Fixed Prices versus Haggling: Optimal Selling Strategies of the Multiproduct Monopolist

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Abstract

In this paper we prove the result that a multiproduct monopolist's optimal selling strategy is deterministic. Therefore, neither lotteries nor any haggling will be beneficial to a monopolist selling multiple products to consumers with unit demands for each component good. This result holds even given any degree of complementarity or substitutability in consumer demand. The result has been achieved by breaking the problem down into two sub-problems. We firstly show that if the monopolist's globally optimal pricing policy is to offer a schedule of lotteries then the monopolist can always improve her profits from the optimal deterministic prices by introducing some single lottery whose price can be perturbed by some small amount from the level at which consumers are just indifferent to it. In other words, we first show that the monopolist's profit function is in a certain sense concave *in an appropriate selling strategy space*. We are then able to go a step further and show that a multiproduct monopolist would never actually benefit from experimenting by adding a single lottery option to the optimal deterministic prices. These two results combine to prove the main result which we capture as Theorem 1.

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1 Introduction

Firms show a great deal of ingenuity in their selling strategies. Many firms post fixed take it or leave it prices but this is only one tool in a large armoury. Examples of firms offering consumers the chance to enter into a 'prize draw' are also common. Still more common are firms entering into bargaining games with potential consumers. Both bargaining and prize draws are all forms of lottery in which either the goods the consumer receives or the price she pays for them is random. The question is which of these strategies, if any, are the most profitable for a multiproduct monopolist: can lotteries enhance monopoly profits or are fixed prices best? In making her choice between fixed prices and lotteries a monopolist is attempting to balance two basic considerations. On the one hand consumers who do not value a product highly might not be prepared to pay for it outright, but are prepared to pay a lot less for the chance of receiving it. Similarly consumers who do value the products will be less keen to lose them through a lottery or by bargaining. The lotteries therefore act to separate the consumers and so would allow the monopolist to increase her profits. On the other hand, the availability of the lottery might encourage some consumers who were going to purchase a product to try and save money by taking a gamble on receiving it through the lottery. This effect would act against the monopolist and tend to depress monopoly profits. Which of these effects dominates has thus far remained an open question. Different aspects of the literature seem to provide circumstantial evidence for either of the two effects being the stronger and so economists are divided as to the reality. This paper will show that we are able to make a great deal of progress towards the heart of the matter.

In essence the monopolist's decision boils down to what is known as a 'screening problem' in the economics literature. The monopolist can use the probability that a good is received as an instrument to allow it to sort consumers into groups: consumers reveal their valuations for a product when they choose one lottery over another. Price discrimination techniques could then be used to extract more surplus from the consumers. The issue here is whether the cost to the monopolist of using the screening instrument at all outweighs any benefits it can then receive through price discrimination.

The first major inroad into this problem was made by Riley and Zeckhauser [12]. They considered a one good monopolist selling to a population of consumers. This simplification meant that consumers were not faced with a choice between different components and had no complementarities in demand: the issue of goods being complementary or substitutable was therefore assumed away. This restriction to one good also had the effect of causing all consumers to differ in only one variable. All the individuals of the population could therefore be ordered on a line. In this context Riley and Zeckhauser [12] were able to tackle the problem directly. The key to their work is to use an integration by parts technique coined in Mirrlees [8] which is common to the screening literature in one dimension. With this tool Riley and Zeckhauser [12] construct an elegant geometrical proof that the optimal probability of sale function takes only the values 0 or 1 and so must correspond to a take-it-or-leave-it offer. In other words, lotteries do not help a single good monopolist. However, the integration by parts technique famously does not hold in multiple dimensions. This immediately led to speculation that the Riley and Zeckhauser [12] result might be specific to the case in which all consumers can be ordered on a line.

The work of Rochet and Choné [14] seemed to support this view. They consider the general multiple dimensional screening problem in which a multiproduct monopolist sells her goods with a continuous range of product qualities to consumers who are characterised by a multiple dimensional type vector. Rochet and Choné [14] are able to find parallels between the monopolist's profit maximisation problem and what is known as an *obstacle problem* in physics. By using the insights afforded from the physics literature they are able to reduce the screening problem to a set of partial differential equations. These partial differential equations often can only be solved numerically. However, Rochet and Choné [14] were able to show that in most such problems the monopolist will choose to bunch some consumers. That is to offer a group of consumers with different valuations the same quality of products. This work however shows the general multidimensional screening problem to be incredibly complicated. It seemed very unlikely indeed that in the complicated multiproduct case lotteries could be ruled out with a Riley and Zeckhauser [12] type result.

Further support for this view came from a number of simple theoretical examples. It can be shown that if a multiproduct monopolist is serving a discrete market with only two types of buyer whose types are carefully chosen then a random delivery rule can be optimal: a specific example of this is given towards the end of this paper on page 36. In other words, using lotteries to screen the consumers in these examples is the monopolist's optimal pricing policy. Many economists felt that examples such as the one on page 36 were not a result of the contrived nature of the market but indicative of a general result: haggling is surely optimal for the *multi*-product monopolist.

Attempts were however made to extend the 'no lottery' result of Riley and Zeckhauser [12] to the multiproduct monopoly case. Possibly the best of these was McAfee and McMillan [6]. McAfee and McMillan [6] attempted to extend the Riley and Zeckhauser [12] proof to multiple consumer characteristics directly. As mentioned above, the integration by parts method of Mirrlees [8] did not extend to multiple dimensions. Instead, McAfee and McMillan [6] ruled out complementarities in demand (such as those between TVs and VCRs) and tried to consider the problem as the sum of one-dimensional elements. However, this proved to be a very complicated process and so the proof is often thought to be opaque in parts. To allow the problem to be broken up in this way McAfee and McMillan [6] make a second concession which is to restrict the distribution of consumers' valuations to satisfy a specific hazard rate condition. Having done this McAfee and McMillan [6] reduce their restricted problem to a series of mathematical conditions. However, they acknowledge that these conditions are difficult to interpret and so restrict the problem further to the case of a two good monopolist. Here, they are able to finally claim that lotteries are not beneficial to the monopolist and so the Riley and Zeckhauser [12] result does extend, albeit in their special case. This result was however unsatisfactory for all the restrictions that were employed and left many economists still dubious as to the merits or otherwise of lotteries in general situations.

Work in a different area by Rasul and Sonderegger [11] tilted the question of lotteries in a different direction. They looked at the context of consumers as agents contracting with the monopolist or, in their work, the principal. What is crucial now is that the consumers (agents) have outside options which depend on their types. This is because Rasul and Sonderegger [11] model the situation in which the agents must make relationship specific investments before they can contract with the principal.[1] Specifically, consumers with high valuations (agents of high type) who do not trade with the principal lose a great deal more than consumers with low valuations. In the Rasul and Sonderegger [11] model this is because of the differing opportunity costs of no trade across the agents. This assumption differs markedly with the standard approach to monopoly situations in which if a consumer receives nothing their utility is zero. Rasul and Sonderegger [11] now find that using lotteries are profit maximising for the principal, even in the one dimensional case. So what has changed? Previously consumers with high valuations might have been tempted to pay less for a monopolist's products and take a gamble on receiving them at all. In the Rasul and Sonderegger [11] context when these high valuation consumers

[1] The Rasul and Sonderegger [11] work is motivated by the automobile industry in which the component manufacturing 'agent' firms make relationship specific investments before they can trade with the principal who actually puts the cars together.

lose the lottery they don't just receive nothing, they are actively hurt. This of course deters high valuation consumers from using the lottery option and so prevents profit loss along this avenue. On the other side of the equation low valuation consumers who wouldn't have participated if no lottery is offered are still tempted to try it. If they lose they are hardly hurt at all as Rasul and Sonderegger [11] have a type dependent outside option. The balance of the two forces we discussed above has therefore clearly been pushed in favour of lotteries and consequently lotteries are found to be beneficial.

The work below seeks to help us to fill in the gap of understanding concerning the standard monopolist's optimal pricing strategy. We analyse the general case of a multiproduct monopolist selling her goods to consumers with unit demands for the component goods and arbitrary complementarities in demand between different components. As is standard in the literature we suppose that any consumer who doesn't make a purchase receives nothing and derives no utility. Importantly we assume that given any combination of the monopolist's goods at the best fixed prices, there exists some (possibly small) probability of a consumer being indifferent between them. We are able to show that in this situation lotteries are not beneficial to the multiproduct monopolist: the Riley and Zeckhauser [12] does extend and the cost to the monopolist of using probabilities as a screening tool outweighs its usefulness. We do not need to make any restrictive assumptions on consumer demand or on the hazard rate of the consumer density function. The full model is established in Section 2. The result is expressed formally as Theorem 1 and is proved in two stages in Sections 3 and 4.

The results of this paper are all derived in the context of a one period model, an approach which is standard in this literature. This therefore specifically rules out repeat interactions: if a consumer purchases a lottery and fails to receive the good she wanted she is unable to make a second purchase in the modelled time period. This is clearly an important restriction and happily also not a very restrictive one. This approach could be said to model situations in which: (a) The multiproduct monopolist is selling a non durable or experience good, such as access to a pay TV program. In this example the consumer will have missed the program when she discovers that she has not been successful with the lottery. (b) The monopolist is selling a durable good but to a transient population. This could be because of substantial travel or contractual costs deterring a consumer from returning. Alternatively, this could be a model of a store in an airport terminal for example. (c) Finally, this model encompasses any haggling or bargaining game in which if the consumer bargains and the negotiations break down then the consumer is unable to return to the monopolist in the modelled time period and restart negotiations. In other words, the monopolist can precommit to any given bargaining strategy.

We will therefore establish that using lotteries is not a beneficial strategy for a multiproduct monopolist in a very general model. Section 5 addresses how this result is changed as the underlying characteristics of the model are altered and considers further extensions. Section 6 concludes.

2 The Model

A multiproduct monopolist sells k individual component goods to a population of consumers. The goods can be sold individually or as part of a bundled package. The monopolist therefore has $K := 2^k - 1$ possible combinations to sell. The figure K is derived from the size of the power set of a set with k elements (that is the number of possible subsets). We remove the empty set (the -1 above) as the firm cannot sell an empty bundle. The consumers have unit demand for each of the k components with each consumer denoted by a type vector $\underline{x} \in \mathbb{R}_+^K$. Each coefficient of the type vector, x_i, gives the consumer's monetary valuation for the bundle $i \in \{1, \ldots, K\}$. These valuations are private information to the consumer. We note that by modelling consumer types in this way we make no assumptions about any complementarities in demand which might exist between one component good and the next. In particular, consumers can see some bundles of goods as complements and others as substitutes.

The consumers are risk neutral and are drawn from a population whose size we normalise to 1. The population is supported on the set $\Omega \subseteq \mathbb{R}_+^K$. We suppose that $\underline{0} \in \Omega$. That is, we require the population of consumers to contain some individuals who do not value the monopolist's products at all. For technical reasons we will require the set Ω to be open, bounded and convex. This guarantees that given consumers of any two distinct types then there will exist consumers in the population who have any intermediate valuations for the bundles of goods. The density of the consumers in the set Ω is given by the function $f(\cdot) : \Omega \to (0, \infty)$. The function f will be assumed to have the standard characteristics of a probabilistic density function and furthermore is bounded and differentiable with bounded derivatives. Finally we assume that the firm, though ignorant of any individual consumer's type, knows the density function $f(\cdot)$.

The model is a one stage game with the population of consumers deciding whether or not to make a purchase from the monopolist. Each consumer can decide to not make a purchase from the monopolist and would receive a utility of 0 in this case. Otherwise each consumer will choose from the options available the one which maximises their utility. There are no repeated interactions in this model. In particular, if a consumer decides to purchase a lottery which only delivers the good with a certain probability then she will not be able (during the modelled time period) to return to the monopolist to make a second purchase should she fail to receive the good. Similarly, if a consumer bargains with a monopolist and the negotiations break down, the consumer can't return to the monopolist during the modelled time period.

On the production side the monopolist is risk neutral, faces a unit cost of c_i for the manufacture of bundle i and has no capacity constraints. This cost structure is very general and allows the firm to enjoy any economies of scope in production. We assume that there are no fixed costs. This is not, however, an important assumption as the presence of a fixed cost would not alter the firm's optimal pricing strategy conditional on it entering the market at all.

The monopolist of this model has two general selling strategies which she can employ:

1. Deterministic Selling Strategies (i.e. Fixed Prices)

   In this situation the monopolist issues a menu of K fixed prices collected into a price vector $\underline{p} \in \mathbb{R}_+^K$ with one price for each of the K individual bundles and p_i denoting the price for bundle i. The monopolist can therefore provide product A for one price, or product B for another or indeed products A and B together for a third price. By choosing bundle i for example, a consumer would pay p_i in return for receiving bundle i with certainty. A consumer of type $\underline{x}$ will therefore derive utility $\max_i \{0, x_i - p_i\}$ and will choose the bundle i which is the $\arg\max_i \{0, x_i - p_i\}$.

2. Stochastic Selling Strategies

   (a) Lotteries

   The monopolist offers the consumers a schedule of lotteries over the K different possible bundles with an associated schedule of prices. In other words, the consumer would choose a lottery, that is a vector of probabilities, $\underline{q}$, lying in a given region $Q \subset [0,1]^K$. Each coefficient, q_i, of the vector $\underline{q}$ gives the probability that the consumer will receive the i^{th} bundle. Crucially we therefore require:

   $$q_i \in [0,1] \ \forall i \quad \text{and} \quad \sum_{i=1}^{K} q_i \leq 1 \tag{1}$$

   The coefficients need not sum to one as there could be a positive probability that the consumer will actually receive nothing. The sum of the probabilities must not exceed one however. The price of the chosen lottery will be given by a transfer function $t : Q \to \mathbb{R}_+$. The consumer must pay $t\left(\underline{q}\right)$ to purchase lottery $\underline{q}$. Once purchased the firm will award the consumer bundle i with probability q_i.

The consumers will choose the lottery which maximises their utility and so we can derive the allocation functions:

$$\begin{aligned} \underline{q}(\underline{x}) &: \left[\text{ Consumer type }\right] \to \left[\text{ Lottery chosen }\right] \\ t(\underline{x}) &: \left[\text{ Consumer type }\right] \to \left[\text{ Price paid }\right] \end{aligned}$$

The monopolist's expected profit will therefore be

$$E(\text{profit}) = \int_{\text{consumer types}} \left[t(\underline{x}) - \underline{q}(\underline{x}) \cdot \underline{c}\right] f(\underline{x})\, d\underline{x} \tag{2}$$

(b) Haggling

The monopolist can announce a (probabilistic) selling strategy. Each consumer will have an optimal response which will depend both on their type, $\underline{x}$, as well as the monopolist's selling strategy. These strategies will determine the probability a consumer receives a particular bundle which can be summarised by the vector $\underline{q}(\underline{x})$. The component $q_i(\underline{x})$ captures the probability a consumer of type $\underline{x}$ receives bundle i. The strategies will also define the expected price vector, $\underline{\widetilde{p}}(\underline{x})$, conditional on sale. Hence a consumer of type $\underline{x}$ would expect to pay $\widetilde{p_i}(\underline{x})$ conditional on receiving bundle i.

The monopolist's expected profit will therefore be

$$\begin{aligned} E(\text{profit}) &= \int_{\text{consumer types}} \underline{q}(\underline{x}) \cdot \left[\underline{\widetilde{p}}(\underline{x}) - \underline{c}\right] f(\underline{x})\, d\underline{x} \\ &= \int_{\text{consumer types}} \left[t(\underline{x}) - \underline{q}(\underline{x}) \cdot \underline{c}\right] f(\underline{x})\, d\underline{x} \end{aligned} \tag{3}$$

where we have set $t(\underline{x}) = \underline{q}(\underline{x}) \cdot \underline{\widetilde{p}}(\underline{x})$ to be the expected price paid by the type $\underline{x}$ consumer.

We therefore see, comparing (2) and (3), that lotteries provide a unified framework within which to consider both haggling and bargaining as well as lottery offers.

The model seeks to address under which demand conditions, if any, the multiproduct monopolist would best be advised to attempt stochastic selling strategies. On the surface this would be an impossibly hard problem: it would require the monopolist to determine what her most profitable stochastic selling strategy is (a multidimensional screening problem which has not yet been solved) and compare the profits it generates against that generated through the most profitable deterministic prices. If we could do this, we would then hope to show that stochastic strategies can never improve the monopolist's profitability. There is, however, a different way to tackle the problem. We first need one further crucial assumption:

Assumption 1. Given any combination of bundles priced at their optimal fixed price levels, there is a non-zero probability of a consumer being indifferent between them and not purchasing at all. ■

This assumption is an important regularity condition on the distribution of consumers. It is satisfied by any distribution with positive support on $\prod_{i=1}^{K}[0, b_i]$ for some $\{b_i\}$ for example. Put most bluntly, this restriction is equivalent to there being at least one consumer in the population indifferent between all the monopolist's different bundles and not purchasing.

Given this assumption the key to solving for the monopolist's optimal strategy is to show that in a certain sense the monopolist's profit function is concave *in an appropriate selling strategy space*. Suppose the monopolist is offering her K bundles of products at the optimal deterministic prices. The simplest type of stochastic selling strategy one could image is for the monopolist to augment her deterministic strategy with one solitary lottery with fixed probabilities. If this lottery is priced high enough, that is at its choke price, then a small subset of the population will just be indifferent between this lottery and some other deterministic option. The assumption above allows us to pin down exactly what this choke price is for any given lottery. A small first order change in any of the prices will then result in a small first order movement of some consumers to the lottery. The concavity type result holds as we show that if the globally optimal selling strategy is stochastic then *there must exist* a single lottery which can be profitably added in the above way to the deterministic prices. This key result reduces the monopolist's problem of globally optimal selling strategies to a consideration of small deviations from optimal deterministic prices through these simple single lotteries. Section 3 constructs this argument in detail and proves the result rigorously through Theorem 2.

Having reduced the monopolist's problem to a consideration of deviations from optimal deterministic prices, Section 4 then considers whether any such simple lottery addition to deterministic prices can ever actually be profit enhancing. To solve this question we must analyse the flow of consumers towards the lottery option when prices are changed marginally. As consumers have K dimensional types the partition of consumer type space between those choosing one good and the next can quickly become very involved. However, by a careful use of dimensionality arguments we are able to show that only certain forms of simple lottery have a hope of being profit enhancing. Some investigation then shows that the local conditions for any of these lotteries to be profit enhancing are identical to the first order conditions for the optimality of deterministic prices. As by definition we began from a benchmark of optimal deterministic prices, these expressions vanish. Therefore, Section 4 shows that no simple lottery priced at its choke price can enhance profits through a marginal price change. This result is captured in Theorem 3.

Sections 3 and 4 therefore combine to prove the following theorem:

Theorem 1 *Suppose a multiproduct monopolist is selling her K goods to risk-neutral consumers with unit demands and valuations $\underline{x}$ supported on $\Omega \subseteq \mathbb{R}_+^K$. Suppose also that $\underline{0} \in \Omega$, Ω is open, bounded and convex. The density of the consumers in the set Ω is given by the function $f(\cdot) : \Omega \to (0, \infty)$. The function f has the standard characteristics of a probabilistic density function and furthermore is bounded and differentiable with bounded derivatives. If Assumption 1 holds then* ***the monopolist can do no better than the optimal deterministic prices.***

Proof. Clear from theorems 2 and 3. ■

We stress that Theorem 1 is not a local result. No stochastic tariff structure satisfying the requirements of the theorem can be strictly more profitable than the best deterministic contract for any population which satisfies the conditions of the theorem.

3 Local Conditions Determining the Globally Optimal Selling Strategy

This section allows us to establish the first milestone in proving Theorem 1. The key is to show that the monopolist's profit is in some sense concave in an appropriate selling strategy space. In particular, we will show that if a monopolist's globally optimal selling strategy is to use lotteries then the monopolist will be able to deviate marginally away from optimal deterministic prices by introducing one single lottery which can be profitably added to the deterministic prices. However, to begin to formalise this argument we must be a great deal clearer about what a 'marginal deviation' might mean for the monopolist.

Let us return to the benchmark of the monopolist offering optimal deterministic prices, $\underline{p} \in \mathbb{R}_+^K$. Each consumer is therefore faced with a menu of K fixed prices and in return for the bundle i of goods must pay p_i. In a theoretical sense this monopolist is identical to one who is offering a menu of deterministic prices $\underline{p}$, but is also offering *every* conceivable lottery $\underline{q} \in Q \subset [0,1]^K$ at a price of $\underline{p} \cdot \underline{q}$. The region Q is the space of all lotteries satisfying conditions (1). This is true as $\underline{p} \cdot \underline{q} = \sum_i p_i q_i$ is the choke price for the lottery $\underline{q}$: it is the lowest price at which no consumer strictly prefers the lottery $\underline{q}$ to the other available options. Any small price reduction for the lottery from $\underline{p} \cdot \underline{q}$ will result in a flow of consumers towards the lottery. Likewise any small price increase in the fixed prices $\underline{p}$ will have the same effect. It is this that will constitute a marginal deviation away from the deterministic selling strategy. This insight is crucial to the work which is to come and is captured in lemma 1 below.

Lemma 1 *Suppose a monopolist is offering the K bundles she has for sale according to a price vector $\underline{p} \in \mathbb{R}_+^K$. The monopolist considers also offering the lottery $\underline{q}$ with fixed probabilities for sale. No group of consumers of positive measure would choose the lottery if it is priced at $\underline{p} \cdot \underline{q}$, though some consumers are indifferent between the lottery and some other deterministic option.*

Proof. The consumers each purchase the option which maximises their utility. A risk neutral consumer of type $\underline{x} \in \Omega \subset \mathbb{R}_+^K$ will derive a utility of $\sum_{i=1}^K q_i (x_i - p_i)$ from the lottery $\underline{q}$ priced at $\underline{p} \cdot \underline{q}$. If a consumer of this type is to choose the lottery then she must derive weakly more utility from it than from the other available options and from the outside option. Suppose a consumer of type $\underline{x}$ satisfies

$$x_1 - p_1 \geq x_2 - p_2 \geq \cdots \geq x_K - p_K \tag{4}$$

If this consumer chooses the lottery option she must derive non-negative expected utility from it so that she prefers it to the outside option. The individual rationality constraint is

$$\sum_i q_i (x_i - p_i) \geq 0 \tag{5}$$

The consumer must therefore be willing to pay for at least some of the bundles, i.e. $x_1 - p_1 \geq 0$. Otherwise, using (4) and the fact that $q_i \in [0,1]$ with not all the q_i equalling 0, the constraint (5) would be violated.

In addition, as the consumer with type $\underline{x}$ chooses the lottery option she must prefer it to the other options (at least weakly). The consumer must therefore satisfy the following incentive compatibility conditions:

$$\sum_i q_i (x_i - p_i) \geq x_j - p_j \quad \forall j \tag{6}$$

Given (4) it is sufficient to have

$$\sum_i q_i\,(x_i - p_i) \geq x_1 - p_1 \geq 0$$

However, the left hand side of the above equation represents a weighted average of the set $\{x_i - p_i\}$ with the weights (q_i) lying in the set $[0,1]$ and summing to less than or equal to 1. This inequality will therefore be violated unless

$$\begin{cases} x_1 - p_1 = x_j - p_j \geq 0 \text{ for all } j \text{ such that } q_j \neq 0 & \text{If } \sum_i q_i = 1 \\ x_1 - p_1 = x_j - p_j = 0 \text{ for all } j \text{ such that } q_j \neq 0 & \text{If } \sum_i q_i < 1 \end{cases}$$

Consumers who satisfy these conditions will be indifferent between the lottery and one of the fixed price options. We also note that neither of these two conditions have any volume in K space. Given any lottery $\underline{q}$, in the first case we will have the intersection of a number of hyperplanes which will have dimension strictly less than K and so is of measure 0. In the second case we have a single point in K which has 0 dimension and is also of measure 0.

Assumption 1 guarantees that there will be some consumers satisfying these conditions. In particular the consumer indifferent between all the bundles and not purchasing at all at optimal fixed prices will also be indifferent to any lottery $\underline{q}$ priced at its choke price of $\underline{p} \cdot \underline{q}$. ■

This result hinges on the fact that the consumers are risk neutral. The trick of looking at deviations of the price of lottery $\underline{q}$ from $\underline{p} \cdot \underline{q}$ has parallels with the work of McAfee et al. [7]. Their paper considers the bundling decision of a two good monopolist facing consumers with independent demand. In particular they consider the monopolist selling good X at p^X and good Y at p^Y with any consumer purchasing both goods paying $p^X + p^Y$. In their paper McAfee et al. [7] seek to determine if the bundled good should be offered at a different price, p^b. They note that the situation is identical to one in which the monopolist is offering a bundled good (X and Y) at a price $p^b = p^X + p^Y$ and so go on to consider local changes in p^b from this level. Similarly we set the lottery price at $\underline{p} \cdot \underline{q}$ and will examine the effect of both local and non-local changes in this price.

We are now in a position to give the main result of this section.

Theorem 2 *Given the consumer type space described, a stochastic selling strategy is globally optimal for the monopolist if and only if there exists some single lottery $\underline{q}$ with fixed probabilities which can be profitably added to the tariff consisting of the K optimal deterministic prices, $\underline{p} \in \mathbb{R}^K_+$, with the price of the lottery option $\underline{q}$ being moved marginally away from $\underline{p} \cdot \underline{q}$.*

This theorem therefore is important for a firm as it provides information about its globally optimal pricing policy drawn from the result of small local experiments. The theorem proves that if some general stochastic selling strategy is optimal for the monopolist, then there will always exist a single lottery strategy which is more profitable than optimal deterministic prices. That is, there will always exist some single lottery $\underline{q}$ which she can profitably add to the optimal deterministic selling strategy at a price close to its choke price of $\underline{p} \cdot \underline{q}$. This result is therefore an existence result. The theorem notes that the price of the lottery option is to be moved locally from $\underline{p} \cdot \underline{q}$ with lemma 1 showing us that no consumers will take the lottery at a price of $\underline{p} \cdot \underline{q}$. Therefore, a monopolist currently charging optimal deterministic prices $\underline{p}$ can experiment by offering the lottery $\underline{q}$ at a price of $\underline{p} \cdot \underline{q} - \varepsilon$. Any change in profit from offering the lottery will at most be of order ε and hence very small. If profit is never increased by these experiments, then Theorem 2 guarantees that there does not exist a *non-local* change in tariffs which will improve profits. In other words, the monopolist's globally optimal selling strategy is the deterministic one.

3.1 Proof of Theorem 2

The 'if' direction of Theorem 2 is trivial. If augmenting the menu of fixed prices with some single lottery with fixed probabilities beats the best deterministic selling strategy then clearly the firm's optimal policy is some form of stochastic selling strategy.

We therefore turn to the 'only if' direction. We suppose that a stochastic selling strategy is strictly more profitable for the monopolist than the best deterministic selling strategy. We begin by introducing a little notation:

Optimal Deterministic Prices

The most profitable menu of take-it-or-leave-it prices is denoted by the vector of prices $\underline{p} \in \mathbb{R}_+^K$, one price for each of the K bundles. Given these prices we can predict how a type $\underline{x}$ consumer will behave. A consumer of type $\underline{x}$ will choose the option which maximises their utility. This defines the utility allocation function $V(\underline{x}) : \Omega \subset \mathbb{R}_+^K \to \mathbb{R}_+$ corresponding to the optimal deterministic tariff.

$$V(\underline{x}) = \begin{array}{l}\text{The utility a type } \underline{x} \text{ consumer receives from} \\ \text{the menu of deteministic prices } \underline{p}.\end{array}$$

Globally Optimal Stochastic Strategy

The monopolist's most profitable strategy is assumed to be to offer the consumers some schedule of lotteries, $\underline{q}$ lying in the set $Q \subset [0,1]^K$, over the K bundles with an associated price for any lottery chosen. The consumer could therefore pick a lottery $\underline{q} \in Q$, would pay some transfer $t\left(\underline{q}\right)$ and would then receive bundle i with probability q_i. Given these options we can again predict how a consumer of type $\underline{x}$ will behave. This will define a utility allocation function, $U\left(\underline{x}\right) : \Omega \subset \mathbb{R}_+^K \to \mathbb{R}_+$. In other words

$$U\left(\underline{x}\right) = \begin{array}{l}\text{The utility a type } \underline{x} \text{ consumer receives from} \\ \text{the monopolist's globally optimal stochastic} \\ \text{strategy.}\end{array}$$

These utility allocation functions such as $U\left(\underline{x}\right)$ and $V\left(\underline{x}\right)$ are sufficient to determine what lotteries a consumer of type $\underline{x}$ receives. To see this we note that the utility allocation function $U\left(\underline{x}\right)$ for a consumer of type $\underline{x}$ is defined by

$$U\left(\underline{x}\right) = \max_{\underline{q} \in Q} \left\{\underline{q} \cdot \underline{x} - t\left(\underline{q}\right)\right\} \tag{7}$$

A consumer of type $\underline{x}$ will opt for the lottery $\underline{q}$ which maximises the right hand side of the above expression. This will therefore define an allocation function $\underline{q}\left(\underline{x}\right) : \Omega \subset \mathbb{R}_+^K \to Q$. An application of the envelope theorem shows that where defined

$$\nabla U\left(\underline{x}\right) = \underline{q}\left(\underline{x}\right) \tag{8}$$

In what follows it is important for us to know which utility allocation functions are allowable ones. In other words, which utility allocation functions can be implemented with a schedule of lotteries $\underline{q} \in Q$ with some associated prices. Rochet [13] answered this question and we repeat his important result in the lemma below:

Lemma 2 (Rochet [13]) *Let $U\left(\underline{x}\right) : \Omega \subset \mathbb{R}_+^K \to \mathbb{R}_+$ be a utility allocation function and $\underline{q}\left(\underline{x}\right) : \Omega \subset \mathbb{R}_+^K \to Q$ a lottery allocation function. There exists a tariff structure $t\left(\underline{q}\right) : Q \to \mathbb{R}_+$ such that $U\left(\underline{x}\right)$ satisfies (7) for a.e. $\underline{x}$ (the maximum being obtained by $\underline{q}\left(\underline{x}\right)$) if and only if*

1. *$\underline{q}\left(\underline{x}\right) = \nabla U\left(\underline{x}\right)$ for a.e. $\underline{x} \in \Omega$,*
2. *$U\left(\cdot\right)$ is convex continuous on Ω.*

We now see that an implementable utility allocation function, such as $U(\underline{x})$ and $V(\underline{x})$, is sufficient for us to determine the entire monopolist's pricing strategy. In particular, suppose that the utility allocation function $U(\underline{x})$ applies. By a direct result of the envelope theorem (or by reference to lemma 2) we have the allocation vector $\underline{q}(\underline{x}) = \nabla U(\underline{x})$. This function tells us which lottery a type $\underline{x}$ consumer receives. In combination with (7) we can therefore find the transfer allocation function, $t(\underline{x}) : \Omega \to \mathbb{R}_+$

$$t(\underline{x}) = \underline{x} \cdot \nabla U(\underline{x}) - U(\underline{x}) \tag{9}$$

This function determines the transfer made by a consumer of type $\underline{x}$ to the monopolist. We therefore know how much each consumer pays $(t(\underline{x}))$ and what lottery she decides to purchase $\left(\underline{q}(\underline{x})\right)$. We therefore have the full tariff structure $\left(t\left(\underline{q}\right) : Q \to \mathbb{R}_+\right)$ as a function of the lottery chosen $\left\{\underline{q}\right\}$.

We now consider the convex combination between the utility allocation function with the optimal stochastic strategy $(U(\underline{x}))$ and the utility allocation function with the optimal deterministic strategy, $(V(\underline{x}))$. We label the convex combination $U_\lambda(\underline{x}) : \Omega \subset \mathbb{R}_+^K \to \mathbb{R}_+$ with the index λ running from 0 to 1.

$$U_\lambda(\underline{x}) := \lambda U(\underline{x}) + (1-\lambda) V(\underline{x}) \qquad \lambda \in [0,1]$$

As λ varies from 0 to 1, $U_\lambda(\underline{x})$ deforms the utility allocation function continuously from $V(\underline{x})$ into $U(\underline{x})$.[2]

The function $U_\lambda(\underline{x})$ satisfies the conditions required of it in lemma 2 and is therefore implementable. In particular:

- As $U(\cdot)$ and $V(\cdot)$ are differentiable then so is $U_\lambda(\cdot)$.
- The affine sum of convex functions is convex and so $U_\lambda(\cdot)$ is convex.
- Individual rationality constraints require each consumer to receive at least a utility of 0. Therefore $U(\underline{x})$ and $V(\underline{x})$ are greater than or equal to 0 for all types $\underline{x} \in \Omega$. We will therefore have $U_\lambda(\underline{x}) \geq 0$ for all $\underline{x} \in \Omega$, and so $U_\lambda(\underline{x})$ satisfies individual rationality.[3]

[2] Note that $U_0(\underline{x}) = V(\underline{x})$ and $U_1(\underline{x}) = U(\underline{x})$.

[3] We note that the consumers at the origin always take the outside option. This is because we have $V(0) = U(0) = 0$. Therefore $U_\lambda(0) = 0\ \forall \lambda \in [0,1]$.

- The probability allocations $\nabla U_\lambda(\underline{x})$ (or lotteries) implied by the utility allocation $U_\lambda(\underline{x})$ in combination with lemma 2 are well defined. In particular we require that the coordinates of $\nabla U_\lambda(\underline{x})$ lie in $[0,1]$ and sum to less than or equal to 1. In short we require $\nabla U_\lambda(\underline{x})$ to lie in the K dimensional simplex which we labelled Q.[4] But Q is convex and $\nabla U(\underline{x})$, $\nabla V(\underline{x})$ lie in Q giving the result.

Given the utility allocation vector $U_\lambda(\underline{x}) : \Omega \subset \mathbb{R}_+^K \to \mathbb{R}_+$ lying between $U(\underline{x})$ and $V(\underline{x})$, expressions (8) and (9) give the transfers and the allocation enjoyed by the monopolist. This allows us to construct the firm's profit function as a function of the utility allocation:

$$\Pi(U_\lambda(\cdot)) = \int_\Omega \{\nabla U_\lambda(\underline{x}) \cdot (\underline{x} - \underline{c}) - U_\lambda(\underline{x})\} f(\underline{x})\, d\underline{x} \tag{10}$$

We have assumed that the monopolist's most profitable pricing policy is stochastic and so we must have $\Pi(U(\cdot)) \gneqq \Pi(V(\cdot))$. Comparing the profit generated by the utility allocation $U_\lambda(\underline{x})$ to that of the optimal deterministic tariff and its associated utility allocation function $V(\underline{x})$ we have:

$$\begin{aligned} \Pi(U_\lambda(\cdot)) - \Pi(V(\cdot)) &= \lambda[\Pi(U(\cdot)) - \Pi(V(\cdot))] \\ &> 0 \qquad \text{for all } \lambda \in (0,1] \end{aligned} \tag{11}$$

Therefore *any* small deformation $U_\lambda(\underline{x})$ will enhance the monopolist's profits. It is this observation which will allow us to move from the global existence of $U(\cdot)$ to the local result captured in Theorem 2. This result also shows us that profit is a concave function of the utility allocation function.[5]

We have therefore constructed an implementable utility allocation function, $U_\lambda(\underline{x})$, which provides the monopolist with more profit than the best menu of deterministic prices captured by the utility allocation function $V(\underline{x}) = U_0(\underline{x})$. In addition we see that $U_\lambda(\underline{x})$ can be made

[4] The K dimensional simplex is the subset of points in $\mathbb{R}_+^K$ which have coordinates which sum to less than or equal 1.

[5] The monopolist's profit function, $\Pi(U(\cdot))$, as a function of the utility allocation $U(\cdot)$ is concave in $U(\cdot)$. To see this recall that the definition of a general concave function $g(x)$ is that

$$\begin{aligned} g(\lambda x + (1-\lambda) y) &\geq \lambda g(x) + (1-\lambda) g(y) \\ &\Leftrightarrow g(\lambda x + (1-\lambda) y) - g(y) \geq \lambda [g(x) - g(y)] \end{aligned}$$

We have shown exactly this condition in (11).

arbitrarily 'close' to $V(\underline{x})$ in a very concrete sense by lowering the index λ towards zero. We ascertained in (8) and (9) that the utility allocation function, $U_\lambda(\underline{x})$, is sufficient to determine what lottery a type $\underline{x}$ consumer receives and how mush she pays for it. Hence, associated with the utility allocation function we have a whole tariff schedule:

$$\begin{aligned} t_\lambda &: \left[\text{ The space of all lotteries }\right] \to \left[\text{ a price }\right] \\ t_\lambda &: Q \subset [0,1]^K \to \mathbb{R}_+ \end{aligned}$$

Note that the 'space of all lotteries' is not composed of a discrete set of individual lotteries but is rather a region. Hence $t_\lambda(\underline{q})$ is the price of lottery $\underline{q}$ purchased when utilities are allocated to consumers according to $U_\lambda(\cdot)$.[6]

Clearly the tariff schedule $t_\lambda : Q \to \mathbb{R}_+$ allows us to determine each utility maximising consumer's behaviour, conditional on their type. We can therefore calculate the monopolist's expected profit and so expression (11) can be repeated as

$$\begin{aligned} \Pi(t_\lambda(\cdot)) - \Pi(t_0(\cdot)) &= \lambda\left[\Pi(t_1(\cdot)) - \Pi(t_0(\cdot))\right] \\ &> 0 \qquad \text{for all } \lambda \in (0,1] \end{aligned} \tag{12}$$

The tariff $t_0(\cdot)$ is that generated by the surplus allocation $V(\underline{x})$ which corresponds to the monopolist offering optimal deterministic prices. We recall from lemma 1 that such a monopolist was identical to a monopolist who, as well as offering bundle i for a fixed price p_i, also offered the consumers the choice of every single possible lottery for sale. Each lottery, $\underline{q}$ say, is however priced at its choke price of $\underline{p}\cdot\underline{q}$. The monopolist's tariff schedule if she is following the optimal deterministic pricing policy can therefore be interpreted explicitly as:[7]

$$t_0(\underline{q}) = \begin{cases} p_i & \text{if } \underline{q} = \underline{e}_i \text{ the standard unit basis vector} \\ \underline{p}\cdot\underline{q} & \text{otherwise} \end{cases} \tag{13}$$

We are now in a position to take stock of our progress. To graphically illustrate our results so far we restrict ourselves to the case of the single product monopolist. The monopolist's best

[6] Hence $t_1(\cdot)$ is the tariff schedule equivalent to the surplus allocation $U(\underline{x})$. Similarly, $t_0(\cdot)$ is the deterministic tariff generated by the surplus allocation $V(\underline{x})$.

[7] Note that the deterministic lotteries are given by the standard basis vectors

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

and so capture that a particular bundle will be given to the paying consumer with a probability of 1.

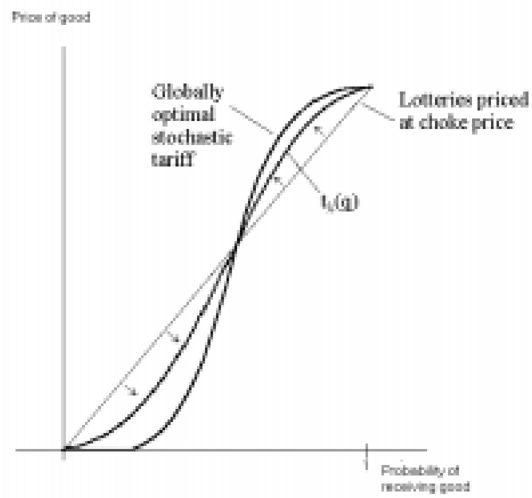

Figure 1: A graph showing the possible tariff schedules of the single good monopolist. The arrows show the deformation of the optimal deterministic tariff into the globally optimal stochastic tariff.

deterministic price, p say, creates the utility allocation function $V(x)$. Lemma 1 showed us that this monopolist is identical to one who is offering any lottery (probability of delivery) $q \in [0,1]$ at its choke price of pq. We can therefore plot the lottery probability, q, against its price and have a straight line (see Figure 1). However, we assumed that the monopolist can maximise her profits by following some stochastic strategy corresponding to the utility allocation function $U(x)$. Expressions (8) and (9) showed us how much a consumer pays and what lottery she receives. We can therefore plot the tariff structure corresponding to the globally optimal pricing strategy. This is also done in Figure 1.

Now note that $U_\lambda(x)$ creates more profit for the monopolist than the best deterministic prices (see (11)) and we can determine the tariff structure associated with $U_\lambda(x)$. This is also drawn in Figure 1. But for small λ's this tariff structure will not be much different from the straight line pq. That is a *small* change in the price of the lotteries on offer (there is a whole region of them, not just a finite number) from pq can be profit increasing. Theorem 2 asserts that we can start the process by picking *one* lottery, some fixed q, whose price we can change from pq and which will be profit increasing for the monopolist. It is this we now need to show.

We therefore now suppose that instead of offering consumers lotteries in the entire region Q, as dictated by the tariff schedule $t_\lambda(\cdot)$, the monopolist only offers a menu of N distinct lotteries: $\{\underline{q}^1, \underline{q}^2, \ldots, \underline{q}^N\}$. These N lotteries are chosen evenly spaced across the space of lotteries $Q \subset [0,1]^K$. The number N will be made to tend to infinity. We require that the lotteries on offer include the outside option of $\underline{0}$ and all of the deterministic lotteries (see footnote 7). The values of the tariff schedule $t_\lambda(\cdot)$ for these given lotteries will be their prices and so determine an N dimensional price vector denoted by $\underline{p}_\lambda^N \in \mathbb{R}_+^N$:

$$\underline{p}_\lambda^N = \left(t_\lambda\left(\underline{q}^1\right), t_\lambda\left(\underline{q}^2\right) \ldots, t_\lambda\left(\underline{q}^N\right)\right)$$

The idea is therefore very simple: we know that with the tariff schedule $t_\lambda(\cdot)$ the monopolist makes more than she can do with deterministic prices alone. We therefore want to approximate the complicated schedule, $t_\lambda(\cdot)$ with a menu of only N lotteries and show that even with these N lotteries, if N is sufficiently large, the monopolist still does better than the best fixed prices can do. From there it is a short jump to prove theorem 2 and show that even with one judiciously chosen lottery added to fixed prices the monopolist's profits are enhanced.

This point is illustrated further in Figure 2 which describes the case for the single good monopolist. The left hand graph is a copy of Figure 1 and shows the deterministic tariff structure $t_0\left(\underline{q}\right)$, the optimal stochastic tariff structure $t_1\left(\underline{q}\right)$ and finally the tariff structure arising from the convex combination $t_\lambda\left(\underline{q}\right)$. We recall that the monopolist setting only fixed prices is identical to one also selling every lottery, $\underline{q}$, at its choke price of $\underline{p} \cdot \underline{q}$ and so $t_0\left(\underline{q}\right)$ has a constant gradient. We have shown that the tariff schedule $t_\lambda\left(\underline{q}\right)$ is more profitable than fixed prices. We now turn to the right hand graph in which the monopolist chooses N probabilities to offer, $\underline{q}^i$, at a price of $t_\lambda\left(\underline{q}^i\right)$. These points are shown in Figure 2 labelled as p_λ^i. As the number of points becomes larger and larger one would expect the profit gained by the monopolist from the complicated t_λ and from the N individual lotteries to become very similar. Hence we will show that eventually this menu of N lotteries must also be profit enhancing for the monopolist.

Before proceeding we therefore need to prove the following technical lemma:

Lemma 3 $\lim_{N \to \infty} \Pi\left(\underline{p}_\lambda^N\right) = \Pi\left(t_\lambda(\cdot)\right)$.

This lemma confirms the intuition above that as the menu of N lotteries we pick becomes larger, we derive a closer and closer approximation to the tariff schedule $t_\lambda(\cdot)$ and so the profits made become identical. This might seem obvious but the result involves an important leap from

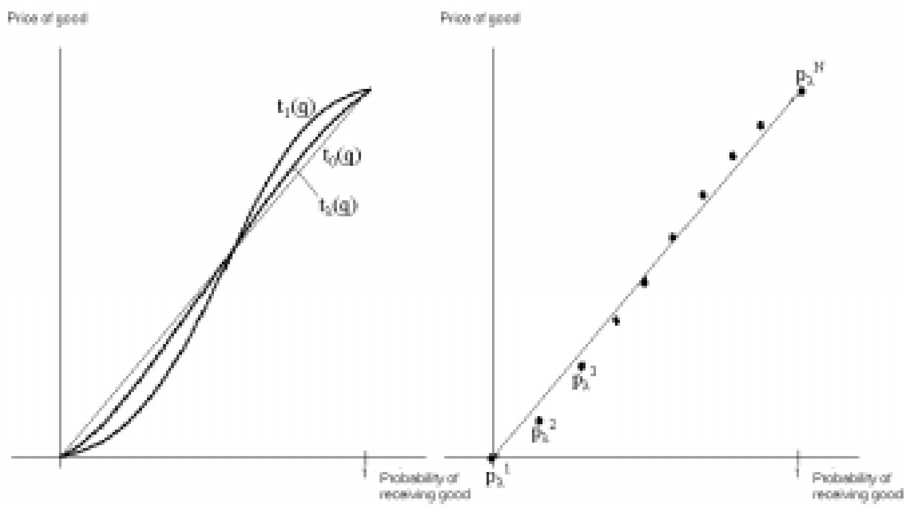

Figure 2: Graphs showing the different tariff structures used to prove Theorem 2.

discrete to continuous considerations. The full proof is given in the appendix and we give a sketch proof here.

Sketch Proof. The price vector of the menu of N lotteries, $\underline{p}_{\lambda}^{N}$ partitions the consumers into regions, each region receiving the same lottery. This creates a map from the consumer type space to the space of lotteries. This map is actually a *step function.* This is because each consumer is linked with one of only N different available lotteries. As we increase N, these step functions approach the lottery allocation function created by $U_{\lambda}(\underline{x})$. To show that the result holds we therefore need:

1. to be able to express the monopolist's profit function in terms of these lottery allocation functions. To do this we need to specify that the consumers at the origin (who do not value any of the monopolist's products) receive zero utility. We can then relate how much utility a consumer receives to those consumers at the origin. We could only do this as the consumers lie in a convex set.

2. We then use a result from Lebesgue Integration Theory known as the Dominated Conver-

gence Theorem to prove the result.[8]

■

We can now return to the price vector $\underline{p}_{\lambda}^{N}$ of the menu of N lotteries. We define the price vector over the same N lotteries corresponding to the monopolist using a fixed price strategy as $\underline{p}_{0}^{N}$. We are again using lemma 1 and so have that the monopolist is selling all lotteries at their choke price. The price vector, $\underline{p}_{0}^{N}$, can therefore be written explicitly as

$$\underline{p}_{0}^{N} = \{\underline{p}\cdot\underline{q}^{1}, \underline{p}\cdot\underline{q}^{2}, \ldots, \underline{p}\cdot\underline{q}^{N}\}$$

We define the difference between these price vectors by:

$$\underline{p}_{\lambda}^{N} = \underline{p}_{0}^{N} + \underline{h}^{N}(\lambda)$$

where $\underline{h}^{N}(0) = \underline{0}$ by construction.

We now note that

$$\begin{aligned} \lim_{N\to\infty}\left[\Pi\left(\underline{p}_{\lambda}^{N}\right) - \Pi\left(\underline{p}_{0}\right)\right] &= \Pi(t_{\lambda}) - \Pi(t_{0}) \qquad \text{by Lemma 3} \\ &= \lambda\left[\Pi(t_{1}) - \Pi(t_{0})\right] \qquad \text{by expression (12)} \end{aligned}$$

and so

$$\lim_{N\to\infty}\frac{\left[\Pi\left(\underline{p}_{\lambda}^{N}\right) - \Pi\left(\underline{p}_{0}\right)\right]}{\lambda} = \underbrace{\left[\Pi(t_{1}) - \Pi(t_{0})\right]}_{\text{const}} > 0 \quad \forall\lambda\in(0,1]$$

Hence

$$\lim_{\lambda\searrow 0}\lim_{N\to\infty}\frac{\left[\Pi\left(\underline{p}_{\lambda}^{N}\right) - \Pi\left(\underline{p}_{0}\right)\right]}{\lambda} > 0 \tag{14}$$

Returning back to the monopolist's profit function, we formally note that if the monopolist only offers the prices $\underline{p}_{\lambda}^{N}$ then it has a profit of:

$$\Pi\left(\underline{p}_{\lambda}^{N}\right) = \sum_{i=1}^{N}\left\{\left[\left(\underline{p}_{\lambda}^{N}\right)_{i} - \underline{c}\cdot\underline{q}^{i}\right]\int_{\left\{\underline{x}|\text{Type }\underline{x}\text{ prefers }\underline{q}^{i}\right\}} f(\underline{x})\,d\underline{x}\right\}$$

[8] The Dominated Convergence Theorem is given on p109 of Weir [19].

The underlying consumer density function $(f(\cdot))$ is differentiable. The final integral in the above sum will therefore be differentiable in the price vector if the boundaries of the sets $\left\{\underline{x}\right|\text{Type } \underline{x} \text{ prefers } \underline{q}^i\}$ vary continuously in the price vector. These sets $\left(\left\{\underline{x}\right|\text{Type } \underline{x} \text{ prefers } \underline{q}^i\}\right)$ are formed by the partition of Ω by hyperplanes formed by the incentive compatibility constraints. The hyperplanes are linear in prices and so vary continuously. This will be sufficient if Ω is open (otherwise a discontinuity can result when the hyperplanes intersect the boundary of Ω). We are assuming that Ω is open. We can therefore conclude that the whole of the above profit function is differentiable in the price vector. Using a Taylor expansion with the notation that $\nabla\underline{h}^N(0)$ implies the component derivative, we have:

$$\begin{aligned} \Pi\left(\underline{p}_\lambda^N\right) = \Pi\left(\underline{p}_0^N + \underline{h}^N(\lambda)\right) &= \Pi\left(\underline{p}_0^N + \underbrace{\underline{h}^N(0)}_{=0} + \lambda\nabla\underline{h}^N(0) + O\left(\lambda^2\right)\right) \\ &\to \Pi\left(\underline{p}_0^N + \lambda\nabla\underline{h}^N(0)\right) \qquad \text{as } \lambda \to 0 \end{aligned} \tag{15}$$

We therefore have that

$$\lim_{\lambda \searrow 0} \frac{\left[\Pi\left(\underline{p}_\lambda^N\right) - \Pi\left(\underline{p}_0^N\right)\right]}{\lambda} = \nabla\Pi\left(\underline{p}_0^N\right) \cdot \left\{\nabla\underline{h}^N(0)\right\}$$

We have shown that interchanging the limits in (14) is valid as the limits exist. Substituting the above result into (14) gives that

$$\begin{aligned} 0 &< \lim_{N\to\infty} \lim_{\lambda \searrow 0} \frac{\left[\Pi\left(\underline{p}_0 + \lambda\nabla\underline{h}^N(0)\right) - \Pi\left(\underline{p}_0\right)\right]}{\lambda} \\ &= \lim_{N\to\infty} \nabla\Pi\left(\underline{p}_0^N\right) \cdot \left\{\nabla\underline{h}^N(0)\right\} \end{aligned}$$

This therefore implies that for sufficiently large N, $\nabla\Pi\left(\underline{p}_0^N\right)$ has some non-zero components. We know that the deterministic prices $\underline{p}$ were optimal by construction, therefore there must be some lottery $\underline{q}$ priced at l such that $\left[\frac{\partial\Pi}{\partial l}\right]_{l=\underline{p}\cdot\underline{q}} \neq 0$. This proves the 'only if' direction of Theorem 2.

To see this final step clearly, let us return to the single product monopolist we have been using to illustrate the results. We have depicted the situation for $N = 4$ in Figure 3.

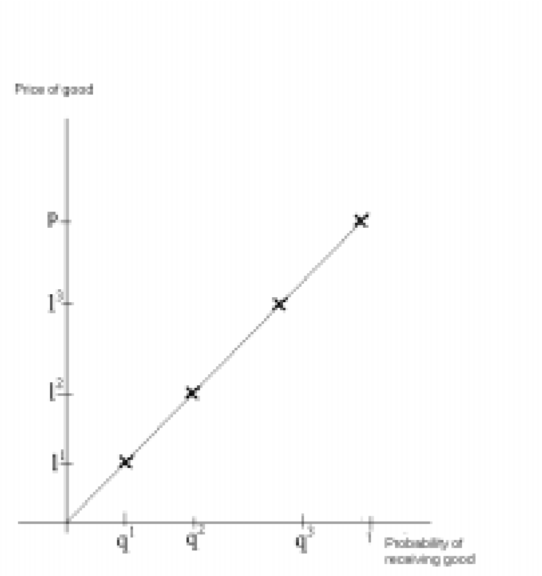

Figure 3: A graph of the one good monopolist's tariff schedule. The crosses show the $N = 4$ lotteries which together allow the monopolist to increase her profits above the best fixed pricces.

The monopolist following the best fixed price strategy sells the 3 lotteries q^1, q^2, q^3 at their choke prices. From the working above we have that

$$0 \neq \nabla\Pi\left(\underline{p}_0^N\right) = \left(\left[\frac{\partial\Pi}{\partial l^1}\right]_{l^1=pq^1}, \left[\frac{\partial\Pi}{\partial l^2}\right]_{l^2=pq^2}, \left[\frac{\partial\Pi}{\partial l^3}\right]_{l^3=pq^3}, \frac{\partial\Pi}{\partial p}\right)$$

If the first three components are all zero, then these lotteries do not help the monopolist though changing the fixed price does. This is a contradiction to the price p being the best fixed price and so we can rule this case out. One of the first three components must therefore be non-zero and so we have proved Theorem 2.

3.2 Discussion

The proof of Theorem 2 has allowed us to make substantial inroads into the relationship between the monopolist's selling strategy and her profit function. These insights move us a great deal closer to our goal of theorem 1. To assess what we have achieved let us define 'single lottery strategies'. These will consist of the situation in which the monopolist offers consumers the menu of K fixed prices for her K different bundles of goods, but also offers one single lottery $\underline{q}$, with fixed probabilities, under which the consumer would receive product i with probability q_i. Lemma 1 showed us that if this lottery $\underline{q}$ were offered to consumers at its choke price of $\underline{p} \cdot \underline{q}$ then a subset of consumers of measure 0 would be indifferent between it and one of the deterministic options. Having established this, it then became clear that if the single lottery strategy is followed and the price of good i were to be raised slightly, then some small set of consumers would begin to buy the lottery. It is this process which we defined to constitute a marginal deviation away from the deterministic selling strategy.

Having defined what constitutes a marginal deviation away from deterministic prices the achievement of Theorem 2 is to establish a pseudo-concavity result for the profit function in the context of such marginal deviations *in selling strategies*. In particular, if the firm's profit is maximised by a stochastic selling strategy, then the single lottery strategy which deforms the deterministic price strategy *in the direction of* the optimal stochastic strategy is profit increasing. In practice the proof of theorem 2 shows that the appropriate single lottery strategy is found by continuously deforming one utility allocation function into another. Theorem 2 is however best understood as an existence result: the theorem confirms that if stochastic strategies are globally optimal then the optimal deterministic strategy can always be profitably perturbed by adding one single lottery to the available consumer options and perturbing its price away from the choke price given in Lemma 1. The next section allows us to take up the story from this point.

4 The optimality of deterministic selling strategies for a multi-product monopolist.

The aim of this paper is to prove Theorem 1 which says that the multiproduct monopolist who is selling her goods to a population of consumers with types $\underline{x} \in \Omega$ satisfying the regularity conditions of the model can do no better than the optimal deterministic tariff. That is, appropriately set fixed prices are best for the monopolist. Theorem 2 above proved that if the monopolist were able to improve its profit using some stochastic selling strategy then it must also be able to improve its profit by locally experimenting by not just offering the consumers fixed price purchases but also some single lottery, $\underline{q}$ say, with fixed probabilities and whose price is set to just below its choke price $\left(\underline{p} \cdot \underline{q}\right)$. To complete the proof of Theorem 1 we therefore need to show that there doesn't exist any single lottery which can be profitably offered in conjunction with the monopolist's best fixed prices when the price of the lottery is just lowered from its choke price. This, together with Theorem 2, will establish a contradiction to our assumption that the globally optimal pricing policy was stochastic and so prove the main result of Theorem 1.

Before we proceed along these lines for the general multiproduct monopolist we can establish a parallel result very easily for the single good monopolist and so gain a valuable insight into how to tackle the general problem.

Example 4 (The case of the single good monopolist) *Let us suppose that consumers have valuation x for the good with x drawn from the distribution $F(\cdot)$ supported on $\left[0, \overline{X}\right]$. If the monopolist were to set a fixed price of t her profits would be:*

$$\Pi(t) = \int_{x=t}^{\overline{X}} (t-c) f(x) \, dx$$

This expression can be differentiated with respect to t and set to zero to give the first order condition for price optimality. We label the optimal fixed price p which will satisfy

$$[1 - F(p)] - (p-c) f(p) = 0 \tag{16}$$

We now recall that Lemma 1 showed us that this monopolist is identical to one who is offering any lottery for the good (probability of delivery) $q \in [0,1]$ at its choke price of pq. The question posed by the work of Theorem 2 is, can the monopolist introduce some lottery with fixed probability, q and lower its price to $pq - \varepsilon$ and *make more profit than offering one fixed price? Instead of answering this problem, we will address a parallel problem and discuss how the two are related in the section below. We wonder whether the monopolist can introduce some lottery with fixed probabilities, q, keep its price at pq whilst raising the fixed price to $p + \varepsilon$* and *make more profit than solely offering a fixed price? We can answer this problem directly. Suppose that the consumers are faced with the options of the good with certainty at a fixed price of $p + \varepsilon$ or the good with probability of delivery q at a price of pq:*

Consumers who choose the lottery must prefer it to the other options and so will have types satisfying

$$\left.\begin{array}{r} qx - pq \geq 0 \\ qx - pq \geq x - p - \varepsilon \end{array}\right\} \Rightarrow p \leq x \leq p + \frac{\varepsilon}{1-q} \tag{17}$$

Consumers who choose the fixed price must prefer it to the other options and so will have types satisfying

$$\left.\begin{array}{c} x-p-\varepsilon \geq qx-pq \\ x-p-\varepsilon \geq 0 \end{array}\right\} \Rightarrow x \geq p+\frac{\varepsilon}{1-q}$$

The monopolist's profit will therefore be

$$\Pi(\varepsilon;p)=\int_{x=p}^{p+\frac{\varepsilon}{1-q}}(pq-qc)\,f(x)\,dx+\int_{x=p+\frac{\varepsilon}{1-q}}^{\overline{X}}(p+\varepsilon-c)\,f(x)\,dx$$

We differentiate this with respect to ε *and then set* $\varepsilon=0$ *to see the effect that introducing the lottery* q *and perturbing the prices slightly has*

$$\begin{aligned} \left[\frac{\partial\Pi}{\partial\varepsilon}\right]_{\varepsilon=0} &= \frac{q}{1-q}(p-c)\,f(p)-\frac{1}{1-q}(p-c)\,f(p)+\int_{x=p}^{\overline{X}}f(x)\,dx \\ &= -(p-c)\,f(p)+[1-F(p)] \qquad (18) \\ &= 0 \qquad \textit{if } p \textit{ is the optimal fixed price} \end{aligned}$$

and so we have the result that:

1. *If the fixed price is originally optimal then (18) vanishes (using (16)) and so the introduction of the lottery doesn't enhance the monopolist's profitability.*

   but

2. *If the monopolist's original fixed price is not optimal then lotteries can enhance profitability, though not as much as changing the fixed price to the optimal level.*

The second point is intuitively clear, though it can seem surprising: we will show that the monopolist's most profitable possible strategy is to charge the *best* fixed prices. If however the monopolist is not charging the best fixed prices then many sorts of deviation in the monopolist's selling strategy are profit enhancing. In particular, lotteries or bargaining can be profit enhancing for the monopolist not charging optimal fixed prices.

This section will prove exactly the result that there doesn't exist any single lottery which can be profitably offered in conjunction with the multiproduct monopolist's best fixed prices when the price of lottery is just lowered from its choke price. Formally, we capture the result in the following theorem:

Theorem 3 *Suppose the multiproduct monopolist of Section 2 is selling her goods to a population of consumers with unit demand for each of the component goods and separate valuations for the goods individually or in any bundle. The monopolist can do no better than the optimal deterministic prices,* $\underline{p}$*, by also offering consumers the choice of a single lottery with a fixed probability vector* $\underline{q}$ *and price close to the choke price of* $\underline{p} \cdot \underline{q}$*.*

Theorem 3 and Theorem 2 will then link up and together prove Theorem 1.

4.1 Proof of Theorem 3

We assume that the monopolist's optimal deterministic price vector is given by $\underline{p} = (p_1, p_2, \ldots, p_K)$ where p_j is the price of good j. The profit function when the single lottery $\underline{q}$ with fixed probabilities is offered at a price of l is denoted by $\Pi^{\text{lot}} = \Pi^{\text{lot}}\left(\underline{p}, l\right)$ and the profit when no lottery is offered is denoted by $\Pi^{\text{no lot}}$. Lemma 1 guarantees that $\Pi^{\text{no lot}}\left(\underline{p}\right) = \Pi^{\text{lot}}\left(\underline{p}, \underline{p} \cdot \underline{q}\right)$ as $\underline{p} \cdot \underline{q}$ is the choke price for the lottery $\underline{q}$. Without any loss of generality we can assume that $q_1 \neq 0$ so that there exists a positive probability of the consumer receiving the first bundle. The first order conditions for price optimality when no lottery is offered therefore give

$$\frac{\partial \Pi^{\text{no lot}}}{\partial p_1} = 0 = \left[\frac{\partial \Pi^{\text{lot}}}{\partial p_1} + q_1 \frac{\partial \Pi^{\text{lot}}}{\partial l}\right]_{l = \underline{p} \cdot \underline{q}}$$

We have assumed that $q_1 \gneqq 0$ and so we have

$$\left[\frac{\partial \Pi^{\text{lot}}}{\partial l}\right]_{l = \underline{p} \cdot \underline{q}} =_{\text{sign}} - \left[\frac{\partial \Pi^{\text{lot}}}{\partial p_1}\right]_{l = \underline{p} \cdot \underline{q}} \tag{19}$$

We therefore see that the change in the monopolist's profit is identical whether the price of the lottery is lowered or the price of good 1 is raised. This only applies if we begin from the benchmark of the best fixed prices.

It will prove analytically simpler for us to consider what happens when the price of good 1 is raised marginally rather than when the price of the lottery is dropped marginally. That is, using expression (19) it is sufficient for us to consider $\left[\frac{\partial \Pi^{\text{lot}}}{\partial p_1}\right]_{l = \underline{p} \cdot \underline{q}}$. This method of focusing on the price of one good rather than on the price of the lottery is similar to a trick used in McAfee, McMillan and Whinston [7]. They consider the bundling decision faced by a two good monopolist serving consumers with independent demand for the component goods. Instead of considering deviations in the bundled good price, they look at deviations in the price of one of the component goods whilst keeping the bundled good price constant.

The derivative of the profit function with respect to the good 1 price is found by using the fact that

$$\frac{\partial \Pi^{\text{lot}}}{\partial p_1}\left(\underline{p}, \underline{p}\cdot\underline{q}\right) = \left[\frac{\partial}{\partial \varepsilon}\Pi^{\text{lot}}\left(p_1+\varepsilon, \underline{p}_{-1}, \underline{p}\cdot\underline{q}\right)\right]_{\varepsilon=0} \tag{20}$$

In other words, we determine the profit of the monopolist when it offers the tariff: $p_1+\varepsilon$ for good 1, p_j for good j with $j>1$ and $l=\underline{p}\cdot\underline{q}$ for the single lottery with probabilities $\underline{q}$. We then consider the rate of change of this profit as $\varepsilon \to 0$.

Clearly the monopolist would hope that she can extract extra surplus from the good 1 consumers whilst some of the good 1 consumers who cease to buy good 1 might be tempted by the lottery. The question is whether these profit enhancing effects counterbalance the loss made by losing the custom of any of the consumers. We will show that there is no local gain from the adding of lotteries: the rate of change of the profit function with respect to ε is zero. This would prove Theorem 3 and hence also Theorem 1.

Theorem 3 therefore hinges on an analysis of the size of the customer flows between different goods weighted by the profit made on each of the goods. To this end we define the demand for product i given the rise in good 1 price of ε as $D^i(\varepsilon)$. The demand for the lottery is similarly denoted $D^l(\varepsilon)$. By definition the flow of consumers as the price perturbation ε is increased is given by the rate of change of the demand functions with respect to ε which is exactly the derivative of the demand functions with respect to ε.

We focus first on the flow of consumers to the lottery option as ε is increased from zero. Formally this is given by $\left[\frac{\partial D^l}{\partial \varepsilon}\right]_{\varepsilon=0}$. If this were to be zero for a particular lottery $\underline{q}$ then the introduction of the lottery is having no first order effect: the price rise in the good 1 price dominates. The first order effect of profit is therefore the same as would have been produced if good 1's price were raised by ε with no lottery introduced. In other words

$$\left[\frac{\partial D^l\left(\underline{p}, \underline{p}\cdot\underline{q}\right)}{\partial \varepsilon}\right]_{\varepsilon=0} = 0 \Rightarrow \frac{\partial \Pi^{\text{lot}}}{\partial p_1} = \frac{\partial \Pi^{\text{no lot}}}{\partial p_1} = 0$$

The final equality follows as, by assumption, the monopolist was perturbing the good 1 price away from the optimal deterministic prices and so the rate of change of profit is 0. This working confirms the unsurprising fact that if the lottery chosen is irrelevant to first order and the monopolist is charging the best fixed prices, then the monopolist cannot increase its profit through a small perturbation in the fixed prices. This observation is however crucial to the following lemma which allows us to simplify the proof of theorem 3 into a small number of tractable cases.

Lemma 5 *Suppose the monopolist chooses the lottery* $\underline{q}$ *with* $q_1 > 0$*. Let* q_0 *denote the probability of receiving nothing so that* $q_0 + \sum_{i=1}^{K} q_i = 1$*. Then if* q_1 *and two or more of the probabilities* $\{q_0, q_2, q_3, \ldots, q_K\}$ *are non zero we must have*

$$\left[\frac{\partial \left[\textit{Demand for the lottery}\right]}{\partial \varepsilon}\right]_{\varepsilon=0} = \left[\frac{\partial D^l}{\partial \varepsilon}\right]_{\varepsilon=0} = 0$$

That is, the lottery is irrelevant to the monopolist's profit to first order.

This lemma allows us to assert that the only types of lottery which could have a beneficial effect to the monopolist's profit under small perturbations to the good 1 price are either (a) the one good lottery where $1 = q_0 + q_1$ with $q_2 = 0 = q_3 = \cdots = q_K$, or (b) the two good lottery with $1 = q_1 + q_2$ with $q_0 = 0 = q_3 = \cdots = q_K$. These cases will have to be analysed individually. In a real sense however, Lemma 5 is the backbone of the problem and so we will try and make the intuition behind it clear through a sketch proof which follows. The formal proof is deferred to the appendix.

Sketch Proof. The aim of the proof is to show that if the conditions of the lemma are satisfied then the volume of consumers, in the consumer type space, who would choose the lottery option is of order $O\left(\varepsilon^2\right)$. In other words, only the types in a volume of order $O\left(\varepsilon^2\right)$ satisfy all the incentive compatibility conditions which guarantee that these consumers prefer the lottery to every other available option. The demand for the lottery, $D^l\left(\varepsilon\right)$, will then be given by

$$D^l\left(\varepsilon\right) = \int \cdots \int_{\text{volume } O(\varepsilon^2)} f\left(\underline{x}\right) d\underline{x}$$

When the demand is then differentiated with respect to ε and evaluated at $\varepsilon = 0$ this contribution will vanish and so prove the lemma.

The question therefore reduces to one of why the conditions in the lemma guarantee that so few people will be tempted by the lottery. The crucial insight here is that every non-zero component of the lottery $\underline{q}$ puts an extra restriction on the types of consumer who can prefer it: if there is a positive probability of receiving bundle $j \neq 1$ then the consumer cannot have a valuation for the bundle j that is too low else she would not take the lottery. In particular, the utility gained through bundle j cannot be much lower than that gained through bundle 1 for the consumer to prefer the lottery to the first bundle. This is because the consumer could well receive bundle j through the lottery. On the other hand, the utility gained from through the bundle j cannot be much larger than that received from getting bundle 1 else the consumer would prefer bundle j delivered with certainty to the lottery as the lottery could well award the consumer bundle 1. These considerations mean that after scaling for price the x_j component of the consumer's type must be within an ε band of the x_1 component of the consumers type.

We need finally consider the effect of having a positive probability that the consumer will receive nothing at all. This is exactly the case in the one good monopolist framework. The types of consumer who would choose the lottery in this context were found in (17). Intuitively, the consumer of the lottery cannot value bundle 1 too highly else she will not be prepared to lose it through the lottery. This therefore restricts the consumer's valuation for bundle 1 to within an ε band of p_1 - the fixed price of good 1.

Putting these insights together we have:

1. Suppose that the lottery $\underline{q}$ on offer provides the consumer with positive probabilities of not just receiving the first bundle but two others, say 2 and 3, as well. In this case the types x_2 and x_3 of any consumers who are tempted by the lottery are both within ε bands of x_1 and so the volume of types who prefer the lottery is indeed of order $O\left(\varepsilon^2\right)$.

2. Similarly, suppose that the lottery $\underline{q}$ on offer provides the consumer with a positive probability of not receiving any good at all. In this case the consumer who prefers the lottery will have a type x_1 component in an ε band. In addition, if the lottery provides the consumer with a positive probability of receiving bundle 2 then any consumer who chooses it will have a valuation for the second bundle, x_2 within an ε band of x_1. Combining these two effects forces the volume of types who prefer the lottery to be of order $O\left(\varepsilon^2\right)$.

Hence, as discussed above, when the demand for the lottery is then differentiated with respect to ε and evaluated at $\varepsilon = 0$ it will vanish and so prove the lemma. ■

The lemma above has therefore shown us that the multiproduct monopolist hoping to benefit from introducing the lottery must focus on two generic types of lottery: (A) $q_0 > 0$, $q_1 > 0$ and $q_2 = 0 = \cdots = q_K$, or (B) $q_1 > 0$, $q_2 > 0$ and $q_0 = 0 = q_2 = \cdots = q_K$. We will show that if the monopolist is beginning from the benchmark of the best fixed prices then neither of these two cases can enhance monopoly profits. Of course, if the monopolist is not charging the best fixed prices then these strategies might be profit enhancing. However, in this case the monopolist would be best advised to alter her menu of fixed prices to the optimal levels. Before continuing it is useful for us to consider possible interpretations of the types of lotteries cases A and B represent:

A. Case A is similar to the one-dimensional case considered in Example 4. Both in that example and here the consumer can elect to purchase a lottery in which consumers receive one bundle with positive probability or nothing. This situation can be interpreted as a very simple bargaining process over bundle 1. The bargaining process is any one such that consumers would have a probability of a deal being struck of q_1 with the expected price conditional on the sale of bundle 1 of p_1. Alternatively the consumers could receive the first bundle with certainty for a price of $p_1 + \varepsilon$, or could receive any of the other bundles with certainty at their fixed prices.

B. Case B is best thought of as a lottery or special offer. In this situation the consumer would buy a ticket in a lottery in which she is guaranteed to win something. The consumer will receive bundle 1 with a probability of q_1 and bundle 2 with a probability of q_2, all for a price of $q_1 p_1 + q_2 p_2$. Alternatively, the consumer could elect to receive the first bundle with certainty, though this will cost $p_1 + \varepsilon$. For example, bundle 1 could be a top of the range TV with bundle 2 a mediocre TV. The lottery consumer would not be willing to purchase the top of the range TV outright at a price of $p_1 + \varepsilon$, but would be prepared to pay less for the chance of receiving it. Either way the lottery consumer will always get a TV from this lottery.

To recap, Theorem 3 aims to show that if the monopolist is beginning from the benchmark of the best fixed prices then neither of these two cases can enhance monopoly profits. We need to calculate the flow of consumers towards the lottery option for small price perturbations ε in these two cases. We introduce the change of variables $\underline{X} = \underline{x} - \underline{p}$ which allows the individual rationality and incentive compatibility conditions to be expressed more succinctly.

Case A: $q_0 + q_1 = 1$, $q_2 = 0 = \cdots = q_K$ **.** The lottery purchaser must satisfy individual rationality and incentive compatibility constraints. These are given by

$$\begin{array}{c} q_1 X_1 \geq 0 \\ \Rightarrow X_1 \geq 0 \end{array} \quad , \quad \begin{array}{rcl} q_1 X_1 & \geq & X_j \ \forall j \geq 2 \\ \\ q_1 X_1 & \geq & X_1 - \varepsilon \Rightarrow X_1 \leq \dfrac{\varepsilon}{q_0} \end{array}$$

Therefore the demand for the lottery option is given by[9]

$$D^l(\varepsilon) = \int_{X_1=0}^{\frac{\varepsilon}{q_0}} \int_{X_2=-p_2}^{q_1 X_1} \cdots\cdots \int_{X_K=-p_K}^{q_1 X_1} f\left(\underline{X}+\underline{p}\right) d\underline{X}$$

The rate of flow of consumers towards the lottery option is therefore given by

$$\left[\frac{\partial D^l}{\partial \varepsilon}\right]_{\varepsilon=0} = \frac{1}{q_0} \int_{X_2=-p_2}^{0} \cdots\cdots \int_{X_K=-p_K}^{0} f\left(\underline{X}+\underline{p}\right)\Big|_{X_1=0} d\underline{X} \qquad (21)$$

Case B: $q_1 + q_2 = 1$, $q_0 = 0 = q_3 = \cdots = q_K$ **.** Once again, the lottery purchaser must satisfy individual rationality and incentive compatibility constraints. These are now given by

$$\begin{aligned} q_1 X_1 + q_2 X_2 &\geq 0 \qquad q_1 X_1 + q_2 X_2 \geq X_2 \qquad q_1 X_1 + q_2 X_2 \geq X_j \ \forall j > 2 \\ q_1 X_1 + q_2 X_2 &\geq X_1 - \varepsilon \end{aligned}$$

There are therefore two cases to consider, $0 \leq X_1 < \varepsilon$ and $X_1 > \varepsilon$. The case $0 \leq X_1 < \varepsilon$ will not contribute to the first order flow of consumers towards the lottery for small price perturbations. To see this note that from the proof of Lemma 5, X_2 must be within an ε band of X_1. If $X_1 \in [0, \varepsilon)$ then the volume of consumers purchasing the lottery from amongst these types is of order $O\left(\varepsilon^2\right)$. This contribution to the flow rate of consumers towards the lottery will therefore vanish as discussed in the proof of Lemma 5. We can therefore confine our attention to $X_1 > \varepsilon$:

$$\begin{aligned} q_1 X_1 + q_2 X_2 &\geq X_2 \Rightarrow q_1 X_1 \geq q_1 X_2 \Rightarrow X_1 \geq X_2 \\ q_1 X_1 + q_2 X_2 &\geq X_1 - \varepsilon \Rightarrow X_2 \geq X_1 - \frac{\varepsilon}{q_2} \end{aligned}$$

Therefore the demand for the lottery option is given by

$$D^l(\varepsilon) = \int_{X_1=\varepsilon}^{\infty} \int_{X_2=X_1-\frac{\varepsilon}{q_2}}^{X_1} \int_{X_3=-p_3}^{q_1 X_1 + q_2 X_2} \cdots\cdots \int_{X_K=-p_K}^{q_1 X_1 + q_2 X_2} f\left(\underline{X}+\underline{p}\right) d\underline{X}$$

The rate of flow of consumers towards the lottery option is therefore given by

$$\left[\frac{\partial D^l}{\partial \varepsilon}\right]_{\varepsilon=0} = \frac{1}{q_2} \int_{X_1=0}^{\infty} \int_{X_3=-p_3}^{X_1} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\Big|_{X_2=X_1} d\underline{X} \qquad (22)$$

[9] Note that the demand for the lottery is not zero due to Lemma 1 which used Assumption 1.

To complete the proof of Theorem 3 we must determine the first order effect on profit of the small price perturbation ε to the good 1 price. That is, we seek to determine the form of $\frac{\partial \Pi^{\text{lot}}}{\partial \varepsilon}\left(p_1+\varepsilon, \underline{p}_{-1}, \underline{p}\cdot\underline{q}\right)$. The proof is completed by showing that this derivative is exactly identical to $\frac{\partial \Pi^{\text{no lot}}}{\partial p_1}$, the first order expression for profit maximising deterministic prices. By assumption the monopolist was initially charging the optimal deterministic prices so that $\frac{\partial \Pi^{\text{no lot}}}{\partial p_1}$ is zero. Then (20) and (19) guarantee that lotteries have no local beneficial effect on profit which proves Theorem 3. We formalise this argument in the form of two further lemmas, the proofs of which are relegated to an appendix.

Lemma 6 *In both Case A and Case B the rate of change of profit with respect to the perturbation ε is given by*

$$\begin{aligned}
&\left[\frac{\partial \Pi^{lot}}{\partial \varepsilon}\right]_{\varepsilon=0} = D^1(0) && (23)\\
&-(p_1-c_1)\left\{\begin{array}{c} \int_{X_2=-p_2}^{0}\cdots\cdot\int_{X_K=-p_K}^{0} f\left(\underline{X}+\underline{p}\right)\big|_{X_1=0}\, d\underline{X} \\ +\sum_{j=2}^{K}\int_{X_1=0}^{\infty}\int_{X_2=-p_2}^{X_1}\cdots\cdot\widehat{\int_{X_j=-p_j}^{X_1}}\cdots\cdot\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1}\, d\underline{X}\end{array}\right\}\\
&+\sum_{j=2}^{K}(p_j-c_j)\int_{X_1=0}^{\infty}\int_{X_2=-p_2}^{X_1}\cdots\cdot\widehat{\int_{X_j=-p_j}^{X_1}}\cdots\cdot\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1}\, d\underline{X}
\end{aligned}$$

where $\widehat{(\cdot)}$ signifies 'is omitted'.

Proof. See appendix. ■

The proof of lemma 6 is essentially an algebraic exercise. For any of the K fixed price options available we can determine which consumers prefer it to all the other options and to the lottery. This is done by writing down all of the individual rationality and incentive compatibility conditions satisfied by consumers who choose the particular bundle considered. We can then explicitly calculate the demand. This will in part depend on ε, that is on the extent to which the first bundle's price is increased. We can then write down the monopolist's profit all as a function of ε. Differentiating with respect to ε and setting ε to zero then gives the result. The expression (23) should be compared to expression (18) of Example 4. This link is formalised in the lemma below.

Lemma 7 *The first order conditions for profit maximising deterministic prices are given by setting expression (23) to zero. In other words*

$$\frac{\partial \Pi^{no\ lot}}{\partial p_1} = \left[\frac{\partial \Pi^{lot}}{\partial \varepsilon}\right]_{\varepsilon=0}$$

Proof. See appendix. ■

These two lemmas therefore complete the proof of Theorem 3.

5 The Implications of Theorem 1 and its Extensions

Theorem 3 has shown us that if a monopolist is offering her products at the optimal deterministic prices then she can never gain by experimenting with augmenting her fixed prices with a single lottery offered at a price near its choke price. In turn, Theorem 2 has shown us that if a monopolist's globally optimal selling strategy is stochastic then she must be able to gain by experimenting with a single lottery deviation from optimal deterministic prices. Between them, theorems 2 and 3 therefore prove Theorem 1 which says that the monopolist's globally optimal selling strategy is the deterministic one. This provides a complete generalisation of the Riley and Zeckhauser [12] result for the one good monopolist. The result of this analysis has far reaching implications. Most importantly it provides an explanation of why few firms actually adopt stochastic selling strategies. At a slightly deeper level the result proves a no haggling result: any haggling game between the monopolist and a consumer will ex ante assign to consumers probabilities of receiving certain goods for certain prices. This is equivalent to a schedule of lotteries and we have shown that the monopolist's optimal strategy is to offer only 0 and 1 probabilities or equivalently take-it-or-leave-it offers.

This result was determined for consumers whose types satisfied the conditions given in Section 2. As might be expected, a result concerning how consumers respond to the offer of *probabilities* of receiving a good will be heavily dependent on how consumer choices are modelled. In this vein, there are perhaps two of the conditions of Section 2 which are worth highlighting:

Consumer Risk Neutrality

Theorem 1 is not robust to a relaxation of the monopolist's screening problem by allowing consumers to be risk averse. This situation is exactly equivalent to allowing the consumers to have decreasing marginal utilities and returns us to the full screening problem. This problem in one dimension has working identical in spirit to that of Mussa and Rosen [9] and has been explicitly derived in Tirole [16] on page 156. Tirole [16] finds that the optimal screening mechanism requires the monopolist to equate each consumers' *virtual type* with an expression involving the rate of change of consumers' marginal utility with respect to the screening instrument. The virtual type is found by adjusting the consumers type to ensure truthful direct revelation. In fact, Riley and Zeckhauser [12] point out that when buyers are risk averse the monopolist can improve her profits by asking the consumers to make a bid for the objects. The consumers are aware that the probability that the monopolist will accept a particular bid increases as the bid increases. The strongly risk averse buyer will therefore bid just a little below her valuation allowing the monopolist to extract almost all of the consumer surplus.[10]

Restrictions on the Consumer Density

Theorem 1 was proved by assuming that consumers types were supported on a set Ω.:

a) The set Ω contained the origin. This implies that there are some consumers in the population who do not value any of the monopolist's products and so will not participate in any offer the monopolist makes and will therefore always have zero surplus.

b) The set Ω was open, convex and with the density function having strictly positive value at all points of Ω. These conditions imply that given any two consumers there exist other consumers with tastes which are any weighted average of the original pair.

c) Given any selection of the monopolist's bundles priced at their optimal fixed prices then there is a non-zero probability of a consumer being indifferent between them and the outside option (Assumption 1). This condition allows us to determine the choke price of any single lottery that the multiproduct monopolist might seek to introduce.

These conditions imply that the utility allocation function determines everything that we need to know about the monopolist's schedule of lotteries and what prices each is charged at. If we break these restrictions then the utility allocation function is no longer sufficient information and the results break down and so Theorem 1 need not hold.

Example 8 gives a market in which conditions (a), (b) and (c) above are violated and in which a stochastic selling strategy is optimal for a two good monopolist.

[10] See Riley and Zeckhauser [12] for further details. In particular, Maskin and Riley [5] have analysed the optimal auctioning mechanism with which to sell one product to n bidders when all of the bidders are risk averse.

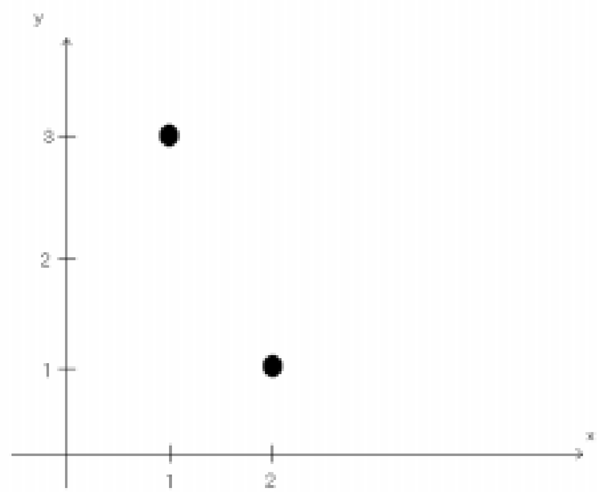

Figure 4: A discrete market which does not satisfy the conditions of Theorem 1. In this case the most profitable tariff is stochastic.

Example 8 *Consider a discrete market with 2 consumers who have additive valuations for the two goods, x and y. Suppose that the firm has no unit costs in production, that one of the consumers is at $(1,3)$ and the other at $(2,1)$. This market is drawn in Figure 4. The optimal deterministic tariff in this case has consumer $(2,1)$ receiving good 1 with probability 1 at a price of 2, and consumer $(1,3)$ receiving both goods with probability 1 at a price of 4. This leaves each consumer with zero surplus. The optimal tariff is however stochastic with the type $(1,3)$ consumer's bundle unchanged and the $(2,1)$ consumer receiving either good 1 only or both goods with equal probability for a price of $2\frac{1}{2}$. This tariff again leaves both consumers with 0 surplus.*

This example fails all three of the conditions (a), (b) and (c) which were given above:

1. The consumer valuation space, $\Omega = \{(1,3)\} \cup \{(2,1)\}$ is closed and not open failing (b). This causes the profit function to not be differentiable in the price vector. In fact, the profit function is not even continuous with respect to the lottery option at the price $l = 2\frac{1}{2}$.
2. The consumer valuation space Ω is not pathconnected (Ω isn't convex) failing (b) for a second reason. The surplus allocation function $U(\underline{x})$ does not therefore determine the

allocation of probabilities. In particular, both the deterministic and the stochastic tariff leave the consumers with zero surplus and yet correspond to completely different probability allocations. The condition that $q(\underline{x}) = \nabla U(\underline{x})$ of lemma 2 doesn't hold and so neither does expression (24) in the proof of Lemma 3.

3. Finally the origin does not lie in the consumer valuation space failing (a). Therefore the construction of (24) breaks down for a second reason.

4. The lottery, $\left\{\Pr(\text{good 1 only}) = \frac{1}{2}, \Pr(\text{both goods}) = \frac{1}{2}\right\}$, introduced in the optimal tariff has a choke price of

$$\text{Choke price} = \frac{1}{2}p^{\text{good 1}} + \frac{1}{2}p^{\text{both}} = \frac{1}{2} \times 2 + \frac{1}{2} \times 4 = 3$$

according to Lemma 1. However, neither of the consumers in Figure 4 is prepared to pay 3 for the lottery. In other words, this is the wrong choke price. The reason is that there is no consumer indifferent between either good 1 alone or both goods together at their fixed prices. This therefore fails Assumption 1 and (c) above.

We are now in a position to assess how the result extends as the underlying characteristics of the model are altered:

Dropping Assumption 1

At some deep level, Assumption 1 ensured that there were enough 'high valuation' type consumers who could switch from buying with certain delivery to buying lotteries to make the potential losses from screening with probabilities exceed the potential benefits. Formally Theorem 2 proved that if a stochastic selling strategy is globally optimal for the multiproduct monopolist then there must exist some lottery $\underline{q}$ which can be introduced alongside the menu of fixed prices which is profit enhancing when the lottery's price is lowered *below its choke price.* Assumption 1 allowed us to calculate this choke price.

If we drop Assumption 1 then there will be some bundles, 1 and 2 say, between which we are *certain* no consumer will be indifferent. The choke price of any lottery between bundles 1 and 2 will therefore be less than $\underline{p} \cdot \underline{q}$. The proof of Theorem 3 now breaks down. Whether this lottery could be profit increasing hinges on how many consumers are indifferent to it at its new choke price. This in turn hinges on the precise shape of Ω, the support of the consumer density function $f(\cdot)$. The question of when, if ever, these lotteries in the absence of Assumption 1 are profit enhancing for the multiproduct monopolist is at present open and is left to future research.

Competition and Strategic Considerations

Perhaps the most important restriction of the model is the fact that we have considered a monopolist. We have therefore abstracted away from any strategic considerations felt by firms. The monopoly case is an important benchmark but can only be a first step towards understanding the screening mechanism in the market. Generalising our results to the full competitive case seems challenging indeed. One might conjecture that lotteries could be used as a way of enticing certain types of consumers away from rivals or as an alternative means of competing, separate from straight price competition.

Furthermore, one would like to analyse how competition might evolve through repeated interaction, extending the one-shot framework we have adopted. Varian [17] analysed the pricing policy of firms selling one good repeatedly in a competitive market to consumers with unit demand. He assumed that each firm offered a take-it-or-leave-it price in every time period, though the level of this price was allowed to vary through time. In a departure from the spirit of our model, he assumed that the population consisted of both informed and uninformed consumers. The uninformed consumers were unaware of the prices available in the market and furthermore all consumers had the same valuation for the good. The only possible screening was therefore between the informed and uninformed consumers. Varian [17] showed that over time firms would randomise their price levels, introducing sales, whilst offering only fixed prices in any particular time period. The motivation for this was strategic: if a firm sets a low price it hopes to set the lowest price and so capture all the informed market. On the other hand, if the firm doesn't capture the informed consumers then it wishes to set a high price and extract as much rent as possible from the uninformed consumers. Firms randomise continuously between these two goals creating equilibrium price dispersion and providing a rationale behind sales. Varian's [17] model neatly therefore highlights the important effects that strategic considerations can have on the firm's optimal pricing policy.

Capacity Constraints and Auctions

In the model above we assumed that the monopolist had no capacity constraints and so was able to meet demand. In this case we have shown that fixed prices are better than auctions. This is because any auctioning mechanism will associate each consumer with a probability of receiving a good and with an expected payment, all dependent on their type. In a technical sense therefore auctions are equivalent to assigning lotteries amongst the consumers. Theorem 1 therefore immediately gives us that optimally set fixed prices are best.

However, Harris and Raviv [4] suppose that the monopolist has capacity constraints and is not able to satisfy all demand realisations. They consider a model in which the monopolist has only one type of good to sell to a finite population of consumers with each consumer having unit demands for the good. Harris and Raviv [4] make direct use of the revelation principal. They note that any selling scheme is equivalent to a direct truthful revelation mechanism. They then order their consumers and show that in this case the monopolist only need ensure that incentive compatibility conditions are satisfied by consumers of adjacent types. Using these insights to simplify the problem they show that the monopolist who is capacity constrained would choose to sell her goods through some form of auction.[11]

Budget Constrained Consumers

In the model above we have assumed that every consumer is able to pay her valuation for a good. The problem faced by the monopolist is completely changed if buyers are not always able to pay their valuations. One might imagine that we could define each consumer's willingness to pay as the minimum of the consumer's valuation and their budget constraint and then proceed with the model as before. This is not however correct as can be shown through the following example:

Example 9 *Suppose a one good monopolist is selling her product to two identical consumers. Each consumer has a valuation of* $\frac{1}{2}$ *for the product and a budget of* 2*. The good is produced at a cost of* 1*. In this case the budget constraint would never bind and the monopolist would not find it profitable to sell the good - there is no market.*

Now suppose that each consumer has a valuation for the product of 2 *but a budget of* $\frac{1}{2}$*. The consumers' willingness to pay is unchanged. The monopolist would now profit by selling each consumer a lottery which awards them the product with a probability of* $\frac{1}{4}$ *for a price of* $\frac{1}{2}$*. The consumers are indifferent between these lotteries and not participating, and the monopolist expects to make a profit of* $\frac{1}{2}$*. Lotteries are therefore optimal for the monopolist facing financially constrained consumers.*

[11] Palfrey [10] extends this work by considering how a multiproduct monopolist with capacity constraints can best auction her products. Palfrey [10] considers whether the monopolist would rather auction the products as one bundle or individually in a number of separate auctions.

This simple example exhibits the intuition that a monopolist can benefit from lotteries if fixed prices would be too expensive for a sufficiently large part of the market. By introducing ability to pay as a consumer type variable we have enlarged the power of lotteries as a screening instrument. Apart from only screening for low valuation consumers as in the above model, lotteries can now also be used to screen out those consumers who have a high valuation for a product but a low ability to pay and extract surplus from these types also. The above example shows that this effect can make lotteries profit enhancing for the monopolist.

A great deal of work has been done on how budget constraints can affect a seller's optimal auctioning format. Che and Gale [2] consider how financial constraints affect the profitability of different auctions for a seller of one good who only has one unit to sell.[12] Che and Gale [2] analyse the effect that budget constraints have on bidding strategies and are therefore able to order first and second price auctions according to expected seller revenue and also on welfare grounds.

Intertemporal Screening

The issue of how the monopolist's optimal pricing policy varies through time has not been addressed by the above model. In the case of repeated sales mentioned above, Varian [17] gives a strategic reason for why firms in competition might not remain offering one fixed price but alter their prices during every time period for each new sale. Importantly however, we have yet to address how the monopolist might attempt to screen consumers through time during the course of one sale period. That is, we have yet to discuss the benefits of intertemporal screening.

a) Intertemporal Screening and Information

One avenue is explored by Courty and Hao [3] who consider the situation in which consumers learn their valuations over time.[13] In their model consumers come into contact with the monopolist knowing *how likely* they are to have a certain value for the products but not yet being certain what their valuation will actually be. The problem now becomes intricately linked to how much information the consumers actually have when they agree on a contract with the monopolist. Courty and Hao [3] show that if the consumers' learnt

[12] Auctions are therefore important as capacity constraints are clearly an issue.

[13] Courty and Hao [3] give a number of examples which are well modelled by this approach. These include car rentals (the consumer's preference over mileage might not be known), telephone pricing (the consumer's requirements for calls and of what length is learnt later) and day passes for public transport (the consumer might not be sure how much transport she will require).

valuations are restricted in a number of ways to differ in only one dimension then indeed the optimal screening mechanism is deterministic. This parallels the results of Riley and Zeckhauser [12]. Courty and Hao [3] go on to point out that the analysis of the full multidimensional problem (the subject of this paper) is very difficult and rarely tractable. They have considered a discrete example of three separate consumers. In Example 8 we considered such discrete populations and showed that one can find such populations which do not satisfy the simple regularity assumptions used in the literature and for whom the results of the model do not hold. In the same way Courty and Hao [3] show that in their discrete example randomisation is used by the monopolist. It seems a promising line of research to investigate the extent to which the results of this paper shed light on the Courty and Hao [3] paper and so determine whether their three consumer example is generic or not.

b) Intertemporal Screening and Commitment

We have shown that if a multiproduct monopolist interacts with consumers only once, then take-it-or-leave-it prices are optimal. However, this observation requires the monopolist to be able to commit to not bargaining should a particular consumer say no to a take-it-or-leave-it offer. The Coase Conjecture highlights how important this commitment assumption is. In particular, if the monopolist cannot commit and offers consumers new prices when an offer is rejected then Coase conjectured that the monopolist would quickly find herself offering the goods to all consumers at their marginal cost and so make zero profit. Tirole [16] gives a heuristic proof of the Coase Conjecture for the single good monopolist.[14] However, through work in a bargaining context, Wang [18] has shown that the *multiproduct* monopolist's plight might not be so dire. Wang [18] considered a risk averse uninformed principal bargaining with an informed agent, who was one of two types, over two dimensions: the wage and the quality of work supplied. Now that bargaining is taking place over multiple dimensions Wang [18] showed that the principal did not lose all of her bargaining power as suggested by the Coase conjecture. The intuition for this result is that the principal can offer two different contracts in a menu: one of which gives the low type of worker a small wage in return for little work quality which just leaves her indifferent and the second provides the high type worker with sufficient rents to just make the contract incentive compatible. As the low type worker accepts the contract intended

[14]See Tirole [16] page 91.

for her so will the high type worker. Failure to do so would separate the worker types and the high type worker would then expect zero surplus from the contract offered in the next round. The monopolist, even without precommitment, will always offer the same set of contracts as is optimal in the one-shot game. This strategy is credible as both types of worker accept their contract in the first round.

Wang [18] abstracts from a discussion of whether a principal would offer random contracts by assuming that the principal is risk averse. However, Wang [18]'s research suggests an escape from the Coase trap for the multiproduct monopolist: by offering a sufficiently large menu of different options to the consumers at fixed prices which induces participation by all consumers who value at least one of the bundles of goods above its marginal cost. Such a menu would remove the need for credibility from the monopolist as there would be no gains from trade left to exploit amongst the non-participating consumers. However, this menu of choices cannot be the optimal fixed prices from the one-shot game. This is because it is optimal for the monopolist to not serve all of the consumers in the one shot game, even if some of them value the good above cost. Without precommitment the monopolist could not guarantee not to lower her price to serve these consumers in the future. This is not relevant when screening with quality (as in Wang [18] with bargaining or Mussa and Rosen [9] with a monopolist) as now agents/consumers' utility does not vary linearly in the quality level and so all the agents/consumers are served.

Therefore, if the monopolist sought to offer no lotteries in a credible menu then she would need to offer a menu of sub-optimal fixed prices to ensure that all consumers who value a good above cost participate. In this case example 4 suggests that randomising may well be optimal. This is because we noted that lotteries (or bargaining) can be profit enhancing if the monopolist is not charging optimal take-it-or-leave-it prices. This intuition is supported by the following discrete example for a single good monopolist:

Example 10 *Suppose a single good monopolist serves two consumers whose valuations for the product are* 1 *and* 2*. The monopolist doesn't know which consumer is which and has a unit cost of production of* $c < 1$*. With precommitment the monopolist's optimal selling strategy is to set a fixed take-it-or-leave-it price of* $p = 2$ *and so not serve the low type consumer. Without precommitment this is not credible. The intuition from Wang [18] highlights that the monopolist must offer a menu of choices from which both consumers participate. The most profitable such menu is for the monopolist to offer the good with probability* ε *at a price of* $\varepsilon > 0$ *(intended for* *the low type consumer) and also offer the good with certainty at a price of* $2-\varepsilon$ *(intended for the high type consumer). Using* Π *to denote profit we have*

$$\begin{aligned}\Pi\,(\textit{With precommitment}) &= 2-c\\ \Pi\,(\textit{Without precommitment}) &= 2-c-\varepsilon c\end{aligned}$$

Wang [18]'s principal lost nothing through having no credibility when using quality as the screening instrument. Here we see that the monopolist with no credibility who can only screen with the probability of delivery is within ε *of her highest profit level when she does have credibility. To achieve this however the monopolist had to offer the consumers a lottery option as well as a fixed take it or leave it price.*

It is therefore important for us to investigate generally the extent to which lotteries are necessary for a monopolist to credibly maintain her prices.

6 Conclusion

In this paper we have been able to prove the theorem that a multiproduct monopolist's optimal selling strategy is deterministic. This has been done by breaking the problem down into two further theorems. We firstly showed that if the monopolist's globally optimal pricing policy is to offer a schedule of lotteries, then the monopolist can always improve her profits from the optimal deterministic prices by introducing some single lottery with fixed probabilities whose price can be perturbed by some small amount from the level at which consumers are just indifferent to it (its choke price). The second theorem was then able to go a step further. We showed that a multiproduct monopolist would not actually benefit from experimenting with adding a single lottery option to the optimal deterministic prices. These two results therefore conspire to prove Theorem 1. That is, we have shown that a multiproduct monopolist can do no better than setting optimal fixed prices: selling lotteries or bargaining with consumers is not a more profitable strategy.

This paper has therefore solved a version of the general multidimensional screening problem and extended the results for the one good monopolist of Riley and Zeckhauser [12]. It is therefore interesting to consider what progress we have made towards the general multidimensional screening problem. The problem we have analysed has become tractable because of an important formal limitation: the firm's cost function and consumers' derived utility both vary exactly proportionately to the screening instruments - the probability vector $\underline{q}$ here - as both parties are risk neutral. The general screening problem would allow consumers to have decreasing marginal utility of consumption and the monopolist to have non-constant unit costs. It is for this fundamental reason that even in the single good case the optimal screening mechanism using quantity of Mussa and Rosen [9] is so different to the optimal screening mechanism using probabilities discussed above and in Riley and Zeckhauser [12]. The general multidimensional screening problem is notoriously complicated. Rochet and Choné [14] have made a very substantial contribution to this area. They have shown that in most such problems the monopolist will choose to bunch some consumers. That is to offer a group of consumers with different valuations the same quality of products. We therefore note that in the context of the Rochet and Choné [14] work our results here seem intuitive: we have shown that in the boundary case of constant returns to scale derived through the sale of lotteries then the monopolist's optimal strategy is to *only* indulge in bunching. In this case the monopolist should offer a menu of fixed prices which partition all the consumers with each group receiving one particular bundle of goods at a fixed price with probability 1 of delivery.

A Proofs of technical results

Proof of Lemma 3. When consumers are offered the N lotteries at prices given by the vector $\underline{p}_\lambda^N$, they pick the lottery which gives them most utility. We suppose that consumers of type $\underline{x} \in J_i \subset \Omega = \operatorname{supp} f$ pick lottery $\underline{q}^i$ $(i \in \{1, \ldots N\})$. Utility maximisation implies that the sets J_i are convex and hence connected. We have that 0 is one of the lotteries offered, all consumers will participate with at least the consumers at the origin opting for the outside option. Therefore

$$\cup_{i=1}^N J_i = \operatorname{supp} f$$

We can therefore associate a step function $q_\lambda^N : \Omega \subset \mathbb{R}_+^K \to Q$ given by

$$q_\lambda^N(\underline{x}) = \sum_{i=1}^N \underline{q}^i \chi_{J_i}(\underline{x})$$

with $\chi(\underline{x})$ being the standard characteristic function.

We earlier assumed that a consumer of type 0 exists (so $0 \in \Omega$) and that Ω was convex. The set $\Omega \subset \mathbb{R}_+^K$ is therefore connected and so given any consumer of type $\underline{x}$ there exists a differentiable path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = 0$ and $\gamma(1) = \underline{x}$. The allocation step function $q_\lambda^N(\underline{x})$ generates a new surplus allocation function, $U_\lambda^N(\underline{x})$ where $\nabla U_\lambda^N(\underline{x}) = q_\lambda^N(\underline{x})$. We therefore have that

$$\begin{aligned} U_\lambda^N(\underline{x}) &= \int_{\operatorname{Im}\gamma} q_\lambda^N \circ \gamma \, d\gamma + U_\lambda^N(\gamma(0)) \\ &= \int_{\operatorname{Im}\gamma} q_\lambda^N \circ \gamma \, d\gamma + \underbrace{U_\lambda^N(0)}_{=0} \end{aligned} \tag{24}$$

and so using (10) the profit of the firm is given by

$$\Pi\left(U_\lambda^N(\cdot)\right) = \int_{\underline{x}\in\Omega} \left\{ q_\lambda^N(\underline{x}) \cdot (\underline{x} - \underline{c}) - \int_{\operatorname{Im}\gamma} q_\lambda^N \circ \gamma \, d\gamma \right\} f(\underline{x}) \, d\underline{x}$$

The step function $q_\lambda^N(\underline{x}) \to q_\lambda$ as N tends to infinity by construction. The profit function Π is however composed of integrals of the allocation function q multiplied by a differentiable f over a bounded, convex region, supp f. Therefore standard results of Lebesgue Integration[15] give us the stated result. (see Weir [19]). ■

Proof of Lemma 5. The mass of consumers who purchase the lottery is given by $D^l(\varepsilon)$. This expression is the integral of the consumer density function $f(\cdot)$ over a polytope in consumer type space. The proof of the lemma proceeds by finding bounds to the dimensions of this polytope. In particular, if the polytope has length of order ε in two dimensions, then the demand for the lottery, $D^l(\varepsilon)$, will be given by

$$D^l(\varepsilon) = \int \cdots \int_{\text{volume } O(\varepsilon^2)} f(\underline{x}) \, d\underline{x}$$

When the demand is then differentiated with respect to ε and evaluated at $\varepsilon = 0$ this contribution will vanish and so prove the lemma.

To formalise this argument, suppose that the monopolist introduces the lottery $\underline{q}$ with $q_1 > 0$ and raises the price of good 1 by ε. The consumers who choose to purchase the lottery must satisfy individual rationality and incentive compatibility constraints. The constraints are most easily written down by using the change of variables $\underline{X} = \underline{x} - \underline{p}$ and are given by

$$\underline{q} \cdot \underline{X} \geq 0 \tag{25}$$

$$\underline{q} \cdot \underline{X} \geq X_1 - \varepsilon \tag{26}$$

$$\underline{q} \cdot \underline{X} \geq X_j \quad \forall j > 1 \tag{27}$$

[15] In particular the result is an application of the Dominated Convergence Theorem given on p109 of Weir [19].

The inequalities (27) form a cone with the coordinates of the vertex (x) satisfying

$$X_2 = \underline{q} \cdot \underline{X} = X_3 = \cdots = X_K = \mathrm{x}$$

Substituting this into a member of (27) gives

$$\begin{aligned} q_1 X_1 + \left(\sum_{i=2}^{K} q_i \right) \mathrm{x} &= \mathrm{x} \\ \Rightarrow \quad \mathrm{x} &= \left(\frac{q_1}{q_1 + q_0} \right) X_1 \end{aligned} \tag{28}$$

We now turn our attention to inequalities (25) and (26). We see that if $X_1 \geq \varepsilon$ then (26) implies (25). In addition we see that (26) becomes easier to satisfy (other things equal) as X_1 decreases. On the other hand the cone defined by (27) becomes easier to satisfy as X_1 increases. Therefore, for the cone of (27) to have an interior at all we must have the vertex, x, satisfying inequality (26). This implies

$$\underline{q} \cdot \underline{X} \geq X_1 - \varepsilon \Leftrightarrow q_1 X_1 + (1 - q_0 - q_1)\,\mathrm{x} \geq X_1 - \varepsilon$$

Substituting in the expression for the vertex, x, gives

$$X_1 \leq \varepsilon \left(1 + \frac{q_1}{q_0} \right)$$

Returning to (27) we see that the right hand side is the geometric average of all the coordinates $\{X_i\}$. In addition it is clear that $\underline{q} \cdot \underline{X} \leq \max\{X_i\}$ with equality implying that $\max\{X_i\} = X_1$. We must therefore have $X_1 \geq X_j$ for all $j > 1$. Now suppose that $X_1 < \varepsilon$ so that (25) implies (26). In this case, as $X_1 \geq X_j$ and $\underline{q} \cdot \underline{X} \geq 0$ we must have $X_1 \geq 0$. Therefore the consumers who purchase the lottery satisfy

$$0 \leq X_1 \leq \varepsilon \left(1 + \frac{q_1}{q_0} \right) \tag{29}$$

Having determined a bound for the lottery consumers in terms of their X_1 variable we move on to the variable X_2. Clearly the cone formed by the inequalities (27) gives the vertex as one bound at $X_2 = \left(\frac{q_1}{q_1 + q_0} \right) X_1$. The second bound is determined by finding the X_2 coordinate of the vertex of the cone given by

$$\left(\underline{q} \cdot \underline{X} \geq X_j \quad \forall j > 2 \right) \text{ and } \begin{cases} \underline{q} \cdot \underline{X} \geq 0 & 0 \leq X_1 < \varepsilon \\ \underline{q} \cdot \underline{X} \geq X_1 - \varepsilon & X_1 \geq \varepsilon \end{cases}$$

Again denoting the vertex by x we have $X_3 = \mathrm{x} = \cdots = X_K$. Which substituting in gives

$$\begin{aligned} q_1 X_1 + q_2 X_2 + (1 - q_0 - q_1 - q_2)\,\mathrm{x} &= \mathrm{x} \\ \Rightarrow q_1 X_1 + q_2 X_2 &= (q_0 + q_1 + q_2)\,\mathrm{x} \end{aligned}$$

If $X_1 \geq \varepsilon$ then at the vertex x we have

$$\begin{aligned} \underline{q} \cdot \underline{X} &\geq X_1 - \varepsilon \\ &\Rightarrow (q_1 X_1 + q_2 X_2) \left[1 + \frac{1 - q_0 - q_1 - q_2}{q_0 + q_1 + q_2} \right] \geq X_1 - \varepsilon \\ &\Rightarrow q_2 X_2 \geq (q_0 + q_2) X_1 - (q_0 + q_1 + q_2)\,\varepsilon \end{aligned}$$

which gives

$$\left(1 + \frac{q_0}{q_2}\right) X_1 - \left(1 + \frac{q_0 + q_1}{q_2}\right) \varepsilon \leq X_2 \leq \left(1 - \frac{q_0}{q_1 + q_0}\right) X_1 \tag{30}$$

If $0 \leq X_1 < \varepsilon$ then at the vertex x we have

$$\begin{aligned} \underline{q} \cdot \underline{X} &\geq 0 \\ &\Rightarrow (q_1 X_1 + q_2 X_2) \left[1 + \frac{1 - q_0 - q_1 - q_2}{q_0 + q_1 + q_2} \right] \geq 0 \\ &\Rightarrow X_2 \geq -\frac{q_1}{q_2} X_1 \end{aligned}$$

which gives

$$-\frac{q_1}{q_2} X_1 \leq X_2 \leq \left(1 - \frac{q_0}{q_1 + q_0}\right) X_1 \tag{31}$$

1. Suppose that $q_1 > 0$, $q_0 > 0$ and $q_2 > 0$. In this case equation (29) gives us that the polytope of consumers who purchase the lottery has X_1 dimension of order ε. Given this, (30) and (31) guarantee that X_2 is also restricted to a region of order ε. Therefore the volume of the consumer polytope of those purchasing the lottery is $O\left(\varepsilon^2\right)$. As the density function of consumers is bounded with bounded derivatives we must have that $\frac{\partial D^l}{\partial \varepsilon} = O(\varepsilon)$ which implies that $\left[\frac{\partial D^l}{\partial \varepsilon}\right]_{\varepsilon=0} = 0$.

2. Now suppose that $q_0 = 0$ but that $q_1, q_2, q_3 > 0$. In this case (29) gives that $X_1 \in [0, \infty)$. However, for $X_1 > \varepsilon$ we have (from (30))

$$X_1 - \left(1 + \frac{q_1}{q_2}\right) \varepsilon \leq X_2 \leq X_1$$

which implies that X_2 is restricted to a region of order ε.[16] Similarly, as $q_3 > 0$ we have that

$$X_1 - \left(1 + \frac{q_1}{q_3}\right)\varepsilon \leq X_3 \leq X_1$$

Therefore the volume of the consumer polytope of those purchasing the lottery is $O\left(\varepsilon^2\right)$. Arguing exactly as above gives us that $\frac{\partial D^l}{\partial \varepsilon} = O(\varepsilon)$ which implies that $\left[\frac{\partial D^l}{\partial \varepsilon}\right]_{\varepsilon=0} = 0.$

■

Proof of Lemma 6. We split the proof of this lemma into its two subcases

Case A: $q_0 + q_1 = 1$, $q_2 = 0 = \cdots = q_K$.

We consider first Case A under which $1 = q_0 + q_1$ with all other lottery probabilities 0. Consumers who demand good 1 will satisfy individual rationality constraints and incentive compatibility constraints given by

$$\begin{aligned} X_1 - \varepsilon &\geq 0 \Rightarrow X_1 \geq \varepsilon \qquad , \qquad X_1 - \varepsilon \geq X_j \ \forall j > 1 \\ X_1 - \varepsilon &\geq q_1 X_1 \Rightarrow X_1 \geq \frac{\varepsilon}{q_0} > \varepsilon \end{aligned}$$

The demand for good 1 is therefore given by

$$D^1(\varepsilon) = \int_{X_1=\frac{\varepsilon}{q_0}}^{\infty} \int_{X_2=-p_2}^{X_1-\varepsilon} \cdots\cdots \int_{X_K=-p_K}^{X_1-\varepsilon} f\left(\underline{X} + \underline{p}\right) d\underline{X}$$

Taking the derivative with respect to ε then gives us the flow of consumers towards good 1 as

$$\begin{aligned} \left[\frac{\partial D^1}{\partial \varepsilon}\right]_{\varepsilon=0} &= -\frac{1}{q_0} \int_{X_2=-p_2}^{0} \cdots\cdots \int_{X_K=-p_K}^{0} f\left(\underline{X} + \underline{p}\right)\big|_{X_1=0} d\underline{X} \\ &\quad - \sum_{j=2}^{K} \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_1}} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X} + \underline{p}\right)\big|_{X_j=X_1} d\underline{X} \quad (32) \end{aligned}$$

where $\widehat{(\cdot)}$ signifies 'is omitted' as is standard.

[16] Clearly if $X_1 \in [0, \varepsilon)$ then (31) guarantees that as X_2 is restricted to a region of order ε. This is sufficient to guarantee that this contribution to $\left[\frac{\partial D^l}{\partial \varepsilon}\right]_{\varepsilon=0}$ vanishes.

The consumers who demand good j with $j > 1$ will have types satisfying the individual rationality and incentive compatibility constraints:

$$\begin{array}{rcl} X_j & \geq & 0 \qquad , \qquad X_i \leq X_j \ \forall i \neq 1, j \\ \left.\begin{array}{r} X_j \geq X_1 - \varepsilon \\ X_j \geq q_1 X_1 \end{array}\right\} & \Rightarrow & X_j \geq X_1 - \varepsilon \text{ is sufficient for small } \varepsilon \end{array}$$

The demand for good j is therefore given by

$$D^j(\varepsilon) = \int_{X_j=0}^{\infty} \int_{X_1=-p_1}^{X_j+\varepsilon} \int_{X_2=-p_2}^{X_j} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_j}} \cdots\cdots \int_{X_K=-p_K}^{X_j} f\left(\underline{X}+\underline{p}\right) d\underline{X}$$

The flow of consumers towards good j is therefore given by

$$\begin{aligned} \left[\frac{\partial D^j}{\partial \varepsilon}\right]_{\varepsilon=0} &= \int_{X_j=0}^{\infty} \int_{X_2=-p_2}^{X_j} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_j}} \cdots\cdots \int_{X_K=-p_K}^{X_j} f\left(\underline{X}+\underline{p}\right)\Big|_{X_1=X_j} d\underline{X} \\ &= \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_1}} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\Big|_{X_j=X_1} d\underline{X} \qquad (33) \end{aligned}$$

where the final equality comes from a change of variable.

The profit function of the firm is given by

$$\Pi^{lot} = (p_1 - c_1 + \varepsilon) D^1(\varepsilon) + \sum_{j=2}^{K} (p_j - c_j) D^j(\varepsilon) + q_1 (p_1 - c_1) D^l(\varepsilon)$$

Taking the derivative with respect to ε and setting ε to 0 gives

$$\left[\frac{\partial \Pi^{lot}}{\partial \varepsilon}\right]_{\varepsilon=0} = D^1(0) + (p_1 - c_1)\left\{\frac{\partial D^1}{\partial \varepsilon} + q_1 \frac{\partial D^l}{\partial \varepsilon}\right\} + \sum_{j=2}^{K} (p_j - c_j) \frac{\partial D^j}{\partial \varepsilon}$$

Now substituting the demand functions into this expression gives

$$\begin{aligned} \left[\frac{\partial \Pi^{\text{lot}}}{\partial \varepsilon}\right]_{\varepsilon=0} &= D^1(0) \\ &- (p_1 - c_1)\left\{\begin{array}{c} \underbrace{\left(\frac{1}{q_0} - \frac{q_1}{q_0}\right)}_{=1} \int_{X_2=-p_2}^{0} \cdots\cdots \int_{X_K=-p_K}^{0} f\left(\underline{X}+\underline{p}\right)\Big|_{X_1=0} d\underline{X} \\ + \sum_{j=2}^{K} \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_1}} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\Big|_{X_j=X_1} d\underline{X} \end{array}\right\} \\ &+ \sum_{j=2}^{K} (p_j - c_j) \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_1}} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\Big|_{X_j=X_1} d\underline{X} \end{aligned}$$

which is precisely the expression given in the statement of lemma 6.

Case B: $q_1 + q_2 = 1$, $q_0 = 0 = q_3 = \cdots = q_K$ **.**

We now turn to Case B under which $q_1 + q_2 = 1$ with all other lottery probabilities (including q_0 the probability of receiving nothing) equal to 0. Consumers who demand good 1 will satisfy individual rationality constraints and incentive compatibility constraints given by

$$\begin{aligned} X_1 - \varepsilon &\geq 0 \Rightarrow X_1 \geq \varepsilon \qquad , \qquad X_1 - \varepsilon \geq X_j \ \forall j > 1 \\ X_1 - \varepsilon &\geq q_1 X_1 + q_2 X_2 \Rightarrow X_2 \leq X_1 - \frac{\varepsilon}{q_2} \end{aligned}$$

Hence, noting that $\frac{\varepsilon}{q_2} > \varepsilon$ we have the demand for good 1 given by

$$D^1(\varepsilon) = \int_{X_1=\varepsilon}^{\infty} \int_{X_2=-p_2}^{X_1-\frac{\varepsilon}{q_2}} \int_{X_3-p_3}^{X_1-\varepsilon} \cdots\cdots \int_{X_K=-p_K}^{X_1-\varepsilon} f\left(\underline{X}+\underline{p}\right) d\underline{X}$$

Taking the derivative with respect to ε gives the flow of consumers towards good 1 as

$$\begin{aligned} \left[\frac{\partial D^1}{\partial \varepsilon}\right]_{\varepsilon=0} = & -\int_{X_2=-p_2}^{0} \cdots\cdots \int_{X_K=-p_K}^{0} f\left(\underline{X}+\underline{p}\right)\big|_{X_1=0} d\underline{X} \\ & -\frac{1}{q_2} \int_{X_1=0}^{\infty} \int_{X_3=-p_3}^{X_1} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_2=X_1} d\underline{X} \\ & -\sum_{j=3}^{K} \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_1}} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1} d\underline{X} \end{aligned}$$

The consumers who demand good 2 will have types which satisfy the conditions

$$X_2 \geq 0 \qquad , \qquad X_2 \geq X_j \ \forall j > 2$$

$$\left.\begin{aligned} X_2 &\geq X_1 - \varepsilon \\ X_2 \geq q_1 X_1 + q_2 X_2 &\Rightarrow X_2 \geq X_1 \end{aligned}\right\} \Rightarrow \qquad X_2 \geq X_1$$

These inequalities are clearly independent of ε and so $\left[\frac{\partial D^2}{\partial \varepsilon}\right] \equiv 0$.

The consumers who demand good j with $j > 2$ will have types which satisfy the individual rationality and incentive compatibility constraints:

$$\begin{aligned} X_j &\geq 0 \qquad , \qquad X_j \geq X_2 \qquad , \qquad X_i \leq X_j \ \forall i \neq 1, j \\ X_j &\geq X_1 - \varepsilon \\ X_j &\geq q_1 X_1 + q_2 X_2 \end{aligned}$$

We again note that for sufficiently small ε the final inequality is satisfied. The demand for good j is therefore given by

$$D^j(\varepsilon) = \int_{X_j=0}^{\infty} \int_{X_1=-p_1}^{X_j+\varepsilon} \int_{X_2=-p_2}^{X_j} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_j}} \cdots\cdots \int_{X_K=-p_K}^{X_j} f\left(\underline{X}+\underline{p}\right) d\underline{X}$$

The flow of consumers towards good j is therefore given by

$$\begin{aligned}
\left[\frac{\partial D^j}{\partial \varepsilon}\right]_{\varepsilon=0} &= \int_{X_j=0}^{\infty} \int_{X_2=-p_2}^{X_j} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_j}} \cdots\cdots \int_{X_K=-p_K}^{X_j} f\left(\underline{X}+\underline{p}\right)\Big|_{X_1=X_j} d\underline{X} \\
&= \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_1}} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\Big|_{X_j=X_1} d\underline{X}
\end{aligned}$$

where the final equality comes from a change of variable.

The profit function of the firm is given by

$$\begin{aligned}
\Pi^{lot} &= (p_1 - c_1 + \varepsilon) D^1(\varepsilon) + (p_2 - c_2) D^2(\varepsilon) + \sum_{j=2}^{K} (p_j - c_j) D^j(\varepsilon) \\
&\quad + [q_1 (p_1 - c_1) + q_2 (p_2 - c_2)] D^l(\varepsilon)
\end{aligned}$$

Taking the derivative with respect to ε and setting ε to 0 gives

$$\begin{aligned}
\left[\frac{\partial \Pi^{lot}}{\partial \varepsilon}\right]_{\varepsilon=0} &= D^1(0) + (p_1 - c_1)\left\{\frac{\partial D^1}{\partial \varepsilon} + q_1 \frac{\partial D^l}{\partial \varepsilon}\right\} \\
&\quad + (p_2 - c_2)\left\{\frac{\partial D^2}{\partial \varepsilon} + q_2 \frac{\partial D^l}{\partial \varepsilon}\right\} + \sum_{j=2}^{K} (p_j - c_j) \frac{\partial D^j}{\partial \varepsilon}
\end{aligned}$$

Now substituting the demand functions into this expression gives

$$\left[\frac{\partial \Pi^{\text{lot}}}{\partial \varepsilon}\right]_{\varepsilon=0} = D^1(0)$$

$$-(p_1-c_1)\left\{\begin{array}{c} \int_{X_2=-p_2}^{0}\cdots\cdots\int_{X_K=-p_K}^{0} f\left(\underline{X}+\underline{p}\right)\big|_{X_1=0}\, d\underline{X} \\ +\underbrace{\left(\frac{1}{q_2}-\frac{q_1}{q_2}\right)}_{=1}\int_{X_1=0}^{\infty}\int_{X_3=-p_3}^{X_1}\cdots\cdots\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_2=X_1}\, d\underline{X} \\ +\sum_{j=3}^{K}\int_{X_1=0}^{\infty}\int_{X_2=-p_2}^{X_1}\cdots\cdots\widehat{\int_{X_j=-p_j}^{X_1}}\cdots\cdots\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1}\, d\underline{X} \end{array}\right\}$$

$$+(p_2-c_2)\underbrace{\left(0+\frac{q_2}{q_2}\right)}_{=1}\int_{X_1=0}^{\infty}\int_{X_3=-p_3}^{X_1}\cdots\cdots\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_2=X_1}\, d\underline{X}$$

$$+\sum_{j=3}^{K}(p_j-c_j)\int_{X_1=0}^{\infty}\int_{X_2=-p_2}^{X_1}\cdots\cdots\widehat{\int_{X_j=-p_j}^{X_1}}\cdots\cdots\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1}\, d\underline{X}$$

which is again precisely the expression given in the statement of lemma 6. ■

Proof of Lemma 7. We consider a monopolist with K possible different options to sell at prices p_j for good j. We suppose that the demand for good j at prices $\underline{p}$ is given by $D^j\left(\underline{p}\right)$. The firms profit function is then given by

$$\Pi^{\text{no lot}}\left(\underline{p}\right) = \sum_{i=1}^{K}(p_i-c_i)\, D^i\left(\underline{p}\right)$$

We suppose that the profit function Π has a unique optimal price vector which satisfies the standard first order conditions. The first order condition for this problem then becomes

$$\begin{aligned} \frac{\partial \Pi^{\text{no lot}}}{\partial p_1} &= D^1\left(\underline{p}\right)+\sum_{i=1}^{K}(p_i-c_i)\frac{\partial D^i}{\partial p_1}\left(\underline{p}\right) \\ &= 0 \text{ at optimum} \end{aligned} \tag{34}$$

A good 1 consumer with type $\underline{x}$ must satisfy individual rationality and incentive compatibility constraints given by

$$x_1-p_1 \geq \max\{0, x_2-p_2, \ldots, x_K-p_K\}$$

Using the transformation $\underline{X} = \underline{x} - \underline{p}$ we see that individual rationality and incentive compatibility constraints can be rewritten as:[17]

$$X_1 \geq 0 \qquad , \qquad X_1 \geq X_j \ \forall j > 1$$

The demand for good 1 is therefore given by

$$D^1\left(\underline{p}\right) = \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X} + \underline{p}\right) d\underline{X}$$

Before we can differentiate this as required by (34) we return to the variables $\underline{x} = \underline{X} + \underline{p}$. This then gives us the demand for good 1 as

$$D^1\left(\underline{p}\right) = \int_{x_1=p_1}^{\infty} \int_{x_2=0}^{x_1-p_1+p_2} \cdots \int_{x_K=0}^{x_1-p_1+p_K} f\left(\underline{x}\right) d\underline{x}$$

As the density function $f(\cdot)$ is bounded with bounded derivatives we can take the derivative through the integral of $f(\cdot)$. We can therefore differentiate the demand for good 1 and determine that

$$\begin{aligned}
\frac{\partial D^1}{\partial p_1} &= -\int_{x_2=0}^{p_2} \cdots \int_{x_K=0}^{p_K} f\left(\underline{x}\right)\Big|_{x_1=p_1} d\underline{x} \\
&\quad -\sum_{j=2}^{K} \int_{x_1=p_1}^{\infty} \int_{x_2=0}^{x_1-p_1+p_2} \cdots \widehat{\int_{x_j=0}^{x_1-p_1+p_j}} \cdots \int_{x_K=0}^{x_1-p_1+p_K} f\left(\underline{x}\right)\Big|_{x_j=x_1-p_1+p_j} d\underline{x}
\end{aligned}$$

and similarly

$$\frac{\partial D^1}{\partial p_j} = \int_{x_1=p_1}^{\infty} \int_{x_2=0}^{x_1-p_1+p_2} \cdots \widehat{\int_{x_j=0}^{x_1-p_1+p_j}} \cdots \int_{x_K=0}^{x_1-p_1+p_K} f\left(\underline{x}\right)\Big|_{x_j=x_1-p_1+p_j} d\underline{x}$$

Changing variables so that $\underline{X} = \underline{x} - \underline{p}$ we can rewrite these expressions as

$$\begin{aligned}
\frac{\partial D^1}{\partial p_1} &= -\int_{X_2=-p_2}^{0} \cdots\cdots \int_{X_K=-p_K}^{0} f\left(\underline{X} + \underline{p}\right)\Big|_{X_1=0} d\underline{X} \\
&\quad -\sum_{j=2}^{K} \int_{X_1=0}^{\infty} \int_{X_2=-p_2}^{X_1} \cdots\cdots \widehat{\int_{X_j=-p_j}^{X_1}} \cdots\cdots \int_{X_K=-p_K}^{X_1} f\left(\underline{X} + \underline{p}\right)\Big|_{X_j=X_1} d\underline{X} \qquad (35)
\end{aligned}$$

[17] The transformation $\underline{X} = \underline{x} - \underline{p}$ was used throughout the proof of Theorem 3 above.

and similarly

$$\frac{\partial D^j}{\partial p_1} = \int_{X_j=0}^{\infty}\int_{X_2=-p_2}^{X_j}\cdots\cdots\widehat{\int_{X_j=-p_j}^{X_j}}\cdots\cdots\int_{X_K=-p_K}^{X_j} f\left(\underline{X}+\underline{p}\right)\big|_{X_1=X_j}\,d\underline{X}$$
$$= \int_{X_1=0}^{\infty}\int_{X_2=-p_2}^{X_1}\cdots\cdots\widehat{\int_{X_j=-p_j}^{X_1}}\cdots\cdots\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1}\,d\underline{X} \qquad (36)$$

where the expression for $\frac{\partial D^j}{\partial p_1}$ comes from that for $\frac{\partial D^1}{\partial p_j}$ by symmetry. The first line then incorporates the change of variables $\underline{X} = \underline{x} - \underline{p}$. The second line follows by relabelling X_j as X_1 which has no effect on the integral's value.

Finally, we substitute (35) and (36) into the profit expression, (34). This gives us the first order condition as

$$\frac{\partial \Pi^{\text{no lot}}}{\partial p_1} = D^1\left(\underline{p}\right)$$
$$-\left(p_1-c_1\right)\left\{\begin{array}{c} \int_{X_2=-p_2}^{0}\cdots\cdots\int_{X_K=-p_K}^{0} f\left(\underline{X}+\underline{p}\right)\big|_{X_1=0}\,d\underline{X} \\ +\sum_{j=2}^{K}\int_{X_1=0}^{\infty}\int_{X_2=-p_2}^{X_1}\cdots\cdots\widehat{\int_{X_j=-p_j}^{X_1}}\cdots\cdots\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1}\,d\underline{X}\end{array}\right\}$$
$$+\sum_{j=2}^{K}\left(p_j-c_j\right)\int_{X_1=0}^{\infty}\int_{X_2=-p_2}^{X_1}\cdots\cdots\widehat{\int_{X_j=-p_j}^{X_1}}\cdots\cdots\int_{X_K=-p_K}^{X_1} f\left(\underline{X}+\underline{p}\right)\big|_{X_j=X_1}\,d\underline{X}$$
$$= 0 \text{ at the optimum}$$

This is exactly expression (18) which completes the proof. ■

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