The Law of Demand and Risk Aversion

By John K.-H. Quah

Abstract: This note proposes a necessary and sufficient condition on a preference to guarantee that the demand function it generates satisfies the law of demand. It shows that the law of demand may be succinctly characterized by differences in an agent's level of risk aversion when she is confronted with different lotteries composed of commodity bundles.

Keywords: law of demand, monotonicity, preference, risk aversion.

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It is commonly assumed in economic theory that demand curves slope downwards, yet it is also well known that this property is not guaranteed by utility maximization. The impact of a price change on demand can be decomposed into the substitution and income effects. The substitution effect is always well behaved, in the sense that in response to a price increase for a good, this effect will always lead to less demand for that good. However, the income effect may cause the agent to buy more or less of that good, so that the overall demand response to a price change is ambiguous.

To prevent this from happening, an agent's preference must be such that income effects, even when they counteract substitution effects, will always be dominated by the latter. The precise conditions on an agent's preference guaranteeing this were established independently by Milleron (1974) and Mitjuschin and Polterovich (1978) (henceforth to be referred to as MMP). Assuming that the commodity space is R^l_{++}, we denote the demand vector at the price vector p and income w by $f(p,w)$. The demand function f satisfies *monotonicity* (or *the law of demand*) if, whenever $p \neq p'$,

$$(p - p') \cdot (f(p,w) - f(p',w)) < 0. \tag{1}$$

Monotonicity clearly implies that the demand for any good decreases with its own price and is the natural multi-variate extension of this property.

It is well-known that under very mild assumptions, a convex preference $\succeq$ over the commodity space R^l_{++} will admit concave (as opposed to merely quasi-concave) utility representations. The theorem of MMP says that the preference $\succeq$ generates a monotonic demand function if it could be represented by a utility function u which is concave and also satisfies the following property:

$$\psi_u(x) \equiv -\frac{x^T \partial^2 u(x) x}{\partial u(x) x} < 4 \tag{2}$$

for all commodity bundles x. Note that $\psi_u(x)$ has a straightforward geometrical interpretation. Defining $H(t) = u(x + tx)$, where t is a scalar, one could check that $\psi_u(x) = -H''(0)/H'(0)$; in other words, $\psi_u(x)$ is a measure of the curvature of u at x and in the direction of x. The MMP theorem requires this value to be less than 4.[2]

These conditions for monotonicity depend on the choice of u and are only sufficient and not necessary for monotonicity; to formulate necessary and sufficient conditions, one should consider at each point in the commodity space, the whole family of concave utility functions which represents the preference in some open and convex neighborhood of x, and then to evaluate the infimum of $\psi_u(x)$ over all possible u. We denote this by $\psi_{\succeq}(x)$ and refer to it as the *MMP coefficient*; the MMP coefficient is a property of the preference and is not dependent on any particular utility representation. If $\psi_{\succeq}(x) < 4$ for all x then the preference generates a monotonic demand function; furthermore, if it is greater than 4 at some point x, then there will be a local violation of monotonicity at x. The significance of $\psi_{\succeq}$ goes beyond the basic MMP result; generalizations of this result to guarantee monotonic market demand in markets with exogenous or endogenous income distributions also rely on conditions involving $\psi_{\succeq}$ (see Quah (1999) and Quah (2000)).

We present in this note a way of characterizing the MMP coefficient of $\succeq$ in terms of any utility function representing the preference. In particular, it enables us to formulate on any utility function representing $\succeq$, a necessary and sufficient condition for the monotonicity of the demand function it generates. This characterization is strikingly natural and also has a nice interpretation in terms of an agent's attitude towards risk.

Let u be a utility function (which is not necessarily concave) representing the convex preference $\succeq$. Fixing the point x, we may consider the functions $H(t;z) = u(x+tz)$, where t is a scalar. Define $\psi_u^z(x) = -H''(0;z)/H'(0;z)$ (where the differentiation is done with respect to t); essentially $\psi_u^z(x)$ measures the curvature of u at x in the direction of z. Clearly $\psi_u(x) = \psi_u^x(x)$ and more generally, $\psi_u^z(x) = -z^T\partial^2 u(x)z/\partial u(x)z$. We define

$$\Delta_u(x) = \psi_u(x) - \inf_{z\in Z_u(x)} \psi_u^z(x); \tag{3}$$

where $Z_u(x) = \{z \in R^l : \partial u(x)z = \partial u(x)x\}$. So $Z_u(x)$ is the family of changes in consumption which have the same value when evaluated with the supporting price at x, i.e., $\partial u(x)$, or equivalently, have the same marginal utility. Figure 1 shows the changes in directions x and z, while Figure 2 shows the curves $H(\cdot;z)$ for different values of z. The curve $H(\cdot;z)$ may be flatter or or more curved than $H(\cdot;x)$; *$\Delta_u(x)$ is (essentially) the gap between the curvature of $H(\cdot;x)$ and the curvature of the flattest possible $H(\cdot;z)$.*

We show in this note that $\Delta_u(x) = \psi_{\succeq}(x)$. Clearly, this also means that if $\tilde{u}$ is another utility function representing $\succeq$, then $\Delta_u(x) = \Delta_{\tilde{u}}(x)$. Among the family of possible utility functions representing $\succeq$ is the agent's Bernoulli utility function.[3] This utility function not only represents the agents preference over the commodity bundles, but also captures the agent's preference over lotteries of these bundles via expected utility; in other words, the agent's utility over a lottery consisting of commodity bundles is the expected value of the utility of the bundles which constitute the lottery. Assuming that the agent is risk averse, the Bernoulli utility will be concave and if u is this utility, then $\psi_u^z(x)$ is just the coefficient of risk aversion at x and in the direction of z.

The original MMP result could be interpreted as saying that an agent generates a monotonic demand function if he is risk averse in commodity space but not *too* risk averse, in the sense that his coefficient of risk aversion at each point x, in the direction x, must not exceed 4. In fact, this condition could be refined to say that $\Delta_u(x)$ must not exceed 4: so it does not matter if the agent's coefficient of risk aversion in the direction of x is, say, 400; monotonicity is guaranteed if and (essentially) only if his coefficient of risk aversion in any other (normalized) direction exceeds 396. In this way, we obtain a succinct characterization of monotonicity in terms of an agent's attitude towards lotteries of commodity bundles.

An important application of the MMP result is to the study of demand for contingent consumption. In that case, we may assume that u has the expected utility form, i.e., $u(x) = \sum_{i=1}^{l} \pi_i u_i(x_i)$, where $x = (x_1, x_2, ..., x_l)$ and $\sum_{i=1}^{l} \pi_i = 1$. So x_i is the level of consumption in state i, u_i is the Bernoulli utility function and π_i is the subjective probability of state i occurring. A straightforward application of the MMP conditions says that demand is monotonic if the coefficient of relative risk aversion $\rho_i(x_i) \equiv -x_i u_i''(x_i)/u_i'(x_i)$ is between 0 and 4. This is a well known application of the MMP theorem, see, for example, Mas-Colell (1991) or Hildenbrand (1994) and it has also been generalized to infinite dimensions and to incomplete markets (see Dana (1995) and Bettzuge (1998)).

This condition, while sufficient, is plainly unnecessary. If ρ_i is identically constant and equals the same constant for all i, then u becomes a CES utility function. Since the preference is then homothetic, we know that demand is monotonic, no matter how big the coefficient of relative risk aversion. The refinement we propose here gives a considerably sharper result: monotonicity is guaranteed provided $|\rho_i(x_i) - \rho_j(x_j)| < 4$ for all i and j. In other words, the coefficient of relative risk aversion must not *vary* by more than 4 across different levels of consumption in different states.

The next section of this note contains formal statements and proofs of the results discussed in this introduction. Section 3 concludes with an outline of the connection between the approach developed in this note and the approach via indirect utility or preference.

2. The MMP Coefficient and Monotonicity

We assume that the commodity space is R^l_{++} and begin with a standard restriction on the preferences being considered.

DEFINITION: A preference $\succeq$ on R^l_{++} is *regular* if it is representable by a *regular utility function*, by which we mean a function $u : R^l_{++} \to R$ with the following properties: it is C^2, its partial derivatives are strictly positive, it is differentiably strictly quasi-concave, and the sets $\mathcal{C}_{\bar{x}} = \{x \in R^l_{++} : u(x) \geq u(\bar{x})\}$ are closed in R^l for any $\bar{x}$ in R^l_{++}.

When a preference $\succeq$ is regular, for any (p, w) in $R^l_{++} \times R_+$, there is a unique $\bar{x}$ in the set $S(p, w) = \{x \in R^l_{++} : p \cdot x \leq w\}$ such that $\bar{x} \succeq x$ for all x in $S(p, w)$. We denote $\bar{x}$ by $f(p, w)$. It is a standard result that the function $f : R^l_{++} \times R_+ \to R^l_{++}$ is C^1 (see Mas-Colell (1985)) and we refer to it as the *demand function generated by* $\succeq$.

It is also known that a regular preference is always representable, on any convex subset of R^l_{++}, by a regular direct utility function that is also concave (see Mas-Colell (1985)). This fact makes it possible for us to define an *MMP coefficient* for a regular preference $\succeq$. At any commodity bundle x, let $U(x)$ be the non-empty collection of concave and regular utility functions which represent $\succeq$ in some open and convex neighborhood of x. For each

u in $U(x)$, we define $\psi_u(x)$ by (2). The MMP coefficient at x, $\psi_{\succeq}(x) = \inf_{u \in U(x)} \psi_u(x)$. Unlike ψ_u, the MMP coefficient is an ordinal concept, and using it, the condition needed to guarantee the monotonicity of demand (as defined by (1)) can be stated precisely.

THEOREM 2.1: *Suppose $\succeq$ is a regular preference generating the demand function f.*
(i) If at some (p, w) in $R^l_{++} \times R_+$, $\psi_{\succeq}(f(p, w)) < 4$, then there exists an open and convex neighborhood around (p, w) in which monotonicity holds, i.e., $(p' - p'') \cdot (f(p', w') - f(p'', w')) < 0$ when (p', w') and (p'', w') are in that neighborhood, with $p' \neq p''$.
(ii) If $\psi_{\succeq}(f(p, w)) < 4$ for all (p, w) in S, a convex subset of $R^l_{++} \times R_+$, then f is monotonic in S.
(iii) If at some (p, w), $\psi_{\succeq}(f(p, w)) > 4$, then there is a price p', which could be chosen arbitrarily close to p, such that $(p - p') \cdot (f(p, w) - f(p', w)) > 0$.

Parts (i) and (ii) are just re-statements of the MMP results in terms of $\psi_{\succeq}$ and are both obvious from the standard proofs (Mas-Colell (1991) or Hildenbrand (1994)). Clearly (ii) implies that if $\psi_{\succeq}(x) < 4$ for all x in R^l_{++}, then the demand function f is monotonic in the whole price-income space. Part (iii) says that the condition on MMP coefficient is also necessary in the sense that an MMP coefficient in excess of 4 at a point implies a local violation of monotonicity. The proof requires Lemmas 2.2 and 2.3 below; since Lemma 2.2 is a standard result we will omit its proof.

LEMMA 2.2: *Suppose that A is a symmetric and negative definite matrix and let (b, r) be an element of $R^l \times R$. Then there is $\bar{z}$ that solves $\max_{b^T z = r} z^T A z$ and the maximum is $r^2 / b^T A^{-1} b$.*

LEMMA 2.3: *Let u be a regular utility function representing $\succeq$ and define for each x in*

R^l_{++}, the set $Z_u(x)$, given by $Z_u(x) = \{z \in R^l : \partial u(x)z = \partial u(x)x\}$. For a fixed x, suppose that $\sup_{z\in Z_u(x)} z^T\partial^2 u(x)z$ exists. (i) For any

$$L \quad > \quad \psi_u(x) + \sup_{z\in Z_u(x)} \frac{z^T\partial^2 u(x)z}{\partial u(x)z}, \tag{4}$$

there is a C^2 function $h : R \to R$ such that $\tilde{u} = h \circ u$ satisfies $\psi_{\tilde{u}}(x) = L$ and $\tilde{u}$ is in $U(x)$. (ii) If there is $\bar{z}$ in $Z_u(x)$ such that $\bar{z}^T\partial^2 u(x)\bar{z}/\partial u(x)\bar{z} = \sup_{z\in Z_u(x)} z^T\partial^2 u(x)z/\partial u(x)z$, then we can find h such that $\tilde{u} = h \circ u$ satisfies (a) $z^T\partial^2\tilde{u}(x)z \leq 0$ for all z, with exact equality at $\bar{z}$ and (b) $\psi_{\tilde{u}}(x) = \psi_u(x) + \sup_{z\in Z_u(x)} z^T\partial^2 u(x)z/\partial u(x)z$.

Proof: Differentiating $\tilde{u} = h \circ u$, we obtain

$$\psi_{\tilde{u}}(x) \quad = \quad -\frac{h''(u(x))}{h'(u(x))}[\partial_x u(x)x] + \psi_u(x). \tag{5}$$

Fixing x, if we choose a function h such that $h' > 0$ and $h''/h' = [-L + \psi_u(x)]/(\partial u(x)x)$, clearly, $\psi_{\tilde{u}}(x) = L$. We need to check that with this choice of h, $\tilde{u}$ is locally concave at x. It is sufficient to check that $z^T\partial^2\tilde{u}(x)z < 0$ for all non-zero z, where

$$z^T\partial^2\tilde{u}(x)z \quad = \quad h'(u(x))\left[\frac{h''(u(x))}{h'(u(x))}[\partial u(x)z]^2 + z^T\partial^2 u(x)z\right]. \tag{6}$$

If $\partial u(x)z = 0$, strict quasi-concavity guarantees that $z^T\partial^2 u(x)z < 0$ and so $z^T\partial^2\tilde{u}(x)z < 0$. If $\partial u(x)z \neq 0$, we may assume without loss of generality that z is in $Z_u(x)$. Since $\partial u(x)x = \partial u(x)z$, (6) says that $z^T\partial^2\tilde{u}(x)z < 0$ if $[-L + \psi_u(x)]\partial u(x)z + z^T\partial^2 u(x)z < 0$ for all z in $Z_u(x)$. This inequality is implied by (4).

To establish (ii), we choose h such that $h' > 0$ and $h''/h' = -\bar{z}^T\partial^2 u(x)\bar{z}/(\partial u(x)\bar{z})^2$. By (6), $\tilde{u} = h \circ u$ satisfies $z^T\partial^2\tilde{u}(x)z \leq 0$ for all z, with $\bar{z}^T\partial^2\tilde{u}(x)\bar{z} = 0$. So property (a) is satisfied, while (b) is obvious once we substitute the value of h''/h' into (5). QED

Proof of Theorem 2.1(iii): Suppose we can find a regular utility function $\tilde{u}$ representing $\succeq$ in some open and convex neighborhood which satisfies property (a) in Lemma 2.3(ii) (specifically, that $\bar{z}^T\partial^2\tilde{u}(x)\bar{z} = 0$ for some $\bar{z}$ in $Z_u(x)$) and, in addition, $\psi_{\tilde{u}}(x) > 4$. Then it is clear from the proofs of the MMP result (Mas-Colell (1991) or Hildenbrand (1994)) that there will be a local violation of monotonicity, so we need only show that such a $\tilde{u}$ exists.

Since $\succeq$ is regular, there is a regular and concave u which represents $\succeq$ in some open and convex neighborhood of x. By definition of $\psi_{\succeq}$, $\psi_u(x) > 4$; if u also satisfies property (a) in Lemma 2.3(ii), we are done. Suppose that it does not satisfy the property (a); since u is concave, $z^T\partial^2 u(x)z < 0$ for all $z \neq 0$. By Lemma 2.2, there is $\bar{z}$ in $Z_u(x)$ such that $\bar{z}^T\partial^2 u(x)\bar{z}/\partial u(x)\bar{z} = \sup_{z\in Z_u(x)} z^T\partial^2 u(x)z/\partial u(x)z$. This allows us to apply Lemma 2.3(ii), which says that there is h such that $\tilde{u} = h\circ u$ satisfies property (a). We claim that $\psi_{\tilde{u}}(x) > 4$. Since $\sup_{z\in Z_{\tilde{u}}(x)} z^T\partial^2\tilde{u}(x)z = 0$, Lemma 2.3(i) tells us that for any $\epsilon > 0$, there is $u*$ in $U(x)$ such that $\psi_{u*}(x) = \psi_{\tilde{u}}(x) + \epsilon$. If $\psi_{\tilde{u}}(x) \leq 4$, then $\psi_{u*}(x) \leq 4 + \epsilon$, which means since ϵ is arbitrary and $u*$ is in $U(x)$ that $\psi_{\succeq}(x) \leq 4$. This is a contradiction. QED

The next result, which follows easily from Lemma 2.3, gives the alternative formulation of $\psi_{\succeq}$ that is the main result of this note. Note that u need not be concave.

Theorem 2.4: *Let u be a regular utility function representing $\succeq$ in some open and convex neighborhood of x. Then $\Delta_u(x) = \psi_{\succeq}(x)$.*

Proof: Assume firstly that u is concave. Then $\sup_{z\in Z_u(x)} z^T\partial^2 u(x)z/\partial u(x)z$ clearly exists and since the right hand side of (4) is in fact $\Delta_u(x)$ (see (3)), Lemma 2.3(i) tells us that $\psi_{\succeq}(x) \leq \Delta_u(x)$. By definition of $\psi_{\succeq}$, for any $\epsilon > 0$, there is $\tilde{u}$ in $U(x)$ such that $\psi_{\tilde{u}}(x) \leq \psi_{\succeq}(x) + \epsilon$. We know that there is a C^2 function $h : R \to R$ such that

$\tilde{u} = h \circ u$. By (5), $h''/h' = [\psi_u(x) - \psi_{\tilde{u}}(x)]/\partial u(x)x$. Substituting this into (6), we see that $[\psi_u(x) - \psi_{\tilde{u}}(x)]\partial u(x)z + z^T\partial^2 u(x)z \leq 0$ for z in $Z_u(x)$ since $z^T\partial^2\tilde{u}(x)z \leq 0$ for all z. Re-arranging this expression give us $\Delta_u(x) \leq \psi_{\tilde{u}}(x) \leq \psi_{\succeq}(x) + \epsilon$. Since ϵ is arbitrary, we obtain $\Delta_u(x) \leq \psi_{\succeq}(x)$. So we have shown that $\Delta_u(x) = \psi_{\succeq}(x)$ when u is concave.

Suppose now that u is not concave. Since $\succeq$ is regular, we know that there is a regular and concave utility function $\hat{u}$ representing $\succeq$ in some open neighborhood of x and there is a function h such that $u = h \circ \hat{u}$. A simple calculation shows that for z in $Z_u(x) = Z_{\hat{u}}(x)$,

$$\psi_u(x) + \frac{z^T\partial^2 u(x)z}{\partial u(x)z} = \psi_{\hat{u}}(x) + \frac{z^T\partial^2\hat{u}(x)z}{\partial\hat{u}(x)z}. \tag{7}$$

By the concavity of $\hat{u}$, the right hand side of this equation has an upper bound as z varies in $Z_u(x) = Z_{\hat{u}}(x)$, so $\sup_{z\in Z_u(x)} z^T\partial^2 u(x)z/\partial u(x)z$ must also exist. It follows that $\Delta_u(x)$ is well defined and by (7), $\Delta_u(x) = \Delta_{\hat{u}}(x)$. We have already established that the latter equals $\psi_{\succeq}(x)$. QED

An important application of Theorem 2.2 is to the class of *additive utility functions*, i.e., functions of the form $u(x) = \sum_{i=1}^{l}\pi_i u_i(x_i)$, where the $\pi_i > 0$, $u_i' > 0$ and $u_i'' < 0$ for $i = 1, 2, ..., l$. This could be interpreted as an agent's expected utility function over consumption in l states of the world. The function $B_u : R^l_{++} \to R$ is defined by

$$B_u(x) = \max_{1\leq i\leq l}\left(-\frac{x_i u_i''(x_i)}{u_i'(x_i)}\right) - \min_{1\leq i\leq l}\left(-\frac{x_i u_i''(x_i)}{u_i'(x_i)}\right). \tag{8}$$

If we interpret u as an expected utility function, then $B_u(x)$ is just the variation in the agent's coefficient of relative risk aversion at the different levels of realized consumption represented by x. (In the case where the agent's utility is state independent, $u_i = u_j$ for all i and j.)

Corollary 2.5: *Suppose u is a regular and additive utility function defined on R^l_{++}, and let $\succeq$ be the preference over R^l_{++} that it represents. Then for any x in R^l_{++}, $\psi_{\succeq}(x) \leq B_u(x)$.*

Proof: For a given value of x, we may assume, without loss of generality, that $\partial u(x)x = 1$. Subsuming π_i into u_i, we can write u as $u(x) = \sum u_i(x_i)$. By definition, $\Delta_u(x) = \psi_u(x) + \sup_{z \in Z_u(x)} z^T \partial u(x) z$ (note that $\partial u(x)z = \partial u(x)x = 1$ if z is in $Z_u(x)$). It is easy to check that $\psi_u(x) \leq \max_{1 \leq i \leq l}(-x_i u_i''(x_i)/u_i'(x_i))$ and by Lemma 2.2,

$$\begin{aligned} \max_{z \in Z_u(x)} z^T \partial^2 u(x) z &= \frac{1}{(\partial u(x))(\partial^2 u(x))^{-1}(\partial u(x))^T} \\ &= \frac{1}{\sum_{i=1}^l u_i'(x_i)^2 / u_i''(x_i)}. \end{aligned}$$

A little work will show that this expression is less than $-\min_{1 \leq i \leq l}(-x_i u_i''(x_i)/u_i'(x_i))$. Therefore, by Theorem 2.4, $\psi_{\succeq}(x) = \Delta_u(x) \leq B(x)$. QED.

It follows immediately from Corollary 2.3 and Theorem 2.1 that monotonicity holds if $B_u(x) < 4$ for all x. In other words, an agent whose coefficient of relative risk aversion does not vary by more than 4 will generate a demand for consumption in different states which is monotonic with respect to the state prices.

Quite naturally we would like the monotonic property to be extended to security prices and their demand. This is not hard to achieve and we will conclude this section by establishing such a result. We assume that there are m securities, with $m \leq l$ (so the market may be incomplete). The $m \times l$ matrix D gives the payoffs of these securities, with the ijth entry being the payoff of the ith security in state j. We say that the payoff matrix D is *well-behaved* if there is θ such that $D^T\theta \gg 0$ and the rank of D is m.

If q in R^m are the security prices, then an agent with a preference $\succeq$ and positive income w chooses a portfolio of securities θ^* in $B(q,w) = \{\theta \in R^m : q \cdot \theta \leq w\}$ such that $D^T\theta^* \succeq D^T\theta$ for all θ in $B(q,w)$. It is well-known that provided $\succeq$ is regular and D is well-behaved, this problem has a solution if and only if q admits no arbitrage, which is equivalent to saying that q is in the set $Q_D = \{Dp : p \in R^l_{++}\}$. Furthermore, for each (q,w) the solution is unique; we denote it by $g(q,w)$ and refer to the function $g : Q_D \times R_+ \to R^m$ as *the securities demand function generated by* $\succeq$ *and* D.

PROPOSITION 2.6: *Suppose that the payoff matrix* D *is well-behaved and that* $\succeq$ *is regular with* $\psi_{\succeq}(x) < 4$ *for all* x *in* R^l_{++}. *Then the securities demand function* g *generated by* $\succeq$ *and* D *is monotonic, i.e., for any* (q,w) *and* (q',w) *in* $Q_D \times R_+$, *with* $q \neq q'$, *we obtain* $(q-q') \cdot (g(q,w) - g(q',w)) < 0$.[4]

Proof: We denote by f the demand generated by $\succeq$. Since $\psi_{\succeq}(x) < 4$ for all x in R^l_{++}, Theorem 2.1(ii) says that f is a monotonic function. For any q in Q_D and $w > 0$, we know that there is $p \gg 0$ such that $q = Dp$ and $D^T g(q,w) = f(p,w)$ (see Duffie (1992) or Magill and Quinzii (1996)). Similarly, there is $p' \gg 0$ such that $q' = Dp'$ and $D^T g(q',w) = f(p',w)$. If $q \neq q'$ then clearly $p \neq p'$, so by the monotonicity of f, $(p-p') \cdot (f(p,w) - f(p',w)) < 0$. Replacing f with $D^T g$ we obtain $(q-q') \cdot (g(q,w) - g(q',w)) < 0$. QED

Obviously, Proposition 2.6 and Corollary 2.3 together tell us that if the payoff matrix D is well-behaved and $\succeq$ is representable by an additive utility function u satisfying $B_u < 4$, then the securities demand function they generate will be monotonic.

3. Conclusion

Our discussion in this note has focussed on the relationship between a preference in commodity space and the monotonicity of the demand it generates. There is in fact another way of characterizing the monotonicity of demand via indirect utility, or more generally, via the indirect preference over price-income situations. This approach is particularly useful in generalizations of the basic MMP result to guarantee monotonicity for *market* (rather than just individual) demand (see Quah (1999) and Quah (2000)).

In this approach one could speak of a concept analogous to the MMP coefficient, called the indirect MMP coefficient, which is defined on an indirect preference. It turns out that the two concepts are related most naturally: the indirect MMP coefficient at (p, w) is equal to the MMP coefficient at $f(p, w)$. This and other results relating the two approaches are established in Martinez-Legaz and Quah (2002) (manuscript under preparation).

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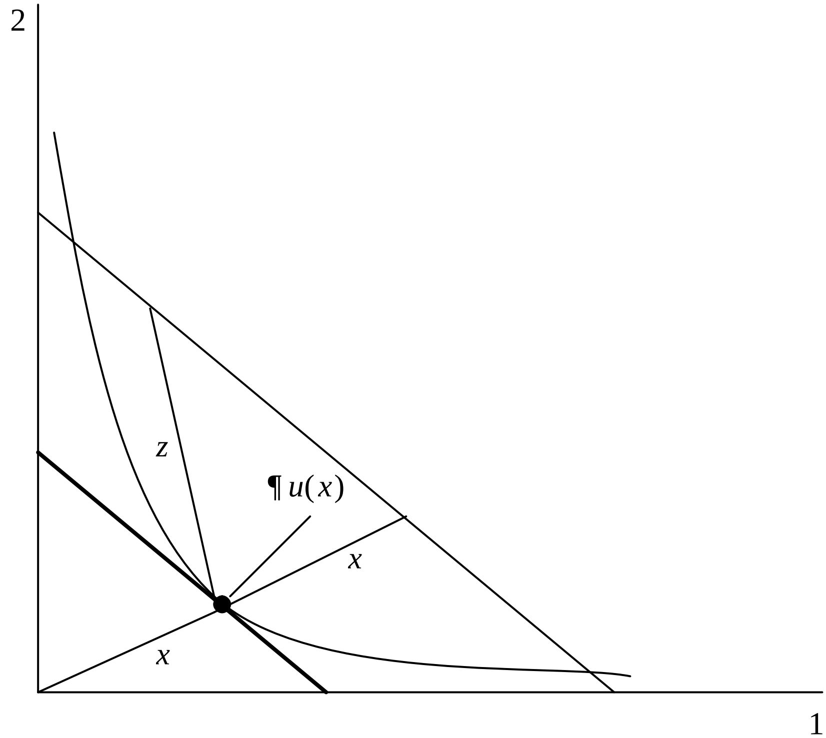

FIGURE 1

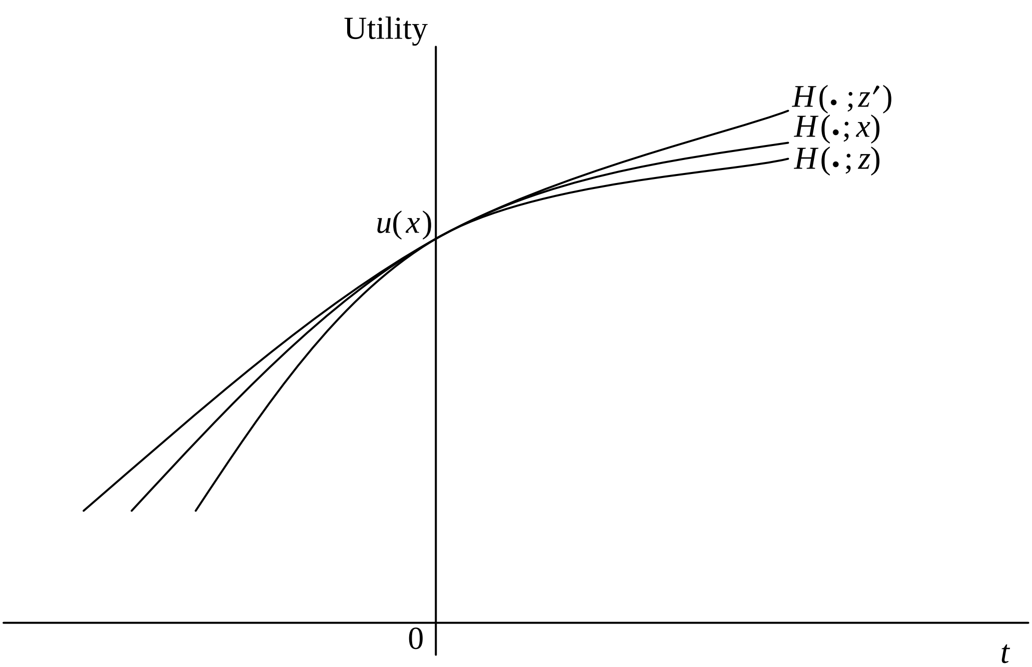

FIGURE 2

Footnotes

1. I would like to thank the Economic and Social Research Council for providing me with financial support through their Research Fellowship Scheme.

2. Milleron's (1974) paper was never published, while Mitjuschin and Polterovich's (1978) paper is in Russian. More accessible and English language versions of the result could be found in Mas-Colell (1991), Hildenbrand (1994), and Mas-Colell et al (1995). All these adopt the method of proof found in Mitjuschin and Polterovich (1978). Mas-Colell (1991) and Mas-Colell et al (1995) also discuss the implications of monotonicity for market demand and general equilibrium theory. Applications of the MMP result or its extensions to address uniqueness and stability issues in general equilibrium models could also be found in Dana (1995), Bettzuge (1998) and Quah (1999, 2000). Quah's papers employ conditions on the indirect preference which are natural analogs to the MMP conditions imposed on the (direct) preference. Kannai (1989) has a characterization of monotonicity via the normalized gradient function defined on the direct preference.

3. The term 'Bernoulli utility function' follows Mas-Colell et al (1995). They point out that the term is non-standard, though their use has probably made it less so. The fact that one could consider an agent's preference over lotteries of commodity bundles, and therefore meaningfully define a Bernoulli utility function over the commodity space (provided that the agent's preference obey, essentially, the von Neumann-Morgenstern axioms) has long been recognized; see, for example, Debreu (1976).

4. If the agent is not endowed with a fixed income w but is instead endowed with bundle of securities ω, then clearly the agent's demand for securities at price q in Q_D is $g(q, q\cdot\omega)$. The assumptions of Proposition 2.6 then guarantee that $(q-q')\cdot(g(q,q\cdot\omega)-g(q',q'\cdot\omega))<0$ provided $q \neq q'$ and $q\cdot\omega = q'\cdot\omega$. The case where the agent is endowed with a bundle of contingent commodities in R^l_+ that is *not* spanned by the available securities is more complicated and has to be handled differently. Bettzuge (1998) gives some conditions under which securities demand could be monotonic in this case.