1}{2}\zeta^2\lambda^{-1} \\ &\quad \cdot \int_0^{\lambda t} (e^{-s}-e^{-\lambda t})\acute{k}'\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-s}-e^{-\lambda t})\right\} \mathrm{d}s \\ &= -\frac{1}{2}\zeta^2\lambda^{-1}(J_0+J_1) \end{aligned}$$

where, by partial integration,

$$\begin{aligned} J_0 &= \int_0^{\lambda t} e^{-s}\acute{k}'\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-s}-e^{-\lambda t})\right\} \mathrm{d}s \\ &= -\acute{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right\} + \int_0^{\lambda t} e^{-s}\acute{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-s}-e^{-\lambda t})\right\} \mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} J_1 &= -e^{-\lambda t} \int_0^{\lambda t} \acute{k}'\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-s}-e^{-\lambda t})\right\} \mathrm{d}s \\ &= e^{-\lambda t}\acute{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right\} \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger x^*(t)\} &= \left(1+\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right)\acute{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right\} \\ &\quad -\frac{1}{2}\zeta^2 \int_0^t e^{-\lambda s}\acute{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(e^{-\lambda s}-e^{-\lambda t})\right\} \mathrm{d}s \end{aligned}$$

But, by the substitution

$$w = -\frac{1}{2}\zeta^2\lambda^{-1}(e^{-\lambda s} - e^{-\lambda t})$$

the last term reduces to

$$-\int_{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})}^{0} \acute{k}(w)\mathrm{d}w = \acute{k}^*(-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t}))$$

where we have introduced

$$\acute{k}^*(x) = \int_0^x \acute{k}(w)\mathrm{d}w$$

The final result is then

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger x^*(t)\} &= \left\{1 + \frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right\} \acute{k}\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right\} \\ &\quad + \acute{k}^*\left\{-\frac{1}{2}\zeta^2\lambda^{-1}(1-e^{-\lambda t})\right\} \end{aligned}$$

Example. Suppose σ^2 is the IG-OU process, i.e. $\sigma^2(t) \sim IG(\delta, \gamma)$, then we have $\acute{k}(\theta) = \delta\gamma\left\{1 - (1 - 2\theta/\gamma^2)^{1/2}\right\}$ and so

$$\acute{k}^*(\theta) = \delta\gamma\theta - \delta\gamma\int_0^\theta (1 - 2w/\gamma^2)^{1/2}\mathrm{d}w.$$

In general for any positive β,

$$\int_0^\theta (1-\beta w)^{1/2}\mathrm{d}w = -\frac{2}{3\beta}(1-\beta\theta)^{3/2} + \frac{2}{3\beta} = \frac{2}{3\beta}\left\{1 - (1-\beta\theta)^{3/2}\right\}.$$

Hence

$$\acute{k}^*(\theta) = \delta\gamma\left[\theta - \frac{\gamma^2}{3}\left\{1 - \left(1 - \frac{2}{\gamma^2}\theta\right)^{3/2}\right\}\right].$$

5 Conclusion

Leverage type effects are empirically important for some stock return data. In this note we have extended the models considered in Barndorff-Nielsen and Shephard (1998a) to deal with this problem. Our model construction allows a great deal of analytic tractability — again giving us the ability to give results on the moments of returns measured in discrete time without any form of approximation. This is not available in more traditionally defined models of these effects.

6 Appendix: Some derivations

The following basic results on OU processes

$$\mathrm{d}\sigma^2(t) = -\lambda\sigma^2(t)\mathrm{d}t + \mathrm{d}\check{z}(\lambda t) \tag{26}$$

are of help in this Appendix. They are

$$\sigma^2(t) = e^{-\lambda t}\sigma^2(0) + e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}\grave{z}(\lambda s)$$

$$\sigma^2(t) = \int_{-\infty}^0 e^s \mathrm{d}\grave{z}(\lambda t + s)$$

$$\sigma^{2*}(t) = \lambda^{-1}\{\grave{z}(\lambda t) - \sigma^2(t) + \sigma^2(0)\}$$

Each is discussed extensively in Barndorff-Nielsen and Shephard (1998a). Then we note that

$$\begin{aligned}
\sigma_n^2 &= \sigma^{2*}(\Delta n) - \sigma^{2*}\left\{\Delta(n-1)\right\} \\
&= \lambda^{-1}\{\grave{z}(\lambda\Delta n) - \grave{z}(\lambda\Delta(n-1)) - \sigma^2(\Delta n) + \sigma^2(\Delta(n-1))\} && (27) \\
&= \lambda^{-1}\left\{\grave{z}_n + (1-e^{-\lambda\Delta})\sigma^2\left\{\Delta(n-1)\right\}\right. \\
&\quad \left. - \int_0^\Delta e^{-\lambda\Delta+\lambda u}\mathrm{d}z\left\{\lambda\Delta(n-1)+\lambda u\right\}\right\} \\
&= \lambda^{-1}\left\{\bar{z}_n + (1-e^{-\lambda\Delta})\bar{\sigma}^2\left\{\Delta(n-1)\right\}\right. \\
&\quad \left. - \int_0^\Delta e^{-\lambda\Delta+\lambda u}\mathrm{d}\bar{z}\left\{\lambda\Delta(n-1)+\lambda u\right\} + \lambda\Delta\xi\right\} \\
&= \lambda^{-1}\left\{\bar{z}_n + (1-e^{-\lambda\Delta})\int_{-\infty}^0 e^u \mathrm{d}\bar{z}\left\{\lambda\Delta(n-1)+u\right\}\right. \\
&\quad \left. - \int_{-\lambda\Delta}^0 e^u \mathrm{d}\bar{z}(\lambda\Delta n + u) + \lambda\Delta\xi\right\} && (28)
\end{aligned}$$

In the rest of this Appendix we let $\lambda = 1$, for the general results are easily derived from those for $\lambda = 1$ by dilation of the time scale. In fact, the general forms follow from those given below simply by substituting Δ by $\lambda\Delta$.

The first result is that

$$\begin{aligned}
\mathrm{Var}\{y_n^2\} &= \mathrm{E}\{y_n^4\} - \mathrm{E}\{y_n^2\}^2 \\
&= \mathrm{E}\{y_{0n}^4\} + 4\rho\mathrm{E}\{y_{0n}^3\bar{z}_n\} + 6\rho^2\mathrm{E}\{y_{0n}^2\bar{z}_n^2\} \\
&\quad +4\rho^3\mathrm{E}\{y_{0n}\bar{z}_n^3\} + \rho^4\mathrm{E}\{\bar{z}_n^4\} - \Delta^2(\xi+\rho^2\grave{\kappa}_2)^2 \\
&= \mathrm{Var}\{y_{0n}^2\} + 6\rho^2\{\Delta^2\xi\grave{\kappa}_2 + (e^{-\Delta}-1+\Delta)\grave{\mu}_3\} \\
&\quad +\rho^4\Delta(\grave{\kappa}_4 + 3\Delta\grave{\kappa}_2^2) - 2\Delta^2\xi\rho^2\grave{\kappa}_2 - \Delta^2\rho^4\grave{\kappa}_2^2 \\
&= \mathrm{Var}\{y_{0n}^2\} + 2\rho^2\{2\Delta^2\xi\grave{\kappa}_2 + 3(e^{-\Delta}-1+\Delta)\grave{\mu}_3\} + \rho^4\Delta(\grave{\kappa}_4 + 2\Delta\grave{\kappa}_2^2).
\end{aligned}$$

Second we have that

$$\begin{aligned}
\mathrm{E}\{y_n y_{n+s}^2\} &= \mathrm{E}\{(y_{0n}+\rho\bar{z}_n)(y_{0n+s}+\rho\bar{z}_{n+s})^2\} \\
&= \rho\mathrm{E}\{\bar{z}_n\sigma_{n+s}^2\}
\end{aligned}$$

$$
\begin{aligned}
&= \rho(1-e^{-\Delta}) \int_{-\infty}^{0} e^{u} \mathrm{E}\{\bar{z}_n \mathrm{d}\bar{z}\,(\Delta(n+s-1)+u)\} \\
&= \rho(1-e^{-\Delta})\grave{\kappa}_2 \int_{-\Delta s}^{-\Delta(s-1)} e^{u} \mathrm{d}u \\
&= \rho\grave{\kappa}_2(1-e^{-\Delta})^2 e^{-\Delta(s-1)}
\end{aligned}
$$

Finally, for the determination of Cov $\left(y_n^2 y_{n+s}^2\right)$ for $s > 0$ we note that

$$
\mathrm{E}\{y_n^2 y_{n+s}^2\} = \sum_{i=1}^{9} T_i
$$

with

$$
T_1 = \mathrm{E}(y_{0n}^2 y_{0n+s}^2)
$$

$$
T_2 = 2\rho \mathrm{E}(y_{0n}^2 y_{0n+s} \bar{z}_{n+s}) = 0
$$

$$
T_3 = \rho^2 \Delta^2 \xi \grave{\kappa}_2
$$

$$
T_4 = 2\rho \mathrm{E}(y_{0n} \bar{z}_n y_{0n+s}^2) = 0
$$

$$
T_5 = 4\rho^2 \mathrm{E}(y_{0n} \bar{z}_n y_{0n+s} \bar{z}_{n+s}) = 0
$$

$$
T_6 = 2\rho^3 \mathrm{E}(y_{0n} \bar{z}_n \bar{z}_{n+s}^2) = 0
$$

$$
T_7 = \rho^2 \left\{\Delta^2 \xi \grave{\kappa}_2 + (1-e^{-\lambda\Delta})^2 e^{-\lambda\Delta(s-1)} \grave{\mu}_3\right\}
$$

$$
T_8 = 2\rho \mathrm{E}(y_{0n}^2 y_{0n+s} \bar{z}_{n+s}) = 0
$$

$$
T_9 = \rho^4 \mathrm{E}(\bar{z}_n^2 \bar{z}_{n+s}^2) = \rho^4 \Delta^2 \grave{\kappa}_2^2.
$$

Here only T_3 and T_7 involve several steps of calculation, as follows

$$
\begin{aligned}
T_3 &= \rho^2 \mathrm{E}(y_{0n}^2 \bar{z}_{n+s}^2) \\
&= \rho^2 \mathrm{E}(y_{0n}^2) \mathrm{E}(\bar{z}_{n+s}^2) \\
&= \rho^2 \Delta^2 \xi \grave{\kappa}_2
\end{aligned}
$$

and

$$
\begin{aligned}
T_7 &= \rho^2 \mathrm{E}(\bar{z}_n^2 y_{0n+s}^2) \\
&= \rho^2 \mathrm{E}\{\bar{z}_n^2 \mathrm{E}\{y_{0n+s}^2 | \mathcal{F}_{\Delta n}\}\} \\
&= \rho^2 \mathrm{E}\{\bar{z}_n^2 \sigma_{n+s}^2\} \\
&= \rho^2((1-e^{-\Delta}) \int_{-\infty}^{0} e^{u} \mathrm{E}\{\bar{z}_n^2 \mathrm{d}\bar{z}\,(\Delta(n+s-1)+u)\} \\
&\quad - \int_{-\Delta}^{0} e^{u} \mathrm{E}\{\bar{z}_n^2 \mathrm{d}\bar{z}(\Delta(n+s)+u)\} \\
&\quad + \Delta^2 \xi \grave{\kappa}_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \rho^2((1-e^{-\Delta})\int_{-\Delta s}^{-\Delta(s-1)} e^u \mathrm{E}\{\bar{z}_n^2 \mathrm{d}\bar{z}\,(\Delta(n+s-1)+u)\} \\
&\quad +\Delta^2\xi\grave{\kappa}_2) \\
&= \rho^2\left(\Delta^2\xi\grave{\kappa}_2) + (1-e^{-\Delta})\int_{-\Delta s}^{-\Delta(s-1)}\int_{-\Delta s}^{-\Delta(s-1)}\int_{-\Delta s}^{-\Delta(s-1)} e^u\right. \\
&\quad \cdot\mathrm{E}\{\mathrm{d}\bar{z}(\Delta(n+s-1)+u)\mathrm{d}\bar{z}(\Delta(n+s-1)+v)\mathrm{d}\bar{z}(\Delta(n+s-1)+w)\}) \\
&= \rho^2\left\{\Delta^2\xi\grave{\kappa}_2 + (1-e^{-\Delta})^2 e^{-\Delta(s-1)}\grave{\mu}_3\right\}
\end{aligned}
$$

All in all

$$
\begin{aligned}
\mathrm{Cov}(y_n^2 y_{n+s}^2) &= \mathrm{E}(y_n^2 y_{n+s}^2) - \mathrm{E}(y_n^2)\mathrm{E}(y_{n+s}^2) \\
&= \mathrm{Cov}(y_{0n}^2, y_{0n+s}^2) + \Delta^2\xi^2 \\
&\quad +2\rho^2\xi\Delta^2\grave{\kappa}_2 + \rho^4\Delta^2\grave{\kappa}_2^2 + \rho^2(1-e^{-\Delta})^2 e^{-\Delta(s-1)}\grave{\mu}_3 \\
&\quad -\Delta^2(\xi+\rho^2\grave{\kappa}_2)^2 \\
&= \mathrm{Cov}(y_{0n}^2, y_{0n+s}^2) + \rho^2(1-e^{-\Delta})^2 e^{-\Delta(s-1)}\grave{\mu}_3
\end{aligned}
$$

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