The Winner's Curse and the Failure of the Law of Demand

Jeremy Bulow

Graduate School of Business, Stanford University,
Stanford, USA

Tel: 650 723 2160
Fax: 650 725 0468

email: jeremy_bulow@gsb-lira.stanford.edu

and

Paul Klemperer

Nuffield College, Oxford University, UK

Int Tel: +44 1865 278588
Int Fax: +44 1865 278557

email: paul.klemperer@economics.ox.ac.uk

January 1998

Abstract

We usually assume increases in supply, allocation by rationing, and exclusion of potential buyers will never raise prices. But all of these activities raise the expected price in an important set of cases when common-value assets are sold. Furthermore, when we make the assumptions needed to rule out these "anomalies" when buyers are symmetric, small asymmetries among the buyers necessarily cause the anomalies to reappear.

Keywords: Auction theory, common value, winner's curse, PCS auction, spectrum auction, airwaves auction, initial public offerings, IPO.

JEL No: D44 (Auctions), L96 (Telecommunications), G30 (Corporate Finance), G24 (Investment Banking)

1 Introduction

Increases in supply lower prices. It is never profitable to commit to rationing at a price at which there is surely excess demand. Excluding potential buyers cannot raise prices. These statements evoke almost universal agreement in our profession. Yet economists from Veblen (1899) to Becker (1991) have sought to explain examples of pricing that appear to contradict these truths.

In fact, it is perfectly reasonable for these statements to be false. This paper shows why, and when this is most likely to happen.

To understand our results, it is important to understand how a bidder determines the maximum he will be willing to pay for an asset. If a buyer's estimate of an asset's value is affected only by his own perceptions and not by the perceptions of others, he should be willing to pay up to his valuation. This is the Adam Smith world, where a buyer can easily maximize his utility given any set of prices, and a firm can easily maximize its profits. In this sort of "private value" model, the statements in the first paragraph are true.

But in many important markets others' perceptions are informative. The extreme cases are "common-value" assets, or assets all buyers would value equally if they shared the same information. Financial assets held by non-control investors may be the best example; oil fields are commonly cited. Most assets have both a private and common value element, particularly if imperfect substitutes exist. For example, a house's value will have both common and idiosyncratic (private) elements.

With common values, buyers may find it prudent to exit an ascending price auction at more or less than their pre-auction estimate of the value. So the statements in the first paragraph are often false, and the "Law of Demand"—that greater supply results in lower prices—is violated.

The reason is the "winner's curse". Buyers must bid more conservatively the more bidders there are, because winning implies a greater winner's curse. This effect can more than compensate for the increase in competition caused by more bidders, so more bidders can *lower* expected prices.[1] Conversely,

[1]Steven Matthews (1984) has already provided an example with symmetric bidders and affiliated common values in which additional bidders reduce expected revenue in a first-price auction. Our paper provides insight into why results like ours and Matthews' can arise, and shows they are surprisingly likely.

adding more supply, and/or rationing, creates more winners, so reduces the bad news learned by winning, and so may raise bids enough to *increase* expected prices. This paper shows when this happens and why it is surprisingly often.

A good example is provided by the market for Initial Public Offerings (IPOs). Rather than being priced to clear the market, many IPOs are made at prices that guarantee excess demand. By pricing low enough so that everyone will want to buy, potential shareowners are absolved of the winner's curse of only being buyers when they are among the most optimistic investors. This allows the pooling price to be quite high and, under quite reasonable conditions, as high or higher than the expected price in a standard auction.[2]

Our results are especially likely in asymmetric "almost common value" markets in which some competitors have a small advantage, because the other bidder(s) then face an exacerbated winner's curse.

This was illustrated in the A and B band spectrum auction held in 1994-95 by the Federal Communications Commission. Pacific Telesis was the natural buyer of the single Los Angeles license available for sale,[3] and was able to acquire it very cheaply.[4] Markets where two licenses were sold generally yielded more competitive prices relative to their demographic characteristics.[5] Even where one bidder had an advantage, the prices of both licenses

[2]A related example is when the value of an asset is not allowed to rise or fall more than a fixed amount in a day. The South Korean won was limited to a 10 percent daily decline through early December of 1997. The limitation prevented the market from fully aggregating bidders' information and the price fell by the maximum on several days. When the limitation was removed and the market was allowed to clear, the price actually rose: it is entirely possible that if a price is artificially fixed only slightly in excess of the expected market clearing price there will be an enormous excess supply, but if the market is allowed to clear the price will rise above the fixed rate. (Obviously, the Korean situation was very complex and relaxing the limit on the amount the won could fall may not have been important for the increase in the market price.)

[3]AT&T was ineligible to bid, and PacTel had the benefit of its name recognition and experience in California, as well as its familiarity with the California wireless market in which it was a duopolist prior to its spinoff of its cellular subsidiary, Airtouch. No one knows what PCS licenses are really worth, but it is fair to say that the LA license was worth more to PacTel than anyone else.

[4]While the FCC's mandate was for economic efficiency rather than revenue, and awarding the license to PacTel was almost certainly efficient, if PacTel had paid more there would have been an efficiency gain to the economy from being able to reduce the deadweight loss from taxation.

[5]The most obvious example is Chicago, where the prices were about $31 per head of population for each of the two licenses, compared with less than $26 per head of population for Los Angeles' single license, in spite of Chicago's inferior demographic characteristics.

were determined by aggressive competition for the second license. So prices were better, even though the third-highest bid set the price in these markets while the second-highest bid set the price in Los Angeles.

In section 2 we set up a simple model of a standard ascending auction[6] among bidders with **"**almost**"**common values. Section 3 shows when the "law of demand"[7] is violated and *higher* prices are associated with selling *more* units in the symmetric case. Section 4 shows that the results are dramatically different when bidders are asymmetric: the "law of demand" is violated precisely when it holds for symmetric bidders! Section 5 shows when rationing, as in Initial Public Offerings, is optimal. It also shows when restricting participation in an auction can raise expected revenues, and considers first-price auctions.[8] Section 6 concludes.[9]

(The famous long commutes of Angelenos and the population density in the area makes it a particularly desirable place to own a wireless telephone franchise.) The single New York license yielded only $17 per head of population.

[6]The spectrum auction was an ascending auction, but also included a number of special features designed to allow licenses for different regions to be sold simultaneously. However, we do not believe these additional features affect our basic argument.

[7]By "law of demand" we mean in this paper that supplying more units yields lower expected equilibrium prices. Nothing in our paper contradicts the result that fixing a lower price yields greater demand.

[8]The first draft of this paper, Bulow and Klemperer (1997), shows how our model can be used to develop one possible explanation of the "Declining Price Anomoly". (See Ashenfelter (1989), Ayres and Cramton (1996), Beggs and Graddy (1997), Black and deMeza (1992), Levin (1997), McAfee and Vincent (1993), and von der Fehr (1994), among others for further discussion of the "anomoly" and other explanations of it.) We plan to pursue this further in subsequent work.

[9]Other recent papers that use similar models to ours are Avery and Kagel's (1997), de Frutos and Rosenthal's (1997), and Krishna and Morgan's (1997) Working Papers. Krishna and Morgan develop important insights about the effects of collusion and joint-bidding in common-value auctions. Independently from the first draft of our paper (Bulow and Klemperer (1997)), they also obtain results that are equivalent to the symmetric case of our section 5.2 about restricting participation. They do not tackle the asymmetric case because their model, unlike ours, is of pure common values, so has a vast multiplicity of equilibria, even when bidders are asymmetric (see note 14). (Nor, since their main focus is different, do they analyse the effects of increasing supply, or of rationing, which are the main focuses of our paper.) Avery and Kagel and de Frutos and Rosenthal address different concerns from ours; Avery and Kagel discuss experimental results in a two-bidder one-prize model, while de Frutos and Rosenthal obtain interesting results about sequential auctions.

2 The Model

We use the simplest possible model to make our points: each of 3 risk-neutral potential bidders observes a private signal t_i independently and identically distributed according to the distribution $F(t_i), i = 1, 2, 3$. We assume $F(\cdot)$ has a strictly positive continuous finite derivative $f(\cdot)$ everywhere on its range, and the lowest possible signal is $\underline{t} > 0$, so $F(\underline{t}) = 0$. Conditional on all the signals, the expected value, v_i, of a unit to i is

$$\begin{aligned} v_1 &= (1+\alpha_1)t_1 + t_2 + t_3 \\ v_2 &= t_1 + (1+\alpha_2)t_2 + t_3 \\ v_3 &= t_1 + t_2 + (1+\alpha_3)t_3 \end{aligned}$$

That is, each unit has a *common value*, $\sum_{i=1}^{3} t_i$,to all the bidders, plus a *private value,* $\alpha_i t_i$, to each bidder i. We will focus on two cases, "the symmetric case" in which $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 0$ and "the asymmetric case" in which $\alpha_1 > \alpha_2 = \alpha_3 = \alpha > 0$. In the latter case we will refer to bidder 1 as the "advantaged" bidder, and bidders 2 and 3 as "disadvantaged" bidders. We are interested in the case in which the private-value components, that is, the α_i's, are all small and so the sizes of bidders' advantages and disadvantages are also small. To make our points most starkly and straightforwardly, we consider an asymmetric case in which α/α_1 is also small, so we state our results throughout for the limits in which $\alpha_i \to 0, \forall_i$, and, for the asymmetric case, $\alpha/\alpha_1 \to 0$.[10]

No bidder wants more than one unit. We consider two cases: the auctioneer has one unit to sell, and the auctioneer has two units to sell. (The number of units is common knowledge.)

We assume a conventional ascending bid "English" auction[11] in which

[10] All we actually need is that the α_i's are small relative to the rates of change of bidders' inverse hazard rates, $\frac{1-F(t_i)}{f(t_i)}$. So the order in which the limits is taken is unimportant.

[11] More formally, we area assuming what auction theorists call a "Japanese auction". Bikhchandani and Riley (1993) describe this as follows (for the single unit case): The auctioneer starts with a very low price and raises it continuously. Bidders indicate, by depressing a button, whether they are interested in buying the object at the current price. Once a bidder withdraws, he cannot reenter the auction. At each price level, the identities of all bidders active at that price are common knowledge. Whenever one or more bidders withdraw at a price, the auctioneer stops raising the price and asks the remaining bidders if they wish to withdraw. If additional bidder(s) withdraw, this is announced by the auctioneer and the remaining bidders are again asked if they wish to withdraw. This

the price, p, starts at zero and rises continuously until the number of bidders who are still willing to pay the current price equals the number of units the auctioneer has for sale. Each bidder observes the price at which any other bidder drops out.

Each player's pure strategy specifies the price level up to which he will remain in the bidding, as a function of his private signal and of the price (if any) at which any other player quit previously. We assume symmetric bidders follow symmetric strategies, and restrict attention to the (Perfect Bayesian) equilibrium in which each bidder stays in the bidding just so long as he would be happy to find himself a winner, and stops bidding at that price at which he would be just indifferent were he to find himself a winner on the assumptions that any opponent(s) who drop out to make him a winner are of their lowest possible types assuming they have followed the equilibrium strategies prior to the current price.[12],[13] The Appendix shows that this yields a unique (Perfect Bayesian) equilibrium.[14]

We write the actual i^{th} highest signal as $t_{(i)}$, write $E(t)$ for the expectation of t_i, and write $E(t \mid t \geq t')$ for the expectation of t_i conditional on it

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process continues until no additional bidders quit. When no additional bidders withdraw, and at least two bidders remain, the auctioneer starts raising the price continuously from the current level. The auction can end in one of two possible ways. If at any price there is only one active bidder, then this bidder is declared the winner and the auction ends. Else, if at any price all the remaining active bidders withdraw (either simultaneously or during the sequential quitting process described above) the auction ends and one of the last active bidders is randomly chosen as the winner. The winner gets the object and pays the current price. The other bidders pay nothing.

[12]That is, strategies specified in this way yield a Perfect Bayesian equilibrium in the space of all strategies; in this equilibrium a player cannot do better by following any other strategy.

[13]Restricting attention to equilibrium of this form both avoids trivialities (although there are other equilibria, they do not seem very natural) and greatly reduces the technical burden: See Bikhchandani and Riley (1993) for an exposition of how cumbersome and lengthy is a fully general analysis of even the completely symmetric version of our model, although they too make assumptions to obtain a unique equilibrium (the same equilibrium as ours, though their model is a special case of ours). See the Appendix for further discussion.

[14]By contrast, in a pure common values model with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ this construction does not define a unique equilibrium. (For example, with just two bidders and $v_1 = v_2 = t_1 + kt_2$, where k is a positive constant, it is an equilibrium for 1 to quit at βt_1 and 2 to quit at $\left(\frac{\beta k}{\beta - 1}\right) t_2$ for any $\beta > 1$.) Hence the need to include the α_i in the model, and to analyze a pure common value model as the limit of almost common value models; focusing on a particular equilibrium of the pure common value model can be misleading.

exceeding t', etc. It will be useful to define bidder i's *marginal revenue*[15] as

$$MR_i = v_i - \frac{1 - F(t_i)}{f(t_i)} \frac{\partial v_i}{dt_i}.$$

Note that since (we assumed) the α_i are all small, $MR_i \approx v_i - h_i$ in which $h_i(t_i) \equiv \frac{1-F(t_i)}{f(t_i)}$ is the reciprocal of i's hazard rate.

3 The Symmetric Case

We begin with the symmetric case in which $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 0$ (but $\alpha \approx 0$).

When three bidders compete for a single object, the lowest bidder quits first in symmetric equilibrium, and the other bidders can then infer (assuming equilibrium behaviour) his actual signal, $t_{(3)}$.[16] The next-lowest bidder then quits when the price reaches the point at which he would just be indifferent about winning were he the marginal winner, that is, were he tied for the highest signal, so he quits at $p = t_{(3)} + (2 + \alpha)t_{(2)}$.[17] We therefore have (since $\alpha \approx 0$):

Lemma 1: *When 3 symmetric bidders compete for 1 object, the bidder with the highest signal wins and the price* $\approx t_{(3)} + 2t_{(2)}$.

Proof: See appendix. □.

If instead, three bidders compete for two objects, the lowest quits in symmetric equilibrium at the price at which he would just be indifferent about

[15] In analyzing our auctions using marginal revenues, we are following Bulow and Roberts (1989) who first showed how to interpret independent private-value auctions in terms of marginal revenues, and Bulow and Klemperer (1996) who extended their interpretation to more general settings such as this one. The marginal revenue of bidder i with signal t_i is exactly the marginal revenue extracted from the customer who is the same fraction of the way down the distribution of potential buyers of a monopolist whose demand is such that it has $q = 1 - F(t_i)$ customers who have values $\geq p = v_i(t_i)$ (i.e. there are $F(t_i)$ customers with values less than $v_i(t_i)$). This allows the direct translation of results from monopoly theory into auction theory, and so facilitates the analysis of auctions and the development of intuition about them.

[16] In fact, the lowest bidder quits at $(3+\alpha)t_{(3)}$, since if he stays in until a slightly higher price he will win only if both other signals are $t_{(3)}$, but this fact is not necessary to our argument.

[17] It is easy to check that if he were to find himself a winner at any higher price he would lose money, since at price $p' = t_{(3)} + (2+\alpha)t'$ with $t' > t_{(2)}$, the inferred value of the unit equals $t_{(3)} + (1+\alpha)t_{(2)} + t'$ conditional on winning at price p', and conversely he would make money at any lower price, so should not quit before p.

winning were he the marginal winner, that is, were he tied with the second-highest signal. So the actual lowest-signal bidder with signal $t_{(3)}$ quits at the value to him if the second-highest-signal bidder has the same signal, $\left(t_{(3)}\right)$, and the remaining signal equals its expected value given the two lowest signals are $t_{(3)}$, that is, $E\left(t|\, t \geq t_{(3)}\right)$. So the lowest-signal bidder quits at $p = (1+\alpha)t_{(3)} + t_{(3)} + E(t \mid t \geq t_{(3)})$.[18] So we have:

Lemma 2: *When 3 symmetric bidders compete for 2 objects, the bidders with the highest signals win and the price* $\approx 2t_{(3)} + E\left(t|\, t \geq t_{(3)}\right)$.

Proof: See appendix. □.

Therefore selling two units rather than one lowers the per-unit price obtained only if $2t_{(3)}+E\left(t|\, t \geq t_{(3)}\right) < t_{(3)}+2t_{(2)}$, that is if $E\left(t|\, t \geq t_{(3)}\right)-t_{(3)} < 2(t_{(2)} - t_{(3)})$, or (in expectation) only if twice the expected distance between the lowest signal and the lower of two higher signals exceeds the expected distance between the lowest signal and a single higher signal. It follows that:

Proposition 1: *With 3 symmetric bidders, the expected price per unit is higher (lower) when 2 units are sold then when 1 unit is sold if hazard rates,* $\left(\frac{1}{h_i}\right)$, *are decreasing (increasing) in the signals,* t_i.

It is now easy to find examples in which the "law of demand" fails. (See Ex.1 below.)

To understand this result better, recall from Bulow and Klemperer (1996) that *the expected price from the auction equals the expected marginal revenue of the winning bidder(s).*[19],[20] Furthermore, when bidders are symmetric, the

[18] Again it is easy to check that if either of the other bidders were to quit and leave him as a winner at any higher price, $p' = (2+\alpha)t' + E(t \mid t \geq t')$ with $t' > t_{(3)}$, he would expect to lose money since he would then infer a unit's value to be $(1+\alpha)t_{(3)} + t' + E(t \mid t \geq t')$ $< p'$, and conversely he would expect to profit from a victory at any lower price.

[19] This result assumes, as is the case here, that a bidder with the lowest possible signal never makes money. Otherwise, expected revenue is reduced by the sum of the expected profits of the bidders conditional on their having their lowest possible signals.

[20] The essential point is that if an English auction ends at a price of $\underline{v}$, the winners will each have an expected value of at least $\underline{v}$ (conditional on all previous actions by bidders and on the auction ending at this price). Therefore, we can draw a probability curve for the bidder's actual value, v, with decumulative probability, $1 - F(v)$, on the quantity axis, and value on the price axis; the curve has its minimum value $\underline{v}$ at a "quantity" of 1. If we

bidder(s) with the highest signal(s) win(s) the unit(s). So when more units are sold, the expected price is lower if and only if the expected marginal revenues of bidders with lower signals are lower, that is, if and only if $v - h$ is lower for bidders with smaller t's than for bidders with larger t's.

Note the contrast between common-value and private-value auctions. In both a pure common-value auction (α arbitrarily small) and a pure private-value auction ($v_i = t_i$), i's marginal revenue equals his value minus his inverse hazard rate. But in the private value case where $v_i = t_i$ we have $MR_i > MR_j \Leftrightarrow t_i - h_i > t_j - h_j$. In the common value case, since $v_i = v_j$, we have $MR_i > MR_j \Leftrightarrow -h_i > -h_j$. So in the private-value case the "law of demand" requires that $t_i - h_i > t_j - h_j \Leftrightarrow t_i > t_j$, which condition is satisfied by many standard distributions $F(\cdot)$, and is often assumed in the literature without comment. However in the common-value case the "law of demand" requires $-h_i > -h_j \Leftrightarrow t_i > t_j$ which is a much more stringent condition on $F(\cdot)$.

In simple terms, the difference is that with private values when a bidder has a higher signal it affects only his own value and marginal revenue. But with common values when a bidder has a higher signal it also raises the other bidders' values and so raises the others' marginal revenues. So it takes a much stronger distributional condition to ensure that bidders with higher signals have higher marginal revenues.

The condition in the private-value case is just that the bidder's marginal revenue is downward sloping, that is, that a monopoly firm with demand $q = 1 - F(p)$ has marginal revenue downward sloping in its *own* output.[21] The condition in the common-value case is that the same firm's marginal revenue

interpret this as a conventional demand curve and draw the associated marginal revenue curve, the expected marginal revenue of the winner has to equal $\underline{v}$ (just as if you knew that a firm sold q units at a price of $\underline{v}$, you would know that each buyer had a value of at least $\underline{v}$, that total revenue was $\underline{v}q$, and therefore that the average marginal revenue from the first q sales was $\underline{v}$). So the expected marginal revenue of the winning bidder(s) always equals the auction price.

[21]The demand curve $q = 1 - F(p)$ is just the conventional demand curve that would be created by a very large number of buyers with values $v_i(t_i)$ when the t_i are drawn independently from the distribution $F(t_i)$. (We hold t_j and t_k fixed; buyers are atomistic with total mass 1.) For more discussion of the analogy between a bidder with signal distributed as $F(t_i)$ and a market with demand curve $1 - F(p)$ see Bulow and Klemperer (1996) and our Working Paper, Bulow and Klemperer (1997).

is steeper than its demand curve,[22] or equivalently that the firm's marginal revenue is downward sloping in a sufficiently small *opponent's* output;[23] this is exactly the condition required to guarantee strategic substitutes in quantity competition in oligopoly—see Bulow, Geanakoplos, and Klemperer (1985a). And the assumption of strategic substitutes, while commonly made, and *perhaps* more plausible than the converse assumption of strategic complements, is not a reasonable general assumption.[24] Indeed, among the most commonly used demand curves, linear demand ($p = A - Bq \Leftrightarrow q = \frac{A-p}{B}$) yields strategic substitutes, constant elasticity demand ($p = Aq^{\frac{1}{\eta}} \Leftrightarrow q = \left(\frac{P}{A}\right)^{\eta}, \eta < -1$) yields strategic complements, and logarithmic demand ($p = A - \frac{1}{\lambda}\log q \Leftrightarrow q = e^{-\lambda(p-A)}$, i.e. quantity is exponential in price) yields strategic independence (neither strategic substitutes nor strategic complements) for a monopolist facing a small new entrant.

Corresponding exactly to the oligopoly cases we have:

Example 1: *With uniformly distributed signals, $F(t) = \left(\frac{t-\underline{t}}{\bar{t}-\underline{t}}\right)$, expected price is decreasing in supply. With constant-elasticity distributed signals, $F(t) = 1 - (\frac{t}{\underline{t}})^{\eta}$, expected price is increasing in supply. With exponentially distributed signals, $F(t) = 1 - e^{-\lambda(t-\underline{t})}$, expected price is constant in supply.*[25]

(With a uniform distribution, twice the expected distance between the lowest signal and the lower of two higher signals exceeds the expected distance between the lowest signal and a single higher signal; with the constant-elasticity distribution this fails; and the exponential distribution is the inter-

[22]Since $-h_i > -h_j \Leftrightarrow t_i > t_j$ implies $((v_i - h_i) - v_i) > ((v_j - h_j) - v_j) \Leftrightarrow t_i > t_j$, implies $(v_i - h_i) - v_i$ increasing in t_i, hence decreasing in q, letting $q \equiv 1 - F(t)$ and $p \equiv v_i(t_i)$.

[23]Assuming the opponent is producing a homogeneous product, see Bulow, Geanakoplos and Klemperer (1985a).

[24]See Bulow, Geanakopolos and Klemperer (1985a) for more discussion, and also Bulow, Genakopolos and Klemperer (1985b) for an example in which a monopolist facing a new entrant views products as strategic complements.

[25]For example, if $F(t) = 1 - t^{-2}$ for $t \geq 1$ (which corresponds probabilistically to a demand curve $q \equiv 1 - F(p) = p^{-2}$, that is, constant elasticity of -2) the expected values of the three signals would be 1.2, 1.6, and 3.2. So the expected price in a 3 for 1 auction would be 1.2+1.6+1.6 = 4.4, and the expected price with 3 for 2 would be $1.2{+}1.2{+}\frac{1.6+3.2}{2} = 4.8$.

If $F(t) = \frac{t}{4}$ for $4 \geq t \geq 0$ (which corresponds probabilistically to a linear demand curve) the expected values of the three signals would be 1,2, and 3. The expected price in a 3 for 1 auction would be $1 + 2 + 2 = 5$, and the expected price in a 3 for 2 auction would be $1 + 1 + \frac{2+3}{2} = 4.5$.

mediate case in which the ratio of the expected distances is exactly one:two.[26])

So, just as in oligopoly it is an empirical matter whether firms' outputs are strategic substitutes or strategic complements, so in symmetric pure common-value auctions it must be an empirical matter whether price is increasing or decreasing in supply.

The next section, however, will show that even (arbitrarily) small asymmetries can make the relationship between supply and price even less predictable.

4 The Asymmetric Case

This section will show that when the "law of demand" is satisfied for the perfectly symmetric case, it can fail when there are even arbitrarily small asymmetries between the bidders. In particular it fails if the item(s) for sale are almost pure common-values but one bidder, say bidder 1, almost certainly has an arbitrarily small private-value advantage. We assume $\alpha_1 > \alpha_2 = \alpha_3 = \alpha > 0$, but $\alpha_1 \approx 0$ and $(\alpha/\alpha_1) \approx 0$.

We begin by analyzing bidding behaviour in more detail:

Lemma 3: *When 3 bidders compete for 1 object in the asymmetric case, the advantaged bidder (almost always) wins and the price* $\approx \underline{t} + t_2 + t_3$ *(in which* t_2 *and* t_3 *are the actual signals of the disadvantaged bidders).*

Proof: See appendix. $\Box$.

[26]With a uniform distribution, $2t_{(3)} + E\left(t \mid t \geq t_{(3)}\right) = 3t_{(3)} + \frac{1}{2}(\bar{t} - t_{(3)})$, while $t_{(3)} + 2E(t_{(2)} \mid t_{(3)}) = 3t_{(3)} + \frac{2}{3}(\bar{t} - t_{(3)})$. With constant elasticity distributed signals, $2t_{(3)} + E\left(t \mid t \geq t_{(3)}\right) = 3t_{(3)} - \frac{1}{1+\eta}t_{(3)}$, while $t_{(3)} + 2E(t_{(2)} \mid t_{(3)}) = 3t_{(3)} - \frac{2}{1+2\eta}t_{(3)}$. For exponentially distributed signals, $2t_{(3)} + E\left(t \mid t \geq t_{(3)}\right) = t_{(3)} + 2E(t_{(2)} \mid t_{(3)}) = 3t_{(3)} + \frac{1}{\lambda}$.

The calculations are straightforward in the first and third cases. In the constant elasticity case, $E\left(t_{(1)} \mid t_{(1)} \geq t_{(2)}\right) = \frac{\eta}{\eta+1}t_{(2)}$ and $(E\left(\frac{t_{(1)}+t_{(2)}}{2}\right) \mid t_{(2)} \geq t_{(3)}) = E\left(t \mid t \geq t_{(3)}\right) = \frac{\eta}{\eta+1}t_{(3)}$. Combining the last two equations yields $\frac{2\eta+1}{1+\eta}E(t_{(2)}) = \frac{2\eta}{1+\eta}t_{(3)}$ so $E(t_{(2)}) = \frac{2\eta}{1+2\eta}t_{(3)}$. By substituting $E\left(t \mid t \geq t_{(3)}\right) = \frac{\eta}{\eta+1}t_{(3)} = t_{(3)} - \frac{1}{1+\eta}t_{(3)}$ we can derive the constant elasticity revenue for when there are two winners, and by substituting $E(t_{(2)} \mid t_{(3)}) = \frac{2\eta}{1+2\eta}t_{(3)}$ we can derive the expected revenue when there is one winner. (The formulae $E\left(t \mid t \geq t_{(3)}\right) = \frac{\eta}{\eta+1}t_{(3)}$ and $E\left(t_{(1)} \mid t_{(1)} \geq t_{(2)}\right) = \frac{\eta}{\eta+1}E(t_{(2)})$ are mathematically identical to the statement that, given constant elasticity demand and a price p, the average buyer has a value of $\frac{p\eta}{\eta+1}$. This must be true since $\frac{\eta}{\eta+1}$ is just the ratio of price to marginal revenue at each point along a constant elasticity curve and therefore the ratio of average value to average revenue (equals price). Here the calculations are the same, except we use $t_{(2)}$ and $t_{(3)}$ instead of p.)

The logic is straightforward. Bidder i quits where he would be just indifferent about finding himself a winner, so his marginal type $\underline{t}_i$ quits at price $p = (1+\alpha_i)\underline{t}_i + \underline{t}_j + \underline{t}_k$, where $\underline{t}_j$ and $\underline{t}_k$ are his expectations of j's and k's signals conditional on his winning at this price. That is, $\underline{t}_j$ is the marginal type of bidder j who is just quitting if any type of j is currently quitting,[27] and similarly for $\underline{t}_k$. Likewise, type $\underline{t}_j$ of j is in fact just quitting iff $p = \underline{t}_i + (1+\alpha_j)\underline{t}_j + \underline{t}_k$. So types $\underline{t}_i$ and $\underline{t}_j$ quit simultaneously iff $(1+\alpha_i)\underline{t}_i + \underline{t}_j + \underline{t}_k = \underline{t}_i + (1+\alpha_j)\underline{t}_j + \underline{t}_k \Leftrightarrow \alpha_i\underline{t}_i = \alpha_j\underline{t}_j$, and $\underline{t}_i$ quits before (after) $\underline{t}_j$ iff $\alpha_i\underline{t}_i < (>)\alpha_j\underline{t}_j$. So since $\alpha_1\underline{t} > \alpha t_2$ and $\alpha_1\underline{t} > \alpha t_3$ for almost all t_2 and t_3 for sufficiently large α_1/α, bidder 1 is almost always the winner. If, for example, in fact $\alpha_1\underline{t} > \alpha t_2 > \alpha t_3$, then bidder 3 quits first at $(1+\alpha)t_3 + t_3 + \underline{t}$ (since at this price he knows $t_2 \geq t_3$ so the current lowest types of bidders 2 and 1 that could remain are $\underline{t}_2 = t_3$ and $\underline{t}_1 = \underline{t}$), and bidder 2 quits next at $(1+\alpha)t_2 + t_3 + \underline{t} \approx t_2 + t_3 + \underline{t}$.

The intuition is that because bidder 1 (almost always) values the asset a little more than bidders 2 and 3, there cannot be any equilibrium where bidder 2 or 3 is willing to pay p and bidder 1 is not willing to pay a little more unless t_1 is almost zero. So bidders 2 and 3 face an enormous winner's curse if bidder 1 ever exits, and they must therefore assume $t_1 \approx \underline{t}$ whenever he bids. So they quit at $\approx t_2 + t_3 + \underline{t}$, and bidder 1 almost always wins.

However, with three bidders competing for two units and increasing hazard rates, bidder 1's advantage is almost eliminated and he wins only when he has one of the two highest signals:

Lemma 4: *When 3 bidders compete for 2 objects, in the asymmetric case (i) if hazard-rates* $\left(\frac{1}{h_i}\right)$ *are increasing in signals, the bidders with the highest signals (almost always) win and the price* $\approx 2t_{(3)} + E\left(t \middle| t \geq t_{(3)}\right)$, (ii) *if hazard-rates are decreasing, the advantaged bidder and the disadvantaged bidder with the higher signal win and the price* $\approx E(t) + 2\min(t_2, t_3)$ *(in which* t_2 *and* t_3 *are the actual signals of the disadvantaged bidders).*

Proof: See appendix. $\square$.

To understand Lemma 4, again begin by observing that bidder i quits

[27] If j has already quit $\underline{t}_j$'s is j's inferred signal, and if j has not quit but no type of j is quitting then $\underline{t}_j$ is j's lowest possible signal consistent with equilibrium.

where he would be just indifferent about finding himself a winner. If $\underline{t}_i, \underline{t}_j$, and $\underline{t}_k$ are the lowest possible signals of bidders i, j and k assuming equilibrium behaviour up to the current price, type $\underline{t}_i$ of bidder i has expected value $(1+\alpha_i)\underline{t}_i+\underline{t}_j+E(t_k \mid t_k \geq \underline{t}_k)$ if j quits now, and expected value $(1+\alpha_i)\underline{t}_i+E(t_j \mid t_j \geq \underline{t}_j)+\underline{t}_k$ if k quits now. So type $\underline{t}_i$ quits at $p = (1+\alpha_i)\underline{t}_i+\underline{t}_j+\underline{t}_k+x_j Prob($$k$ quits now$\mid$ j or k quits now$)+x_k Prob($$j$ quits now$\mid$ j or k quits now$)$ in which $x_j \equiv E(t_j - \underline{t}_j \mid t_j \geq \underline{t}_j)$ and x_i and x_k are defined similarly. Since $\alpha_2 = \alpha_3 = \alpha < \alpha_1$, some types of bidders 2 and 3 quit (symmetrically) before any types of bidder 1 quits. Now note that for small enough α and α_1 the differences between $\alpha_1\underline{t}_1$ and $\alpha\underline{t}_2(= \alpha\underline{t}_3)$ are very small relative to differences between x_1 and $x_2(= x_3)$.[28] So if hazard rates are increasing, so x_i is decreasing in $\underline{t}_i$, then if $\underline{t}_1$ were to fall much behind $\underline{t}_2(= \underline{t}_3)$ then x_2 would become small relative to x_1 and $\underline{t}_1$ would wish to quit at a lower price than $\underline{t}_2$. So types of bidder 1 would have to quit until $\underline{t}_1$ roughly caught up to the value of $\underline{t}_2(= \underline{t}_3)$. Therefore increasing hazard rates require $\underline{t}_1 \approx \underline{t}_2 = \underline{t}_3$. So bidder i quits at (approximately) $(1+\alpha_i)t_i + t_i + E\left(t_k \middle| t_k \geq t_j = t_i\right)$, just as in symmetric equilibrium with symmetric bidders, and the bidder with the lowest signal, $t_{(3)}$, (approximately) quits first, and we have part (i) of the Lemma.

The intuition is that even if bidder 1 had a large advantage, bidders 2 and 3 would compete against each other, symmetrically, for the second unit, and in that competition they would not face an abnormally large winner's curse. Because the prices of both units will be the same, the more aggressive bidding by bidders 2 and 3 will force bidder 1 to pay more, and may cause bidder 1 to exit if his signal is low enough (which further reduces the other bidders' winner's curse). When bidder 1's advantage is small, it becomes irrelevant with increasing hazard rates.[29]

[28] The exception is when hazard rates are constant, $h_1 = h_2 = h_3 = h$, in which case $x_k = h, \forall \underline{t}_k, \forall_k$.

[29] A numerical example may help some readers: assume, counterfactually, that bidder 1 almost always wins when there are two units and t is distributed uniformly on $[0, 10]$. Then bidders 2 and 3 will not learn anything about bidder 1's signal through the auction and will assume that $t_1 = 5$ (its average value). So bidder 2 will bid up to $\approx 2t_2 + 5$ and bidder 3 will bid to $\approx 2t_3 + 5$. If, for example, $t_2 = 2 < t_3$ then bidder 2 would exit at a price of 9. But if this happened, and bidder 1 had a signal close to 0, bidder 1 would suffer regret: he would estimate the value as only $\approx t_1 + t_2 + E(t_3 \mid t_3 \geq 2) = 0 + 2 + \frac{2+10}{2} = 8$. So bidder 1 will have already dropped out. Bidder 1's small private value advantage is overwhelmed by the difference between his expectation of the other winner's signal in

On the other hand, if hazard rates are decreasing, x_i is increasing in $\underline{t}_i$, so once $\underline{t}_1$ falls behind $\underline{t}_2 (= \underline{t}_3)$ then x_2 becomes large relative to x_1 so $\underline{t}_1$ wishes to quit at a still higher price relative to $\underline{t}_2$, so (since some types of bidders 2 and 3 start quitting first) no type of bidder 1 ever quits. As the auction proceeds and more types of bidders 2 and 3 quit, $x_2 \equiv E(t_2 - \underline{t}_2)$ and $x_3 \equiv E(t_3 - \underline{t}_3)$ increase while $x_1 \equiv E(t_1 - \underline{t}_1)$ remains unchanged and so even the lowest type of bidder 1 expects a larger and larger surplus conditional on winning; the higher the bidding goes, the more *under*priced bidder 1 thinks the object is. Since bidders 2 and 3 are symmetric, the bidder with the lower of their two signals loses, and if this signal is $\widehat{t}$ he quits at price $p = E(t_1) + (1+\alpha)\widehat{t} + \widehat{t} \approx E(t) + 2\widehat{t}$.

Lemmas 3 and 4 yield:

Proposition 2: *In the asymmetric case, the expected price per unit is higher (lower) when 2 units are sold than when 1 unit is sold if hazard rates,* $\left(\frac{1}{h_i}\right)$ *,are increasing (decreasing) in the signals,* t_i.

Proof: Note that $E\left(t \middle| \, t \geq t_{(3)}\right) = \frac{1}{2}\left\{E\left(t_{(1)} \middle| t_{(3)}\right) + E\left(t_{(2)} \middle| \, t_{(3)}\right)\right\}$,since the expectation of a single signal above $t_{(3)}$ equals the average of the expectations of the higher of two signals above $t_{(3)}$ and the lower of two signals above $t_{(3)}$. So

$$
\begin{aligned}
& E(\underline{t} + t_2 + t_3) < E\left(2t_{(3)} + E\left(t \middle| \, t \geq t_{(3)}\right)\right) \\
\iff \quad & E\left(\underline{t} + \tfrac{2}{3}t_{(1)} + \tfrac{2}{3}t_{(2)} + \tfrac{2}{3}t_{(3)}\right) < E\left(2t_{(3)} + \tfrac{1}{2}t_{(1)} + \tfrac{1}{2}t_{(2)}\right) \\
\iff \quad & 0 < \tfrac{1}{6}\left\{\left[3E\left(t_{(3)} - \underline{t}\right) - E\left(t_{(1)} - t_{(2)}\right)\right] + \left[3E\left(t_{(3)} - \underline{t}\right) - 2E\left(t_{(2)} - t_{(3)}\right)\right]\right\}.
\end{aligned}
$$

But increasing hazard-rates implies that each of the expressions in square brackets is strictly positive (they would be zero with constant hazard-rates[30]), which proves the result for increasing hazard-rates.

excess of its minimum possible value ($\frac{2+10}{2} - 2 = 4$ in this case) and the similar expectation for bidders 2 and 3 about bidder 1 ($\frac{0+10}{2} - 0 = 5$). Equilibrium will require that bidder 1 exit at almost exactly the same rate as bidders 2 and 3, so bidders win as often and at (approximately) the same prices as if they were all symmetric.

[30]With constant hazard rates, where $F(t) = 1 - e^{-\lambda(t-\underline{t})}$, $E(t_{(3)}) = \underline{t} + \frac{1}{3\lambda}$, $E(t_{(2)}) = \underline{t} + \frac{1}{3\lambda} + \frac{1}{2\lambda}$, and $E(t_{(1)}) = \underline{t} + \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$.

Note that $E(\min(t_2, t_3)) = \frac{1}{3}E(t_{(2)}) + \frac{2}{3}E(t_{(3)})$. So

$$
\begin{aligned}
& E(\underline{t} + t_2 + t_3) > E(E(t) + 2\min(t_2, t_3)) \\
\iff \quad & E\left(\underline{t} + \tfrac{2}{3}t_{(1)} + \tfrac{2}{3}t_{(2)} + \tfrac{2}{3}t_{(3)}\right) > E\left(\tfrac{1}{3}t_{(1)} + \tfrac{3}{3}t_{(2)} + \tfrac{5}{3}t_{(3)}\right) \\
\iff \quad & \tfrac{1}{3}E(t_{(1)} - t_{(2)}) > E(t_{(3)} - \underline{t})
\end{aligned}
$$

which is true with decreasing hazard-rates (the last expression would hold with equality with constant hazard-rates) and so proves the result for decreasing hazard-rates. □.

Notice how much bidder 1's position is weakened by the sale of the second unit in the "normal" increasing hazard-rates case. When just one unit is for sale, bidder 1 always wins it. But when there are two units for sale, his opponents' winners' curses are weakened, so his own winner's curse is strengthened and he wins barely more often than they do.

As with the symmetric case, marginal revenues help us understand these results better: When a single unit is sold it always goes to bidder 1, so the expected price equals the expected marginal revenue of bidder 1 equals the expected marginal revenue a randomly drawn signal.

When two units are sold, and hazard-rates are increasing, the winners are the bidders with the two highest signals, and the expected per-unit price equals the expected average marginal revenues of these bidders. Since increasing hazard-rates imply that the bidders with the highest signals have the highest marginal revenues, it follows that two units sell at a higher per-unit price, on average, than one unit.

When hazard-rates are decreasing, on the other hand, and two units are sold, they are won by bidder 1, whose expected marginal revenue equals that of a randomly drawn signal, and by the other bidder who has the higher of the other signals and the *lower* of the other marginal revenues. So the expected marginal revenue of a winner is lower when two units are sold than when just one unit is sold. [31]

[31] Actually this is only half the story for why with decreasing hazard-rates two units yields a lower per-unit price than a single unit. Remember that the result that the expected revenue from an auction equals the expected marginal revenue of the winner

In short, the "law of demand" fails when bidders are asymmetric and hazard rates are increasing, and when bidders are symmetric and hazard rates are decreasing.

5 Extensions

5.1 Rationing and Initial Public Offerings

Selling two half-units yields the same per-unit prices as selling two whole units in our model. So rationing by permitting each bidder to buy only a half-unit yields a higher expected price than selling a single unit in those cases in which the "law of demand" fails.[32] The intuition is that creating additional winners reduces the winners' curse that any of them face, and so elicits more aggressive bidding behaviour. By the same logic, the seller can do better still in the decreasing hazard rate case by simply offering each buyer one-third of a unit at a fixed price, or alternatively by choosing the winner randomly among those prepared to pay the fixed price.[33]

Proposition 3*:* *Rationing to all 3 bidders at the fixed price* $\underline{t} + 2E(t)$ *is more profitable than raising the price to clear the market when hazard rates are decreasing.*[34]

Proof: See appendix. $\square$.

Our proposition favoring rationing among all bidders requires decreasing

assumes any bidder with the lowest feasible signal receives no expected surplus, just as the Revenue Equivalence Theorem applies only to auctions where bidders with the worst possible signals make no money. In most standard auctions the assumption holds trivially, but our decreasing hazard rate asymmetric auction is an exception in which even the lowest type of bidder 1 always wins, makes zero expected surplus at the lowest feasible sale price, and an ever increasing expected surplus at higher sale prices. So the two-unit decreasing hazard-rate auction is even less profitable relative to any of our other auctions which do all satisfy the standard assumption. (See note 19.)

[32]Equivalently, choosing the winner randomly when two bidders remain does better than selling a single unit to the highest bidder in these cases.

[33]In the asymmetric, increasing hazard-rate, case the optimal number of bidders to ration among is two.

[34]Rationing is *strictly* more profitable than auctioning in the symmetric case, and/or when two units are available. In the asymmetric case, rationing a single unit to all three bidders at the fixed price is equally profitable in expectation as the standard ascending auction (independent of whether hazard rates are increasing or decreasing).

hazard rates. The more general point, however, is that the difference between the expected revenues from choosing a price that guarantees an immediate sell-out and from searching for the best possible price may be small; because searching for a high price may reveal some negative information (about where low bidders quit), it can lead to either a higher or a lower price than the pooling equilibrium that rationing induces. If the seller is risk averse, it may prefer the sure price that rationing guarantees.

In many finance and oil-lease models, signals are assumed to be distributed lognormally, so hazard rates are first increasing and then decreasing. In these cases, with symmetric bidders, the seller does best to gradually raise price to eliminate the buyers with the lowest signals but then ration when a high enough price is reached. This fits closely with practice in Initial Public Offerings where a range of prices may be explored, but the final price is often fixed at a point where excess demand is most likely.[35]

5.2 Restricting the number of bidders

It is also evident that when increasing supply raises price, so can restricting demand.[36]

Again the intuition is that reducing the number of bidders reduces each bidder's winner's curse. Since in a private-value ascending auction bidders follow the same strategy regardless of the number of bidders (they bid up to their true value), it should be no surprise that with common values each bidder bids more aggressively when there are fewer of them.[37] This effect can dominate the winner having a lower signal, on average, when there are fewer bidders.

As before it is quickest to see the results using marginal revenues, though we will offer proofs using more traditional methods.

When n symmetric bidders compete for one unit, the expected price equals the expected marginal revenue of the winner, equals the expected

[35]This is true even when the final IPO price is set above the initially specified range.

[36]Krishna and Morgan's (1997) Working Paper developed the results of the symmetric case of this subsection independently and simultaneously with the first draft, Bulow and Klemperer (1997), of our own paper.

[37]In the symmetric case, with two bidders i bids up to $2t_i + E(t)$. With three bidders i bids up to $2t_i + t_{(3)}$.

marginal revenue of the bidder with the highest signal among the n bidders.[38] So if hazard rates are decreasing, that is, the bidders with the higher signals have the lower marginal revenues, then the expected price is decreasing in n.

On the other hand, when bidders are asymmetric and all three bidders are present, bidder 1 is the winner and, in expectation, has the marginal revenue of a randomly selected bidder. However, when only two bidders are selected, the winner will be the bidder with the higher of their two signals when bidders 2 and 3 are selected (and the winner will be bidder 1 otherwise). So if marginal revenues are higher (lower) for the higher-signal bidders, the expected price will be higher (lower) when the number of bidders is arbitrarily restricted to two. That is, the results for the asymmetric case are again opposite to those for the symmetric case.

Proposition 4: *In the symmetric case the expected price when 1 unit is sold is higher (lower) when only 2 bidders are allowed to participate than when all 3 compete if hazard rates,* $\left(\frac{1}{h_i}\right)$, *are decreasing (increasing) in the signals.*

In the asymmetric case the opposite results apply.

Proof: See appendix. □.

In sum, restricting the number of bidders who are allowed to participate is likely to be a profitable strategy when bidders are asymmetric if hazard rates are increasing, or when bidders are completely symmetric (in what is publicly known about them) but hazard rates are decreasing.[39] So our model can explain strategies such as, for example, a merger-target opening

[38] Note that any bidder's actual marginal revenue is a function of all the other active bidders' signals, so depends on n. But with independent signals a bidder's marginal revenue, $\widetilde{MR}_i(t_i, t_j)$, when i and j are active, equals his expected marginal revenue conditional on t_i and t_j, $E_{t_k}\{MR_i(t_i, t_j, t_k)\}$, when an additional bidder k is active. So with 2 bidders and decreasing hazard rates profits are $E\min(\widetilde{MR}_1(t_1,t_2), \widetilde{MR}_2(t_1,t_2)) = E\min(E_{t_3}\{MR_1(t_1,t_2,t_3)\}, E_{t_3}\{MR_2(t_1,t_2,t_3)\}) \geq E\min(MR_1(t_1,t_2,t_3), MR_2(t_1,t_2,t_3)) > E\min(MR_1(t_1,t_2,t_3), MR_2(t_1,t_2,t_3), MR_3(t_1,t_2,t_3))$ which last expression equals expected profits when all 3 bidders are present.

[39] Note that we are assuming that the participants are chosen randomly when their numbers are restricted. Restricting numbers by requiring bidders to pay an entry fee after learning their signals would be very unprofitable for the seller, since it would select precisely those bidders (higher signals in the symmetric case, and advantaged bidders in the asymmetric case) that the seller wishes to exclude. Of course if entry fees can be imposed before bidders learn their signals, then almost all the surplus can be extracted by the seller.

negotiations with only a limited number of potential acquirers.[40]

5.3 Sealed-Bid Auctions

How are our results affected if the other of the two most common auction forms, that is, a sealed-bid or first-price auction, is used?[41]

The answer is hardly at all when bidders are symmetric, since the highest-signal bidder(s) win in any standard auction, so it follows from the Revenue Equivalence Theorem that expected revenues are the same in any standard auction. So Lemmas 1 and 2 apply in expectation, and Proposition 1 applies exactly as before.

However the outcome of a sealed-bid auction, in stark contrast to that of an ascending auction is, it is believed, almost unaffected by small asymmetries between the bidders.[42] Assuming this is true, it then follows easily that (see Appendix), *in the asymmetric case, the expected revenue is more (less) for 1 unit and is the same (more) for 2 units from a sealed-bid auction than from an ascending auction, if bidders' hazard-rates are increasing (decreasing); in the symmetric case the two auction types are always equally profitable.*

In sum, without detailed information about the distribution of bidders' signals, it is very hard to make any predictions about which of sealed-bid and ascending auctions are more profitable.[43]

[40]Of course these results contrast with our earlier work, Bulow and Klemperer (1996), which emphasised conditions under which restricting bidding is not merely undesirable for the seller, but is even a bad idea for a seller who can gain additional negotiating power by limiting participation. The point of this section is that while the conditions specified in our earlier work are very natural for private-value auctions with symmetric bidders, they are less compelling for symmetric common-value or almost-common-value auctions, and perhaps even unnatural for asymmetric almost-common-value auctions.

[41]In a sealed-bid or first-price auction for two units, bidders simultaneously and independently submit bids. The winners are the two high bidders and each pays his actual bid.

[42]To our knowledge there is no general theorem proving this, although Avery and Kagel (1997) theorem 2.6 demonstrates the results for a model that is almost a special case of ours, and Bulow, Huang, and Klemperer (1996) proves the result in a related model.

[43]Klemperer (1997) builds on the first draft of this paper, Bulow and Klemperer (1997), to discuss how auctions of PCS licenses and auctions of companies can be designed to capture the benefits that first-price auctions offer.

6 Conclusions

Economists' intuition about supply and demand comes from the partial equilibrium analysis of fully-informed buyers and sellers. These agents know the value they place on assets. In these "private-value" models, more buyers raise prices, more quantity implies a lower price, and if demand exceeds supply it always makes sense for a seller to try to raise price.

We have shown this intuition does not carry over to "common-value" settings such as financial markets where buyers have differential assessments of assets that would be valued similarly by all if they shared their information.

With symmetric agents, the standard results only occur with a rather strong distributional assumption, equivalent to what is needed for strategic substitutes in Cournot competition. When this assumption fails, setting a price that guarantees excess demand and rationing, as in Initial Public Offerings, may be more profitable than finding the price that clears the market. Furthermore, restricting entry to an auction may increase expected revenues.

With asymmetric agents the "law of demand" is violated under exactly the conditions for which it holds under symmetry. This may explain why, in the FCC's initial PCS auction, prices seemed to be low in some regions where a single license was sold, relative to markets where two licenses were available.

Appendix

A. Proof of Propositions 3 and 4

Proof of Proposition 3

When bidders are symmetric and there are two objects, the standard auction to clear the market yields per-unit profits of $2t_{(3)} + E(t \mid t \geq t_{(3)})$, by Lemma 2. But the expectation of this quantity is less than $\underline{t} + 2E(t)$ (when hazard rates are decreasing) by the argument of Proposition 2. By Proposition 1, the expected profit from auctioning a single unit in the symmetric case is still lower.

When bidders are asymmetric, auctioning two objects yields per-unit profits of $E(t) + 2\min(t_2, t_3)$ (by Lemma 4) the expectation of which is less than $\underline{t} + 2E(t)$ by the argument of Proposition 2. By Lemma 3, the profit from auctioning a single unit in the asymmetric case is $\underline{t} + t_2 + t_3$, the expectation of which equals $\underline{t} + 2E(t)$. $\square$.

Proof of Proposition 4

In the symmetric case, with three bidders the price $p^{[3]} \approx t_{(3)} + 2t_{(2)}$, by Lemma 1. When only two bidders, say i and j, are permitted to participate the loser, say bidder i, quits at price $p^{[2]} \approx 2t_i + E(t_k)$, since this is the point at which he would just be indifferent about winning conditional on being tied with bidder j. Since t_i is the lower of two signals drawn from three bidders, $E(t_i) = E\left(\frac{2}{3}t_{(3)} + \frac{1}{3}t_{(2)}\right)$ while $E(t_k) = \frac{1}{3}E\left(t_{(1)} + t_{(2)} + t_{(3)}\right)$. So $E(p^{[3]}) < E(p^{[2]}) \Leftrightarrow \frac{1}{3}E\left(3t_{(3)} + 6t_{(2)}\right) < \frac{1}{3}E\left(5t_{(3)} + 3t_{(2)} + t_{(1)}\right) \Leftrightarrow 2E\left(t_{(2)} - t_{(3)}\right) < E\left(t_{(1)} - t_{(2)}\right)$ which holds if hazard rates are decreasing (and the converses hold if hazard rates are increasing).

In the asymmetric case, $p^{[3]} \approx \underline{t} + t_2 + t_3$. If bidder 2 is excluded $p^{[2]} \approx \underline{t} + t_3 + E(t_2)$; $p^{[2]} \approx \underline{t} + t_2 + E(t_3)$ if bidder 3 is excluded; and $p^{[2]} \approx 2t_i + E(t_1)$ if bidder 1 is not permitted to participate, and bidder i is the loser (as for the symmetric case for this last case). So $E(p^{[3]}) < E(p^{[2]}) \Leftrightarrow E(t - \underline{t}) < 2E(t_i - \underline{t})$, which is true (false) if hazard-rates are increasing (decreasing). $\square$.

B. Proof of Lemmas

At a given price p and for a given history (i.e. the first quitter's quit price if there has been a quit), we write $\underline{t}_i$ for the lowest (or infimum), i.e. marginal, type of bidder i remaining in equilibrium, or write $\underline{t}_i$ for bidder i's expected signal if he has already exited, and write $w_i = (1 + \alpha_i)\underline{t}_i + \underline{t}_j + \underline{t}_k$. Write $x_i = E(t_i - \underline{t}_i \mid t_i \geq \underline{t}_i)$. (Thus $\underline{t}_i$, w_i and x_i are all functions of p and the history, but we will not usually write this dependence explicitly.) It will be convenient to write $\underline{x} = E(t_i - \underline{t})$.

Analysis of the 1 unit auction

We are looking for an equilibrium in which i stays in the bidding iff $p < w_i$. Now $\alpha_i \underline{t}_i \geq \alpha_j \underline{t}_j \Rightarrow w_i \geq w_j \Rightarrow$ type $\underline{t}_i$ of i cannot quit if type $\underline{t}_j$ of j remains in the bidding. So types t_i of i and t_j of j quit simultaneously iff $\alpha_i t_i = \alpha_j t_j$. So if $\alpha_1 \geq \alpha_2 = \alpha_3 = \alpha$, bidders $i = 2, 3$ quit according to $\underline{t}_2 = \underline{t}_3$ with t_i quitting at price $p = \underline{t} + (1+\alpha)t_i + t_i$ for $p < (1+\alpha_1)\underline{t} + \frac{\alpha_1}{\alpha}\underline{t} + \frac{\alpha_1}{\alpha}\underline{t}$, and if bidder i quits in this range, then type t_j of the other of these two bidders quits at price $p = \underline{t} + \underline{t}_i + (1+\alpha)t_j$ for $p < (1+\alpha_1)\underline{t} + \underline{t}_i + \frac{\alpha_1}{\alpha}\underline{t}$, and beyond this price bidder j and bidder 1 both quit according to $\alpha_1 \underline{t}_1 = \alpha \underline{t}_j$ and $p = (1+\alpha_1)\underline{t}_1 + \underline{t}_i + \frac{\alpha_1}{\alpha}\underline{t}_1 = \underline{t}_1 + \underline{t}_i + (1+\alpha)\underline{t}_j$ $(j = 2, 3;\ j \neq i)$. (Bidders 1 and j infer i's actual signal $\underline{t}_i$ from the price at which he quit.) No type of bidder 1 quits until $p = (1+\alpha_1)\underline{t}_1 + \underline{t}_2 + \underline{t}_3$. If neither of the other bidders quit before this price (so then $p = (1+\alpha_1)\underline{t} + \frac{\alpha_1}{\alpha}\underline{t} + \frac{\alpha_1}{\alpha}\underline{t}$ and $\alpha_1 \underline{t} = \alpha \underline{t}_2 = \alpha \underline{t}_3$), then all three bidders quit according to $\underline{t}_2 = \underline{t}_3 = \frac{\alpha_1}{\alpha}\underline{t}_1$ (and $p = w_1 = w_2 = w_3$) thereafter, and after one bidder has quit the remaining bidders l and m quit according to $\alpha_l \underline{t}_l = \alpha_m \underline{t}_m$ and $p = w_l = w_m$. It is straightforward that this is a (Perfect Bayesian) equilibrium and is unique under our assumptions.[44] Thus if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ the final price is $(1+\alpha)t_{(2)} + t_{(2)} + t_{(3)} \approx 2t_{(2)} + t_{(3)}$ for small α. If $\alpha_1 > \alpha_2 = \alpha_3 = \alpha$, then as $\left(\frac{\alpha}{\alpha_1}\right) \to 0$ the probability of bidder 1 winning approaches 1 so the final price is (almost always) $\underline{t} + t_2 + t_3 + \alpha \max(t_2, t_3) \approx \underline{t} + t_2 + t_3$. This proves Lemmas 1 and 3.

Analysis of the 2 unit auction

We look for an equilibrium in which a bidder quits when he would be just indifferent were he to find himself a winner. In such an equilibrium let $H_i(p)$

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[44] There are other equilibria that do not satisfy our assumptions. In particular,

(i) when three bidders compete for a single unit, and player 2 or 3 receives a sufficiently low signal ($\alpha_2 t_2 < \alpha_1 \underline{t}$ or $\alpha_3 t_3 < \alpha_1 \underline{t}$) that he knows he will surely lose to player 1, different equilibria can be constructed by making different assumptions about how far he bids up the price in such a case (however different assumptions then ours would not importantly affect our results).

(ii) even when just two players i and j compete for a single unit, it is an equilibrium for i to quit immediately while j never quits. (With unbounded supports of the signals, this is a Perfect Bayesian Nash equilibrium supported by j believing that if he were to observe the out-of-equilibrium behavior that i stays in to price p then i's signal is at least p; such equilibria can be ruled out by having a largest possible signal, or by insisting each player bids up at least as far as his minimum possible value given his own information.) Obviously these kinds of equilibria also arise when three bidders compete for either one or two units.

(iii) when three players compete for two units and hazard rates are decreasing, equilibria can be constructed in which the first-order conditions fail because a player initially expects to lose money conditional on winning, but he expects to make up these losses (in expectation) if the bidding continues for a while. These equilibria seem particularly unnatural since they require symmetric players to behave asymmetrically.

be the hazard-rate with which i quits at price p, that is,

$$H_i(p) = \frac{\underline{t}_i'(p) f(\underline{t}_i(p))}{1 - f(\underline{t}_i(p))}.^{45}$$

So type t_i of i quits when

$$p = (1+\alpha_i)t_i + \underline{t}_j + \underline{t}_k + x_j \left(\frac{H_k}{H_j + H_k}\right) + x_k \left(\frac{H_j}{H_j + H_k}\right)$$

that is, the price equals i's expected value conditional on winning since, in this case, with probability $\left(\frac{H_j}{H_j+H_k}\right)$ it is j who has quit so $t_j = \underline{t}_j$ and $E(t_k) = \underline{t}_k + x_k$.

We begin with the asymmetric case. Let $\alpha_1 > \alpha_2 = \alpha_3 = \alpha$.

Increasing hazard rates

Begin with the standard case in which the hazard rate, $\frac{f(t_i)}{1-f(t_i)}$ is increasing in t_i, so x_i is decreasing in $\underline{t}_i$. No-one quits until $p = (3+\alpha)\underline{t}+\underline{x}$, at which price the lowest types of bidders 2 and 3 quit. Since 2 and 3 behave symmetrically, $\underline{t}_2 = \underline{t}_3$, and $p = \underline{t} + (2+\alpha)\underline{t}_2 + \underline{x}$ until $\alpha_1\underline{t} + x_2 = \alpha\underline{t}_2 + \underline{x}$ at which price, say $\underline{p}$, bidder 1 is also just indifferent about finding himself a winner. For types of all three bidders to be quitting simultaneously, we require

$$\begin{aligned}
\alpha_1\underline{t}_1 + x_2\left(\frac{H_3}{H_2+H_3}\right) + x_3\left(\frac{H_2}{H_2+H_3}\right) &= \alpha\underline{t}_2 + x_1\left(\frac{H_3}{H_1+H_3}\right) + x_3\left(\frac{H_1}{H_1+H_3}\right) \\
&= \alpha\underline{t}_3 + x_1\left(\frac{H_2}{H_1+H_2}\right) + x_2\left(\frac{H_1}{H_1+H_2}\right),
\end{aligned}$$

so using the symmetry of bidders 2 and 3 yields

$$\alpha_1\underline{t}_1 + x_2 = \alpha\underline{t}_2 + x_1\left(\frac{H_2}{H_1+H_2}\right) + x_2\left(\frac{H_1}{H_1+H_2}\right)$$

$$\Rightarrow \quad \frac{\alpha_1\underline{t}_1 - \alpha\underline{t}_2}{x_1 - x_2} = \frac{H_2}{H_1+H_2} = \frac{1}{\frac{d\underline{t}_1}{d\underline{t}_2}\frac{h_2}{h_1} + 1} \Rightarrow \frac{d\underline{t}_1}{d\underline{t}_2} = \frac{h_1}{h_2}\left[\left(\frac{x_1 - x_2}{\alpha_1\underline{t}_1 - \alpha\underline{t}_2}\right) - 1\right]$$

in which $h_i(\underline{t}_i) \equiv \frac{1-f(\underline{t}_i)}{f(\underline{t}_i)}$ (i.e., as defined in the Model Section). Since $h_i(\underline{t}_i)$ is finite everywhere this yields $\underline{t}_1$ as a continuous upward-sloping function of $\underline{t}_2$, that is, $\infty > \frac{d\underline{t}_1}{d\underline{t}_2} \geq 0$ everywhere, and $\underline{t}_2 > \underline{t}_1, x_1 > x_2$ and $\alpha_1\underline{t}_1 > \alpha\underline{t}_2$ everywhere. (As $(\alpha_1\underline{t}_1 - \alpha\underline{t}_2) \to 0$, $\frac{d\underline{t}_1}{d\underline{t}_2} \to \infty \Rightarrow \alpha_1\underline{t}_1 > \alpha\underline{t}_2$ everywhere. As

[45] More precisely, the equilibrium functions $\underline{t}_i(p)$ are defined by the equilibrium hazard-rates $H_i(p)$, since otherwise $\underline{t}_i'(\cdot)$ might not be defined by $\underline{t}_i(\cdot)$. The condition that a bidder quits when he is just indifferent about winning ensures that $H_i(p)$ is finite, that is, $\underline{t}_i(p)$ is single-valued and continuous.

$(x_1 - x_2) - (\alpha_1 \underline{t}_1 - \alpha \underline{t}_2) \to 0, \frac{dt_1}{d\underline{t}_2} \to 0$ so also $\frac{dx_1}{d\underline{t}_2} \to 0$ while $\frac{dx_2}{d\underline{t}_2} < 0, \Rightarrow$ $(x_1 - x_2) - (\alpha_1 \underline{t}_1 - \alpha \underline{t}_2) > 0$ everywhere when $\underline{t}_1 > \underline{t}$. So also $\infty > \frac{d\underline{t}_1}{d\underline{t}_2} > 0$ when $\underline{t}_1 > \underline{t}$.)

Does the (unique) solution to this differential equation define a (Perfect Bayesian) Nash equilibrium? To see that it does, first note that having $\underline{t}_1$ as a function of $\underline{t}_2$ (uniquely) defines $\underline{t}_1(p)$ and $\underline{t}_2(p)$ using $p = (1+\alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2$ (since $\underline{t}_1$ and $\underline{t}_2 + x_2$ and hence p are all continuous and upward-sloping functions of $\underline{t}_2$). Now assume bidders 2 and 3 bid according to $\underline{t}_2(p)$ (and $\underline{t}_3(p) = \underline{t}_2(p)$). Then type t_1 of bidder 1's profits from finding himself a winner at price p are $p - ((1+\alpha_1)t_1 + 2\underline{t}_2 + x_2) = (1+\alpha_1)(\underline{t}_1 - t_1)$ and we have shown $\underline{t}_1$ is continuous and increasing in p for $p \geq \underline{p}$ (i.e. where $\underline{t}_1 > \underline{t}$) so type t_1's uniquely optimal strategy is to quit at $t_1 = \underline{t}_1$. Similarly, assume bidders 1 and 3 bid according to $\underline{t}_1(p)$ and $\underline{t}_2(p)$, respectively. Then type t_2 of bidder 2's profits from finding himself a winner at price p are

$$p - \left((1+\alpha)t_2 + \underline{t}_1 + \underline{t}_2 + x_1\left(\frac{H_2}{H_1+H_2}\right) + x_2\left(\frac{H_1}{H_1+H_2}\right)\right) = (1+\alpha)(\underline{t}_2 - t_2)$$

and $\underline{t}_2$ is continuous and increasing in p (for $p \geq \underline{p}$ from our analysis of the differential equation, and for $p \in ((3+\alpha)\underline{t} + \underline{x}, \underline{p})$ from our earlier argument). So type t_2 quits at $t_2 = \underline{t}_2$. So our equations define a (Perfect Bayesian) Nash equilibrium.[46]

Finally, it is easy to check that there are no other candidate equilibria in the increasing hazard-rate case. At any price after $\underline{p}$ (at which price

$$p - (\underline{t}_1 + \underline{t}_2 + \underline{t}_3) = \alpha_1 \underline{t}_1 + x_2 = \alpha \underline{t}_2 + x_1\left(\frac{H_2}{H_1+H_2}\right) + x_2\left(\frac{H_1}{H_1+H_2}\right)$$

is first satisfied) it yields a straightforward contradiction for there to be no types of 1 quitting, or no types of 2 and 3 quitting, or no types of any of 1, 2 and 3 quitting, as the price rises.[47]

So the equilibrium we have found is unique under our assumptions. Finally, note from the differential equation that $x_1 - x_2$ is of order $\alpha_1 \underline{t}_1 - \alpha \underline{t}_2$. (Since h_i is an inverse hazard rate, $\frac{h_1}{h_2} > 1$, so $\frac{d\underline{t}_1}{d\underline{t}_2} > \left(\frac{x_1 - x_2}{\alpha_1 \underline{t}_1 - \alpha \underline{t}_2} - 1\right)$, so $x_1 - x_2$ cannot become much larger than $\alpha_1 \underline{t}_1 - \alpha \underline{t}_2$ without $\frac{d\underline{t}_1}{d\underline{t}_2}$ becoming large so reducing $x_1 - x_2$.) So as $\alpha_1 \to 0$, $x_1 \to x_2$ and so $\underline{t}_2 \to \underline{t}_1$ along the equilibrium path.[48] So the winners are almost always the bidders with the higher signals,

[46] If there is an upper bound, $\bar{t}$, on t_i, then above the price where $\underline{t}_2 = \underline{t}_3 = \bar{t}$, we can continue to define 1's strategy according to $p = (1+\alpha)\underline{t}_1 + \bar{t} + \bar{t} + 0$.

[47] If 2 and 3 alone stop quitting, their marginal types would earn $(\underline{t}_1 + \underline{t}_2 + \underline{t}_3) + \alpha_2 \underline{t}_2 + x_2 - p$, i.e. strictly lose money in expectation, if they found themselves winners; if 1 alone stops quitting the marginal types of 2 and 3 would earn $(\underline{t}_1 + \underline{t}_2 + \underline{t}_3) + \alpha_2 \underline{t}_2 + x_1 - p$ by winning so would also stop quitting; if all stopped quitting 1 would earn $(\underline{t}_1 + \underline{t}_2 + \underline{t}_3) + \alpha_1 \underline{t}_1 + x_2 - p$ so 1 would instead continue to quit.

[48] More precisely, $\forall \varepsilon, \forall K, \exists\ \delta\ s.t.\{\alpha_1 < \delta \Rightarrow \mid \underline{t}_2 - \underline{t}_1 \mid < \varepsilon \quad \forall \quad \underline{t}_2 < K\}$. To see this let $\min_{0 \leq t_i \leq K}\{-x_i'(t_i)\} = \phi > 0$ (this minimum exists since $-x_i'(t_i) = 1 - x_i(t_i)/(h_i(t_i))$ and

and the price is almost always set by the bidder with the lowest signal, $t_{(3)}$, who quits at $\approx (1+\alpha)t_{(3)} + t_{(3)} + E(t \mid t \geq t_{(3)})$.

Decreasing hazard rates

As for the increasing hazard-rate case, no-one quits until $p = (3+\alpha)\underline{t} + \underline{x}$ at which price the lowest types of bidders 2 and 3 start quitting symmetrically according to $\underline{t}_2 = \underline{t}_3$ and $p = \underline{t} + (2+\alpha)\underline{t}_2 + \underline{x}$. Now with decreasing hazard rates as $\underline{t}_2$ increases so does x_2 so if, as we assume, α is small, $(1+\alpha_1)\underline{t} + 2\underline{t}_2 + x_2 > \underline{t} + (2+\alpha)\underline{t}_2 + \underline{x} = p$ for all $\underline{t}_2$. That is, for bidder 1 to never quit while bidders 2 and 3 quit symmetrically satisfies the first-order conditions for equilibrium everywhere. It is straightforward that this also defines a (Perfect Bayesian) Nash equilibrium: if bidders 2 and 3 bid according to $\underline{t}_2(p)$ no types of player 1 ever wish to quit. If no type of bidder 1 ever quits, while bidder 3 bids according to $\underline{t}_2(p)$, then the expected profits of type t_2 of bidder 2 if he finds himself a winner at price p are $p - ((1+\alpha)t_2 + \underline{t} + \underline{t}_2 + \underline{x}) = (1+\alpha)(\underline{t}_2 - t_2)$ which is continuous and increasing in p so t_2 optimally quits at $t_2 = \underline{t}_2$.

Are there any other equilibria in the decreasing hazard-rate case? Clearly, as the price rises with $p = \underline{t} + (2+\alpha)\underline{t}_2 + \underline{x}$ there is no point at which some types of 1 start quitting. (Their expected values from being a winner always exceed the price.) However we need to consider the possibility that at some price at or above $(3+\alpha)\underline{t} + \underline{x}$ no types of any players are quitting. This would require beliefs that conditional on the out-of-equilibrium event that player 2 does find himself a winner, he believes that player 3 quit with probability $\leq \lambda$ where $p = \underline{t} + (2+\alpha)\underline{t}_2 + \lambda\underline{x} + (1-\lambda)x_2$. Note that as p rises, λ falls since $x_2 > \underline{x}$. So no types of players 2 and 3 can ever start quitting again unless types of player 1 also do, since if the marginal types of 2 and 3 (but not 1) quit their expected value conditional on being a winner is $\underline{t} + (2+\alpha)\underline{t}_2 + \underline{x} - p = (1-\lambda)(\underline{x} - x_2) < 0$, so an atom of types of 2 and 3 wish to quit, so (almost) all of these types lose money conditional on winning, which is a contradiction. Now one possibility is $\alpha_1\underline{t} > \alpha\underline{t}_2$, so no types of player 1 would ever quit since their expected values from being a winner exceed $(1+\alpha_1)\underline{t} + 2\underline{t}_2 + x_2 > \underline{t} + (2+\alpha)\underline{t}_2 + \lambda\underline{x} + (1-\lambda)x_2 = p$, $\forall\lambda \in [0,1]$. In this case we have a contradiction at the price that yields $\lambda = 0$.[49] (The price cannot rise above this price without at least the marginal types of 2 wishing to quit, so an atom of types 2 and 3 wish to quit (as above), so (almost) all of these lose money conditional on winning which is a contradiction.) Another possibility is $\alpha_1\underline{t} = \alpha\underline{t}_2$. In this case the marginal types of 1 also wish to quit at $\lambda = 0$. But the price cannot then rise higher without any types of 2 and 3 quitting since 1's marginal condition would imply $p = (1+\alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2 \geq \underline{t}_1 + (2+\alpha)\underline{t}_2 + x_2$ which implies types

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$f_i(t_i)$ and hence $-x_i'(t_i)$ is continuous, and $\phi > 0$ since $x_i(t_i) < h_i(t_i)$ follows from the increasing hazard-rate). So $x_1 - x_2 > \phi(\underline{t}_2 - \underline{t}_1)$. So if $\delta < \phi\varepsilon/4K$ then $(\underline{t}_2 - \underline{t}_1) > \varepsilon/2 \Rightarrow \frac{d\underline{t}_1}{d\underline{t}_2} > \frac{\phi(\varepsilon/2)}{\phi\varepsilon/4} - 1 = 1 \Rightarrow \underline{t}_2 - \underline{t}_1$ can never increase above $\varepsilon/2$.

[49] This price is reached with positive probability (as are all prices) since the hazard rate is decreasing.

of 2 and 3 must quit, but for any (actual) relative probability $\lambda = \left(\frac{H_3}{H_1+H_3}\right)$ with which player 2 believes that another player who quits is player 3, player 2's expected value from winning is $\underline{t}_1 + (2+\alpha)\underline{t}_2 + \lambda x_1 + (1-\lambda)x_2 < p$ so an atom of types of 2 and 3 must quit, which is a contradiction, as before. Finally, we may have $\alpha_1 \underline{t} < \alpha \underline{t}_2$ at the price at which types of 2 and 3 stop quitting. In this case types of player 1 start quitting at price $p = (1+\alpha_1)\underline{t} + 2\underline{t}_2 + x_2 = \underline{t} + (2+\alpha)\underline{t}_2 + \lambda \underline{x} + (1-\lambda)x_2$ for some $\lambda \in (0,1)$. At this price the marginal types of players 2 and 3 must also start quitting at hazard rates such that $\lambda = \left(\frac{H_2}{H_1+H_2}\right)\left(= \left(\frac{H_3}{H_1+H_3}\right)\right)$.(If not, the marginal types of players 2 and 3 would either be strictly losing or strictly making money (in expectation) conditional on winning. Both are contradictions; the latter because the types just below the current marginal types of 2 and 3 would not have been willing to quit earlier where their first-order conditions were satisfied.) Now where types of 1 start quitting we have $\underline{t}_1 = \underline{t} < \left(\frac{\alpha}{\alpha_1}\right)\underline{t}_2$ and $x_2 > x_1$ so when α_1 is small we require λ small, that is, $\frac{H_1}{H_2}$ large, hence $\frac{d\underline{t}_1}{d\underline{t}_2}$ is large. So $\alpha_1 \underline{t}_1 - \alpha \underline{t}_2 \to 0$ and $\alpha_1 \underline{t}_1 = \alpha \underline{t}_2$ is achieved for finite $\underline{t}_2$. (Until this point $\underline{t}_1$ and $\underline{t}_2$ must just be following the differential equation determined by $p = (1+\alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2 = \underline{t}_1 + (2+\alpha)\underline{t}_2 + \left(\frac{H_2}{H_1+H_2}\right)x_1 + \left(\frac{H_1}{H_1+H_2}\right)x_2$, that is, the same differential equation as in the increasing hazard-rate case.) But at $\alpha_1 \underline{t}_1 = \alpha \underline{t}_2$, and hence $\left(\frac{H_2}{H_1+H_2}\right) = 0$, we have the same contradiction that we had with $\alpha_1 \underline{t} = \alpha \underline{t}_2$ and $\lambda = 0$. (Any finite rate of quitting of player 2 would imply all types close to $\underline{t}_2$ strictly wished to quit which is a contradiction, but if no types of player 2 quit as the price, and hence $\underline{t}_1$, rises, then we will have $p = (1+\alpha_1)\underline{t}_1 + 2\underline{t}_2 + x_2 > \underline{t}_1 + (2+\alpha)\underline{t}_2 + x_2$ which is also a contradiction.)

So the equilibrium we found, in which player 1 never quits while players 2 and 3 quit symmetrically according to $\underline{t}_2 = \underline{t}_3$ and $p = \underline{t} + (2+\alpha)\underline{t}_2 + \underline{x}$, is the unique (Perfect Bayesian) Nash equilibrium satisfying our assumptions, and the final price is $\underline{t} + \underline{x} + (2+\alpha)\min(t_2, t_3) = E(t) + (2+\alpha)\min(t_2, t_3) \approx E(t) + 2\min(t_2, t_3)$ in which t_2 and t_3 are the actual signals of bidders 2 and 3.

The Symmetric Case

When $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ it is straightforward that it is a (Perfect Bayesian) equilibrium for bidders to quit according to $\underline{t}_1 = \underline{t}_2 = \underline{t}_3$ and $p = (3+\alpha)\underline{t}_1 + x_1$, and that this is the unique equilibrium satisfying our assumptions. In this case the final price is $(3+\alpha)t_{(3)} + x_{(3)} \approx 2t_{(3)} + E(t \mid t \geq t_{(3)})$.

Thus we have proved Lemmas 2 and 4.

C. Comparison of Sealed-Bid and Ascending Auctions

In the asymmetric case, when 1 unit is sold, the ascending auction yields $\approx \underline{t} + 2E(t) = \underline{t} + \frac{2}{3}E\left(t_{(1)} + t_{(2)} + t_{(3)}\right)$ in expectation (Lemma 3). The sealed-bid auction yields $\approx E\left(t_{(3)} + 2t_{(2)}\right)$ in expectation — we assume the conjecture in Section 5.3 that the expected revenue from the sealed-bid auction is almost unaffected by the small asymmetries between the bidders,[50] so is almost Revenue Equivalent to the situation in Lemma 1. Furthermore, $E\left(t_{(3)}+2t_{(2)}\right) > \frac{2}{3}E\left(t_{(1)}+t_{(2)}+t_{(3)}\right)+\underline{t} \Leftrightarrow E\left(9t_{(3)}+6\left(t_{(2)}-t_{(3)}\right)\right) > E\left(6t_{(3)}+4\left(t_{(2)}-t_{(3)}\right)+2\left(t_{(1)}-t_{(2)}\right)\right)+3\underline{t} \Leftrightarrow E\left(3(t_{(3)}-\underline{t})+2\left(t_{(2)}-t_{(3)}\right)\right) > E\left(2\left(t_{(1)}-t_{(2)}\right)\right)$ which is always true (false) if hazard-rates are increasing (decreasing).[51]

When 2 units are sold the sealed-bid auction yields $\approx E\left(2t_{(3)}+E\left(t|\, t \geq t_{(3)}\right)\right)$ in expectation, assuming approximate Revenue Equivalence to the situation in Lemma 2. The ascending auction yields the same in expectation if hazard-rates are increasing, but $\approx E(E(t)+2\min(t_2,t_3))$ in expectation if hazard-rates are decreasing (Lemma 4). But $E\left(2t_{(3)}+E\left(t|\, t \geq t_{(3)}\right)\right) = E(2t_{(3)}+\frac{1}{2}t_{(1)}+\frac{1}{2}t_{(2)})$ and $E(E(t)+2\min(t_2,t_3)) = E(\frac{1}{3}t_{(1)}+\frac{3}{3}t_{(2)},+\frac{5}{3}t_{(3)})$ (see the proof of Proposition 2) and $E(2t_{(3)}+\frac{1}{2}t_{(1)}+\frac{1}{2}t_{(2)}) > E(\frac{1}{3}t_{(1)}+\frac{3}{3}t_{(2)}+\frac{5}{3}t_{(3)}) \Leftrightarrow E(t_{(1)}-t_{(2)}) > E(2(t_{(2)}-t_{(3)}))$ which is always true with decreasing hazard rates.

[50] See note 42 of the text.

[51] Thinking about marginal revenues is the quickest way to see the result for the 1 unit case, since the sealed bid and ascending auctions yield the expected marginal revenue of the highest-signal bidder and the average bidder, respectively. For the 2 unit, decreasing hazard-rate case, however, the calculations are trickier since this is the special case in which the ascending auction gives positive expected surplus to the lowest type of bidder 1 (see notes 19 and 31 of the text), so the expected revenue from the ascending auction is the sum of the expected marginal revenues of the winning bidders minus this expected surplus.

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