Econometrics of testing for jumps in financial economics using bipower variation

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Abstract

In this paper we provide an asymptotic distribution theory for some non-parametric tests of the hypothesis that asset prices have continuous sample paths. We study the behaviour of the tests using simulated data and see that certain versions of the tests have good finite sample behaviour. We also apply the tests to exchange rate data and show that the null of a continuous sample path is frequently rejected. Most of the jumps the statistics identify are associated with governmental macroeconomic announcements.

Keywords: Bipower variation; Jump process; Quadratic variation; Realised variance; Semimartingales; Stochastic volatility.

1 Introduction

In this paper we will measure the contribution of jumps to the variation of asset prices and form robust tests for the presence of jumps on individual days in financial markets. Being able to distinquish between jumps and continuous sample path price movements is important as it has implications for risk management and asset allocation. A stream of recent papers in financial econometrics has addressed this issue using low frequency return data (e.g. the parametric models of Eraker, Johannes, and Polson (2003), Andersen, Benzoni, and Lund (2002), Chernov, Gallant, Ghysels, and Tauchen (2003) and the Markovian, non-parametric analysis of Aït-Sahalia (2002), Johannes (2003) and Bandi and Nguyen (2003)) and options data (e.g. Bates (1996), Carr and Wu (2003) and the review by Garcia, Ghysels, and Renault (2003)). Our approach will be non-parametric and exploit high frequency data. Monte Carlo results suggest that it performs well when based on empirically relevant sample sizes. Furthermore, empirical work points us to the conclusion that jumps are very common, suggesting the vast and very elegant literature on the use of continuous sample path processes built out of Brownian motion may be based on an ill fitting assumption.

Traditionally in the theory of financial economics the variation of asset prices is measured by looking at sums of outer products of returns calculated over very small time periods. The mathematics of this is based on the quadratic variation process (e.g. Chamberlain (1988) and Back (1991)). Asset pricing theory links the dynamics of increments of quadratic variation to the increments of the risk premium. The recent econometric work on this topic, estimating quadratic variation using discrete returns, under the general heading of realised quadratic variation, realised volatility and realised variances was discussed in independent and concurrent work by Andersen and Bollerslev (1998a), Barndorff-Nielsen and Shephard (2001) and Comte and Renault (1998). It was later developed in the context of the methodology of forecasting by Andersen, Bollerslev, Diebold, and Labys (2001), while a central limit theory for realised variances was developed by Barndorff-Nielsen and Shephard (2002). Multivariate generalisations to realised covariation are discussed by, for example, Barndorff-Nielsen and Shephard (2004a) and Andersen, Bollerslev, Diebold, and Labys (2003). See Andersen, Bollerslev, and Diebold (2004) for an incisive survey of this area and references to related work.

In a recent paper Barndorff-Nielsen and Shephard (2004d) introduced a partial generalisation of the quadratic variation process called the bipower variation (BPV) process. They showed that in some cases relevant to financial economics BPV can be used, in theory, to split up the individual components of quadratic variation into that due to the continuous part of prices and that due to jumps. In turn the bipower variation process can be consistently estimated using an equally spaced discretisation of financial data. This estimator is called the realised bipower variation process.

In this paper we study the difference or ratio of realised BPV and realised quadratic variation. We show we can use these statistics to construct non-parametric tests for the presence of jumps. We derive the asymptotic distributional theory for the tests under surprisingly weak conditions. This is the main contribution of the paper. We will also illustrate the jump tests using both simulations and exchange rate data. We relate some of the jumps to macroeconomic announcements by Government agencies.

A by-product of our research is an Appendix which records a proof of the consistency of realised BPV under substantially weaker conditions than those used by Barndorff-Nielsen and Shephard (2004d) and a joint limiting distribution for realised BPV and the corresponding realised quadratic variation process. The latter result demonstrates the expected conclusion that realised BPV is slightly less efficient than realised quadratic variation as an estimator of quadratic variation in the case where prices have a continuous sample path.

In the next Section we will set out our notation and recall the definitions of quadratic variation and BPV. In Section 3 we will give the main Theorem of the paper, which is the asymptotic distribution of the proposed tests. In Section 4 we will extend the analysis to cover the case of a time series of daily statistics for testing for jumps. In Section 5 we study how the jump tests behave in simulation studies, while in Section 6 we apply the theory to two exchange rate series. In Section 7 we discuss various additional issues, while Section 8 concludes. The proofs of the main results in the paper are given in the Appendix.

2 Definitions and previous work

2.1 Notation

Let the log-price of an asset be written as Y_t for $t \geq 0$. Here t represents continuous time. Extensions to deal with the multivariate case will be discussed in Section 7. We assume Y is a semimartingale ($\mathcal{SM}$), which means it can be decomposed as $Y = A + M$, where A is a process with *finite variation* ($\mathcal{FV}$) paths and M is a *local martingale* ($\mathcal{M}_{loc}$). For a very accessible discussion of probabilistic aspects of this see Protter (2004), while its attraction from an economic viewpoint is discussed by Back (1991). We will often restrict various classes of processes to those with continuous or purely discontinuous sample paths. We generically denote this with superscripts c and d respectively, e.g. $\mathcal{M}_{loc}^c$ stands for the class of continuous local martingales, while M^c denotes the continuous component of M.

2.2 Quadratic variation

For all $Y \in \mathcal{SM}$ the *quadratic variation* (QV) process can be defined as

$$[Y]_t = \text{p}-\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2, \tag{1}$$

(e.g. Jacod and Shiryaev (1987, p. 55)) for any sequence of partitions $t_0 = 0 < t_1 < ... < t_n = t$ with $\sup_j\{t_{j+1} - t_j\} \to 0$ for $n \to \infty$. It is well known that

$$\begin{aligned} [Y]_t &= [M^c]_t + \sum_{0\leq s\leq t} \Delta Y_s^2 \\ &= [M^c]_t + [Y^d]_t, \end{aligned} \tag{2}$$

where $\Delta Y_t = Y_t - Y_{t-}$ are the jumps in the process. (2) means that the QV of Y aggregates the QV of M^c and the QV of Y^d. This tells us that if we could disaggregate QV into $[M^c]$ and $[Y^d]$ then we can test for jumps by asking if $[Y] = [M^c]$? This will be at the kernel of our approach to testing for jumps.

Our econometric analysis of QV will be based on a discretised version of Y based on intervals of time of length $\delta > 0$. The resulting process, which we write as $Y_{\delta,}$ is

$$Y_{\delta\lfloor t/\delta \rfloor}, \quad t \geq 0, \tag{3}$$

recalling that $\lfloor x \rfloor$ is the integer part of x. This allows us to construct δ-returns

$$y_j = Y_{j\delta} - Y_{(j-1)\delta}, \quad j = 1, 2, ..., \lfloor t/\delta \rfloor .$$

The *realised quadratic variation* process is

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2,$$

the QV of the discretised process. Clearly the QV theory means that

$$[Y_\delta]_t \xrightarrow{p} [Y]_t$$

as $\delta \downarrow 0$. The daily increments of the realised QV process, using $\hbar$ to denote the period of a day, will be written as

$$\widehat{v}_i = [Y_\delta]_{\hbar i} - [Y_\delta]_{\hbar(i-1)} \xrightarrow{p} [Y]_{\hbar i} - [Y]_{\hbar(i-1)}, \quad i = 1, 2, ...$$

The estimator $\widehat{v}_i$ is called the realised variance in financial economics, as we briefly discussed in the first Section, and will play a significant role in Section 4 onwards in this paper. The square root of $\widehat{v}_i$ is called the realised volatility.

2.3 Bipower variation

The quadratic variation process always exists when $Y \in \mathcal{SM}$. This is not necessarily the case for the *bipower variation* (BPV) process. It is defined for $r \geq 0$ and $s \geq 0$ as

$$\{Y\}_t^{[s,r]} = \text{p}-\lim_{\delta \downarrow 0} \delta^{1-(r+s)/2} \sum_{j=2}^{\lfloor t/\delta \rfloor} |y_{j-1}|^s |y_j|^r ,$$

when the probability limit exists. In this paper our focus will be entirely on the $r = s = 1$ case for this will allow us to derive limit theorems under rather weak assumptions. In that situation we have that

$$\{Y\}_t^{[1,1]} = \text{p}-\lim_{\delta \downarrow 0} \sum_{j=2}^{\lfloor t/\delta \rfloor} |y_{j-1}| |y_j| .$$

The existence of this limit can be established when log-prices obey the following process.

Definition 1 *Y is a member of the stochastic volatility plus finite activity jump semimartingale ($\mathcal{SVFASM}$) class if*

$$Y_t = A_t + \int_0^t \sigma_s \mathrm{d}W_s + \sum_{j=1}^{N_t} c_j, \tag{4}$$

where $A \in \mathcal{FV}^c$, the spot volatility process σ is càdlàg and W is a standard Brownian motion. We also assume that (for all $t < \infty$) $\int_0^t \sigma_s^2 \mathrm{d}s < \infty$. The counting process N has (for all $t < \infty$) $N_t < \infty$ and the c_j are non-zero random variables. When $N = 0$ we write the class as $\mathcal{SVSM}^c$. On the other hand if $A = 0$ we write $Y \in \mathcal{SVFA}$. If both $N = A = 0$ then $Y \in \mathcal{SV}$.

A well known key feature of the $\mathcal{SVFASM}$ model class is that

$$[Y]_t = \int_0^t \sigma_s^2 \mathrm{d}s + \sum_{j=1}^{N_t} c_j^2.$$

Barndorff-Nielsen and Shephard (2004d) showed that if $Y \in \mathcal{SVFA}$ and σ is independent from W (a no leverage assumption, which we will return to in Section 7.1) then

$$\{Y\}_t^{[1,1]} = \mu_1^2 \int_0^t \sigma_s^2 \mathrm{d}s,$$

where

$$\mu_1 = \mathrm{E}\,|u| = \sqrt{2}/\Gamma\left(\frac{1}{2}\right) = \sqrt{2}/\sqrt{\pi} \simeq 0.79788 \tag{5}$$

and $u \sim N(0,1)$. Hence $\mu_1^{-2}\{Y\}^{[1,1]}$ and QV are the same in the $\mathcal{SV}$ case but differ when there are jumps. This result is quite robust as it does not depend on any other assumptions on the structure of N, the distribution of the jumps or the relationship between the jump process and the SV component. Further, clearly, $\{Y\}_t^{[1,1]}$ can be consistently estimated by the *realised BPV* process

$$\{Y_\delta\}_t^{[1,1]} = \sum_{j=2}^{\lfloor t/\delta \rfloor} |y_{j-1}|\,|y_j|\,,$$

as $\delta \downarrow 0$. One would expect these results on BPV to continue to hold when we extend the analysis to allow $A \neq 0$. Subject to a mild regularity condition on A this is indeed the case, as will be discussed in the next Section.

Barndorff-Nielsen and Shephard (2004d) point out that

$$[Y]_t - \mu_1^{-2}\{Y\}_t^{[1,1]} = \sum_{j=1}^{N_t} c_j^2 = [Y^d]_t.$$

This can be consistently estimated by $[Y_\delta]_t - \mu_1^{-2}\{Y_\delta\}_t^{[1,1]}$. Hence, in theory, the realised BPV process can be used to consistently estimate the continuous and discontinuous components of

QV or, if augmented with the appropriate asymptotic distribution theory, as a basis for testing the hypothesis that prices have continuous sample paths.

The only other work we know which tries to split QV into that due to the continuous and jump components is Mancini (2003a) and Mancini (2003b). She does this via the introduction of a jump threshold whose absolute value goes to zero as the number of observations within each day goes to infinity. Following Barndorff-Nielsen and Shephard (2004d), Woerner (2003) has studied the robustness of realised power variation $\delta^{1-r/2}\sum_{j=1}^{\lfloor t/\delta\rfloor}|y_j|^r$ to an infinite numbers of jumps in finite time periods showing that the robustness property of realised power variation goes through in that case. A related paper is Aït-Sahalia (2004), which shows that maximum likelihood estimation can disentangle a homoskedastic diffusive component from a purely discontinuous infinite activity Lévy component of prices. Outside the likelihood framework, the paper also studies the optimal combinations of moment functions for the generalised method of moment estimation of homoskedastic jump-diffusions.

3 A theory for testing for jumps

3.1 Infeasible tests

In this Section we give the main contribution of the paper, Theorem 1. It gives the asymptotic distribution for a *linear jump* statistic, G, based on $\mu_1^{-2}\{Y_\delta\}_t^{[1,1]}-[Y_\delta]_t$ and a *ratio jump* statistic, H, based on[1] $\mu_1^{-2}\{Y_\delta\}_t^{[1,1]}/[Y_\delta]_t$. Their distributions, under the null of $Y\in\mathcal{SVSM}^c$, will be seen to depend upon the unknown integrated quarticity $\int_0^t\sigma_s^4\mathrm{d}s$ and so we will say the results of the Theorem are statistically infeasible. We will overcome this problem in the next subsection.

Theorem 1 *Let $Y\in\mathcal{SVSM}^c$ and let t be a fixed, arbitrary time. Suppose the following conditions are satisfied:*

(a) *That*

$$\delta^{-1}\int_{\delta(j-1)}^{\delta j}\sigma_s^2\mathrm{d}s$$

is bounded away from 0 and infinity, uniformly in j and δ.

(b) *The mean process A satisfies, (pathwise) as $\delta\downarrow 0$,*

$$\delta^{-1}\max_{1\le j\le\lfloor t/\delta\rfloor}|A_{j\delta}-A_{(j-1)\delta}|=O(1). \tag{6}$$

(c) *The joint process (A,σ) is independent of the Brownian motion W.*

[1]In private correspondence Xin Huang has informed me that following Barndorff-Nielsen and Shephard (2004d) Huang and George Tauchen have independently and concurrently used simulations to study the behaviour of this type of ratio, although they do not provide the corresponding asymptotic theory.

Recall the definition $\mu_1 = \sqrt{2}/\sqrt{\pi}$ *in (5) and let*

$$\vartheta = \left(\pi^2/4\right) + \pi - 5 \simeq 0.6090. \tag{7}$$

Then as $\delta \downarrow 0$

$$G = \frac{\delta^{-1/2}\left(\mu_1^{-2}\left\{Y_\delta\right\}_t^{[1,1]} - [Y_\delta]_t\right)}{\sqrt{\int_0^t \sigma_s^4 \mathrm{d}s}} \xrightarrow{L} N\left(0, \vartheta\right), \tag{8}$$

and

$$H = \frac{\delta^{-1/2}\left(\frac{\mu_1^{-2}\left\{Y_\delta\right\}_t^{[1,1]}}{[Y_\delta]_t} - 1\right)}{\sqrt{\frac{\int_0^t \sigma_s^4 \mathrm{d}s}{\left\{\int_0^t \sigma_s^2 \mathrm{d}s\right\}^2}}} \xrightarrow{L} N\left(0, \vartheta\right). \tag{9}$$

Further, if $Y \in \mathcal{SVFASM}$ *and* ***(a)****-****(c)*** *hold, then*

$$\left\{Y\right\}_t^{[1,1]} = \mu_1^2 \int_0^t \sigma_s^2 \mathrm{d}s. \tag{10}$$

Remark 1 *(i) Condition* ***(a)*** *in Theorem 1 essentially means that, on any bounded interval,* σ^2 *itself is bounded away from 0 and infinity. This is the case, for instance, for the square root process (due to it having a reflecting barrier at zero) and the Ornstein-Uhlenbeck volatility processes considered in Barndorff-Nielsen and Shephard (2001). More generally* ***(a)*** *does not rule out jumps, diurnal effects, long-memory or breaks in the volatility process.*

(ii) Result (10) is a generalisation of Barndorff-Nielsen and Shephard (2004d) which showed this result in the case where $A = 0$*.*

(iii) It is clear from the proof of Theorem 1 that in realised BPV we can replace the subscript $j+1$ *with* $j+q$ *where* q *is any positive but finite integer.*

(iv) Condition ***(c)*** *rules out leverage effects (e.g. Nelson (1991)) and is an unfortunate limitation of the result.*

(v) Result (10) means that under the alternative hypothesis of jumps

$$\mu_1^{-2}\left\{Y_\delta\right\}_t^{[1,1]} - [Y_\delta]_t \xrightarrow{p} -\sum_{j=1}^{N_t} c_j^2 \leq 0$$

and

$$\frac{\mu_1^{-2}\left\{Y_\delta\right\}_t^{[1,1]}}{[Y_\delta]_t} - 1 \xrightarrow{p} -\frac{\sum_{j=1}^{N_t} c_j^2}{\int_0^t \sigma_s^2 \mathrm{d}s + \sum_{j=1}^{N_t} c_j^2} \leq 0.$$

This implies that the linear and ratio tests will be consistent.

(vi) A biproduct of the Proof of Theorem 1 is Theorem 3, given in the Appendix, which is a joint central limit theorem for scaled realised BPV and QV processes. This is proved under the assumption that $Y \in \mathcal{SVSM}^c$ *and shows that they, of course, both estimate* $\int_0^t \sigma_s^2 \mathrm{d}s$ *with realised QV having a slightly smaller asymptotic variance.*

3.2 Feasible tests

In order to construct computable linear and ratio jump tests we need to be able to estimate the integrated quarticity $\int_0^t \sigma_s^4 \mathrm{d}s$ under the null hypothesis of $Y \in \mathcal{SVSM}^c$. However, in order to ensure the test has power under the alternative it is preferable to have an estimator which is also consistent under the alternative $\mathcal{SVFASM}$. This is straightforward. Barndorff-Nielsen and Shephard (2004d) generalised bipower variation to multipower variation, a special case of which is the robust estimator called realised quadpower variation

$$\{Y_\delta\}_t^{[1,1,1,1]} = \delta^{-1} \sum_{j=4}^{\lfloor t/\delta \rfloor} |y_{j-3}| \, |y_{j-2}| \, |y_{j-1}| \, |y_j| \xrightarrow{p} \mu_1^4 \int_0^t \sigma_s^4 \mathrm{d}s.$$

Following early drafts of the work reported in this paper our central limit theory for the linear jump statistic has been used by Huang and Tauchen (2003) and Andersen, Bollerslev, and Diebold (2003). They favoured using the robust realised tripower variation

$$\{Y_\delta\}_t^{[4/3,4/3,4/3]} = \delta^{-1} \sum_{j=3}^{\lfloor t/\delta \rfloor} |y_{j-2}|^{4/3} \, |y_{j-1}|^{4/3} \, |y_j|^{4/3} \xrightarrow{p} \mu_{4/3}^3 \int_0^t \sigma_s^4 \mathrm{d}s,$$

where $\mu_r = \mathrm{E}\,|u|^r$ and $u \sim N(0,1)$. Both of these estimators are consistent under the $\mathcal{SVFASM}$ hypothesis. From now on we will focus solely on the quadpower case.

The above discussion allows us to define the feasible *linear jump test* statistic, $\widehat{G}$, which has the asymptotic distribution

$$\widehat{G} = \frac{\delta^{-1/2} \left(\mu_1^{-2} \{Y_\delta\}_t^{[1,1]} - [Y_\delta]_t \right)}{\sqrt{\mu_1^{-4} \{Y_\delta\}_t^{[1,1,1,1]}}} \xrightarrow{L} N(0, \vartheta), \tag{11}$$

where we would reject the null of a continuous sample path if (11) is significantly negative. Likewise, the *ratio jump test* statistic, $\widehat{H}$, defined as[2]

$$\widehat{H} = \frac{\delta^{-1/2}}{\sqrt{\{Y_\delta\}_t^{[1,1,1,1]} \Big/ \left\{ \{Y_\delta\}_t^{[1,1]} \right\}^2}} \left(\frac{\mu_1^{-2} \{Y_\delta\}_t^{[1,1]}}{[Y_\delta]_t} - 1 \right) \xrightarrow{L} N(0, \vartheta), \tag{12}$$

rejects the null if significantly negative.

An interesting feature of the ratio jump test is that $\{Y_\delta\}_t^{[1,1]} / [Y_\delta]_t$ is asymptotically equivalent to

$$\widehat{\rho}_{1,t} = \frac{\sum_{j=2}^{\lfloor t/\delta \rfloor} |y_{j-1}||y_j|}{\sqrt{\sum_{j=2}^{\lfloor t/\delta \rfloor} y_{j-1}^2 \sum_{j=2}^{\lfloor t/\delta \rfloor} y_j^2}} \xrightarrow{p} \mu_1^2 \frac{\int_0^t \sigma_s^2 \mathrm{d}s}{\int_0^t \sigma_s^2 \mathrm{d}s + \sum_{j=1}^{N_t} c_j^2},$$

[2]It is somewhat tempting to look at the log-linear version of this test statistic, studying $\log\left(\mu_1^{-2} \{Y_\delta\}_t^{[1,1]}\right)$ minus $\log\left([Y_\delta]_t\right)$. Its asymptotic behaviour follows immediately from (12) via the delta method. Simulations suggest that this does not improve the finite sample performance of the test and so we will not discuss it further in this paper.

under $\mathcal{SVFASM}$. $\widehat{\rho}_{1,t}$ is a correlation like measure between $|y_{j-1}|$ and $|y_j|$. It converges to $\mu_1^2 \simeq 0.6366$ under $\mathcal{SVSM}^c$. Estimates below μ_1^2 provide evidence for jumps. Its asymptotic distribution under the null follows trivially from (12).

4 Time series of realised quantities

We remarked in the introduction of this paper that considerable attention has recently been given to daily discretisations of the realised QV process

$$\widehat{v}_i = [Y_\delta]_{\hbar i} - [Y_\delta]_{\hbar(i-1)}, \quad i = 1, 2, ..., T.$$

This produces a time series of daily realised variances. Here we briefly discuss the corresponding results for the realised BPV process and then discuss the asymptotic theory for a time series of such sequences. These results follow straightforwardly from our previous theoretical results.

Clearly we can define, for a fixed time interval $\hbar > 0$ which we will refer to as a day for concreteness, a sequence of T daily *realised bipower variations*

$$\begin{aligned} \widetilde{v}_i^{[1,1]} &= \{Y_\delta\}_{\hbar i}^{[1,1]} - \{Y_\delta\}_{\hbar(i-1)}^{[1,1]}, \quad i = 1, 2, ..., T, \\ &= \sum_{j=2}^{\lfloor t/\delta \rfloor} |y_{j-1,i}| \, |y_{j,i}|, \end{aligned}$$

where we assume δ satisfies $\delta \lfloor t/\delta \rfloor = t$ for ease of exposition and

$$y_{j,i} = Y_{\delta j + \hbar(i-1)} - Y_{\delta(j-1)+\hbar(i-1)}.$$

Clearly

$$\mu_1^{-2} \widetilde{v}_i^{[1,1]} \xrightarrow{p} [Y^c]_{\hbar i} - [Y^c]_{\hbar(i-1)}.$$

In order to develop a feasible limit theory it will be convenient to introduce a sequence of daily *realised quadpower variations*

$$\begin{aligned} \widetilde{q}_i^{[1,1,1,1]} &= \{Y_\delta\}_{\hbar i}^{[1,1,1,1]} - \{Y_\delta\}_{\hbar(i-1)}^{[1,1,1,1]}, \quad i = 1, 2, ..., T \\ &= \delta^{-1} \sum_{j=4}^{\lfloor t/\delta \rfloor} |y_{j-3,i}| \, |y_{j-2,i}| \, |y_{j-1,i}| \, |y_{j,i}|. \end{aligned}$$

The above sequences of realised quantities suggest constructing a sequence of non-overlapping infeasible and feasible daily jump test statistics

$$G_{\delta i} = \frac{\delta^{-1/2} \left(\mu_1^{-2} \widetilde{v}_i^{[1,1]} - \widehat{v}_i \right)}{\sqrt{\int_{\hbar(i-1)}^{\hbar i} \sigma_s^4 \mathrm{d}s}}, \tag{13}$$

$$H_{\delta i} = \frac{\delta^{-1/2}}{\sqrt{\int_{\hbar(i-1)}^{\hbar i} \sigma_s^4 \mathrm{d}s \Big/ \left\{ \int_{\hbar(i-1)}^{\hbar i} \sigma_s^2 \mathrm{d}s \right\}^2}} \left(\frac{\mu_1^{-2} \widetilde{v}_i^{[1,1]}}{\widehat{v}_i} - 1 \right), \tag{14}$$

$$\widehat{G}_{\delta i} = \frac{\delta^{-1/2} \left(\mu_1^{-2} \widetilde{v}_i^{[1,1]} - \widehat{v}_i \right)}{\sqrt{\mu_1^{-4} \widetilde{q}_i^{[1,1,1,1]}}}, \tag{15}$$

$$\widehat{H}_{\delta i} = \frac{\delta^{-1/2}}{\sqrt{\widetilde{q}_i^{[1,1,1,1]} \Big/ \left\{ \widetilde{v}_i^{[1,1]} \right\}^2}} \left(\frac{\mu_1^{-2} \widetilde{v}_i^{[1,1]}}{\widehat{v}_i} - 1 \right). \tag{16}$$

By inspecting the proof of Theorem 1 it is clear that as well as each of these individual tests is converging to $N(0, \vartheta)$ as $\delta \downarrow 0$, they converge as a sequence in time jointly to a multivariate normal distribution. In particular, focusing solely on the feasible versions of the tests, define a sequence of tests based on T consecutive days

$$\widehat{G}_{\delta} = \left(\widehat{G}'_{\delta 1}, ..., \widehat{G}'_{\delta T} \right)' \quad \text{and} \quad \widehat{H}_{\delta} = \left(\widehat{H}'_{\delta 1}, ..., \widehat{H}'_{\delta T} \right)',$$

as a collection of T days of separate tests, then under the null, as $\delta \downarrow 0$ so

$$\widehat{G}_{\delta} \xrightarrow{L} N(0, \vartheta I_T) \quad \text{and} \quad \widehat{H}_{\delta} \xrightarrow{L} N(0, \vartheta I_T).$$

5 Simulation study

5.1 Simulation design

In this section we document some Monte Carlo experiments which assess the finite sample performance of our asymptotic theory for realised QV and BPV processes. Throughout we work with a $\mathcal{SVFASM}$ model and $\hbar = 1$. In particular we assume that

$$Y_t = \int_0^t \sigma_s \mathrm{d}W_s + \sum_{j=1}^{N_t} c_j.$$

Throughout our simulations the component processes σ, W, N and c are assumed to be independent. Before we start we should mention that in independent and concurrent work Huang and Tauchen (2003) have also studied the finite sample behaviour of our central limit theory using an extensive simulation experiment. Their Monte Carlo work was based around the empirical results found in Andersen, Benzoni, and Lund (2002) and Chernov, Gallant, Ghysels, and Tauchen (2003). They studied the effectiveness of the feasible linear and log-based theory based on realised quadpower (which we will also do here) and tripower variation (which we will not use). Their conclusions are broadly in line with the ones we reach in this Section. We should mention that their studies allow the processes to exhibit leverage effects, which our theory does not cover and we rule out in our simulations. Previous results in the context of realised variances suggest leverage effects make no difference to the asymptotic distribution of realised variation objects. See Barndorff-Nielsen and Shephard (2004b) for details.

Our model for σ is derived from some empirical work reported in Barndorff-Nielsen and Shephard (2002) who used realised variances to fit the spot variance of the DM/Dollar rate from 1986 to 1996 by the sum of two uncorrelated processes

$$\sigma^2 = \sigma_1^2 + \sigma_2^2.$$

Their results are compatible with using CIR processes for the σ_1^2 and σ_2^2 processes. In particular we will write these, for $s = 1, 2$, as the solutions to

$$\mathrm{d}\sigma_{t,s}^2 = -\lambda_s \left\{ \sigma_{t,s}^2 - \xi_s \right\} \mathrm{d}t + \omega_s \sigma_{t,s} \mathrm{d}B_{\lambda_s t, s}, \quad \xi_s \geq \omega_s^2/2, \tag{17}$$

where $B = (B_1, B_2)'$ is a vector standard Brownian motion, independent from W. The process (17) has a gamma marginal distribution

$$\sigma_{t,s}^2 \sim Ga(2\omega_s^{-2}\xi_s, 2\omega_s^{-2}) = Ga\left(\nu_s, a_s\right), \quad \nu_s \geq 1,$$

with a mean of ν_s/a_s and a variance of ν_s/a_s^2. The parameters ω_s, λ_s and ξ_s were calibrated by Barndorff-Nielsen and Shephard (2002) as follows. Setting $p_1 + p_2 = 1$, they estimated

$$\mathrm{E}(\sigma_s^2) = p_s 0.509, \quad \mathrm{Var}(\sigma_s^2) = p_s 0.461, \quad s = 1, 2,$$

with

$$p_1 = 0.218, \quad p_2 = 0.782, \quad \lambda_1 = 0.0429, \quad \text{and} \quad \lambda_2 = 3.74,$$

which means the first, smaller component of the variance process is slowly reverting with a half-life of around 16 days while the second has a half-life of around 4 hours. Bollerslev and Zhou (2002) have found similar results using a shorter span of this type of exchange rate data.

When we add jumps to the prices we will take N as a stratified Poisson process so that there are always K jumps in each unit of time. We specify $c_j \overset{i.i.d.}{\sim} N(0, \sigma_c^2)$, so the jump process is a stratified compound Poisson process. The contributions of the long-run variation in log-prices of the jump process and the continuous component are $tK\sigma_c^2$ and $t0.509$ respectively. In our experiments we will vary K and σ_c^2, which allows us to see the impact of the frequency of jumps and their size on the behaviour of the realised bipower variation process. To start off we will fix $K = 2$ and $\sigma_c^2 = 0.2 \times 0.509$, which means that the jump process will account for 33% of the variation of the process. Clearly this is a high proportion. Later we will study the cases when $K = 1$ and $\sigma_c^2 = 0.1 \times 0.509$ and 0.05×0.509.

Throughout our calculations we will index the results by

$$n = 1/\delta.$$

Here n denotes the number of high frequency observations per unit of time. As n increases the Y_δ process becomes closer to the Y process.

5.2 Infeasible limit theory

5.2.1 Null distribution

Our Monte Carlo experiments will assess the finite sample behaviour of the infeasible jump tests given in (13), (14) and their corresponding feasible versions (15), (16). We start with the infeasible limit theory for (13) and (14), looking at their null distributions. Throughout we fix the duration of the jump test to $\hbar = 1$ and use $5,000$ simulated days.

The left hand side of Figure 1 shows the results from the first 300 days in the sample. The crosses depict $\mu_1^{-2}\widetilde{v}_i^{[1,1]} - \widehat{v}_i$, while the infeasible 95% one sided critical values (roughly -1.6 times the asymptotic standard errors in (13)) of the statistics are given by the solid line. As we go down the graph n increases and so, as the null hypothesis is true, the magnitude of $\mu_1^{-2}\widetilde{v}_i^{[1,1]} - \widehat{v}_i$ and corresponding critical values tend to fall towards zero. The most important part of these graphs is that the critical values of the tests change dramatically through time, reflecting the volatility clustering in the data.

The middle part of Figure 1 repeats this analysis, but now using the ratio jump test. Again the magnitude of $\mu_1^{-2}\widetilde{v}_i^{[1,1]}/\widehat{v}_i - 1$ tends to fall as n increases. The infeasible critical values of this test hardly change through time, reflecting the natural scaling of the denominator for the ratio jump test. The right hand part of Figure 1 shows the QQ plots of the two t-tests. On the y-axis are the ranked values of the simulated t-tests, while on the x-axis are the corresponding expected values under Gaussian sampling. For very small values of n the linear jump test has too much mass in its extreme right hand tail. Overall both tests have quite good QQ plots for moderate values of n.

In the upper part of Table 1 we show the biases and standard deviations of (13), (14). The estimated standard deviations are close to one and the one side 95% coverage rates are quite accurate, even for small values of n. Overall we can see that there is very little to choose between these two infeasible tests.

As a final check on the null distribution of the jump tests, we will repeat the above analysis but increasing λ_2, the memory parameter of the fast decaying CIR volatility process, by a factor of five. This reduces its half life down to 20 minutes. This case of an extremely short half-life is quite a challenge as a number of econometricians view very short memory SV models as being

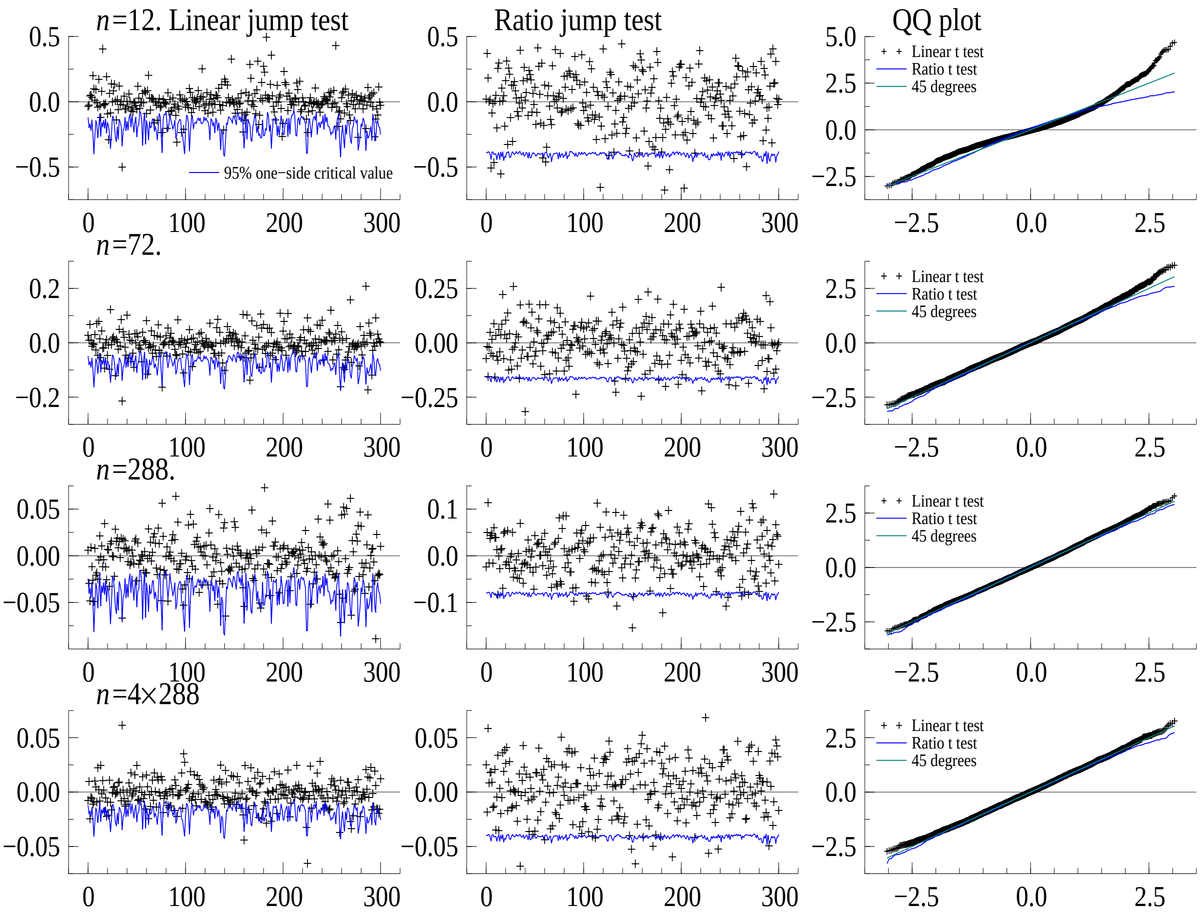

Figure 1: *Simulation from the null distribution of the infeasible limit theory for the linear (left hand side) and rato (middle) jump tests for a variety of values of* $\delta^{-1} = n$*. Right hand side gives the QQ plots of the t-statistics. Code:* `jump_RV_daily.ox`.

good proxies for processes with jumps. Table 1 shows the results. The linear test has a small negative bias. This effect falls as n becomes very large. The ratio test is has a small negative bias, which causes the test to very slightly over reject. However, the degree of overrejection is modest but more important than before. Hence this testing procedure can be challenged by very fast reverting volatility components.

5.2.2 Impact of jumps: the alternative distribution

We now introduce some jumps into the process and see how the tests react. The Poisson process is stratified so there are either 1 or 2 jumps per day, while the variance of the jumps is either 5%, 10% or 20% of the expectation of σ^2.

In the infeasible case the results are given in Table 2. The results are the expected ones. There is little difference in the power of the linear and ratio tests. As the number of jumps

	Standard setup						Robustness check $\lambda_2 = 5 \times 3.74$					
	linear test			ratio test			linear test			ratio test		
n	bias	S.D.	Cove	bias	S.D.	Cove	bias	S.D.	Cove	bias	S.D.	Cove
12	-.032	0.93	.974	-.012	0.93	.941	-.059	0.90	.979	-.039	0.89	.949
72	-.017	1.00	.955	-.014	0.99	.943	-.093	0.97	.972	-.097	0.96	.943
288	-.010	1.00	.957	-.008	1.00	.945	-.053	0.99	.960	-.053	0.99	.941
4×288	-.023	0.98	.956	-.024	0.98	.946	-.042	1.01	.954	-.041	1.01	.942

Table 1: *Finite sample behaviour of the infeasible linear and ratio tests. Cove denotes coverage, designed level is 0.95. Based on 5,000 seperate days. Left is the simulation based on the standard setup with* $\lambda_2 = 3.74$. *Right changes* λ_2 *to* 5×3.74 *to check for robustness to very fast decaying components.* Code: `jump_RV_daily.ox`.

increases, so the rate of accepting the null falls. The same effect happens as the variance of the jumps falls. However, in the extreme case where there is only a single jump a day and the jump is 5% of the variability of the continuous component of prices, we reject the null hypothesis 20% of the time when $n = 288$.

	$N(1) = 1$						$N(1) = 2$					
20%	linear test			ratio test			linear test			ratio test		
n	mean	S.D.	Cover	mean	S.D.	Cover	bias	S.D.	Cover	mean	S.D.	Cover
12	-0.30	1.56	.878	-0.14	1.00	.907	-0.51	1.95	.832	-0.23	1.03	.896
72	-1.45	3.38	.707	-0.84	1.58	.748	-2.81	4.49	.526	-1.47	1.79	.587
288	-3.69	6.91	.530	-2.21	2.98	.553	-7.18	9.26	.288	-3.89	3.41	.313
10%												
12	-0.12	1.15	.920	-0.06	0.96	.929	-0.19	1.33	.895	-0.10	0.98	.924
72	-0.61	1.86	.819	-0.43	1.26	.848	-1.19	2.36	.700	-0.78	1.41	.751
288	-1.68	3.48	.668	-1.22	2.09	.687	-3.27	4.58	.462	-2.22	2.47	.490
5%												
12	-.061	1.01	.938	-.029	.940	.937	-0.08	1.09	.926	-0.04	.959	.931
72	-.253	1.26	.885	-.200	1.08	.907	-0.48	1.46	.828	-0.37	1.15	.860
288	-.749	1.90	.792	-.612	1.48	.807	-1.44	2.38	.651	-1.14	1.71	.682

Table 2: *Infeasible case. Effect of jumps on the linear and ratio tests. On the right hand side we show results for the case where there are 2 jumps per day. On the left hand side, there is a single jump per day. The variance of the jumps are 20%, 10% and 5% respectively of the expectation of the variance process* σ^2, *with the results for the 20% case given at the top of the Table.* Code: `jump_RV_daily.ox.`

One of the interesting features of Table 2 is that the probability of accepting the null is roughly similar if $N = 2$ and each jump is 10% of the variation of σ^2 compared to the case where $N = 1$ and we look at the 20% example. This is repeated when we move to the $N = 2$ and 5% case and compare it to the $N = 1$ and 10% case. This suggests the rejection rate is heavily influenced by the variability of the jump process, not just the frequency of the jumps or the size of the individual jumps.

5.3 Assessing the performance of the feasible asymptotic theory

How do these conclusions change when we move from the infeasible to feasible limit theories (15) and (16)? Table 3 shows the results for the null distribution. It indicates that the linear test statistic is quite upset by moving to the feasible version, while the ratio statistic is reasonably robust for moderate values of n. Both statistics have a negative mean, leading to overrejection of the null. Even when $n = 288$ the linear test rejects the null around 8% of the time, which is quite some way from the nominal value.

	Standard setup						Robustness check $\lambda_2 = 5 \times 3.74$					
	linear test			ratio test			linear test			ratio test		
n	bias	S.D.	Cove	bias	S.D.	Cove	bias	S.D.	Cove	bias	S.D.	Cove
12	-.597	2.68	.813	-.102	1.41	.877	-.637	2.61	.804	-.151	1.41	.865
72	-.169	1.18	.891	-.053	1.07	.919	-.257	1.23	.875	-.133	1.09	.906
288	-.084	1.05	.918	-.029	1.02	.935	-.132	1.06	.908	-.077	1.03	.926
4×288	-.059	1.00	.935	-.035	0.99	.943	-.098	1.00	.932	-.073	0.99	.939

Table 3: *Finite sample behaviour of the feasible linear and ratio tests. Cove denotes coverage, designed level is* 0.95*. Based on* 5,000 *seperate days. Left is the simulation based on the standard setup with* $\lambda_2 = 3.74$*. Right changes* λ_2 *to* 5×3.74 *to check for robustness to very fast decaying components.* Code: `jump_RV_daily.ox`

These results are backed up by Figure 2, which shows a very poor QQ plot for the linear test even when $n = 72$. For larger values of n the asymptotics seems to have some substantial bite. The ratio test has quite good QQ plots for n being 72 or above. Table 3 also shows that when we boost the value of λ_2, so the second component of the spot variance process is very fast reverting, the results do not change very much. Again the ratio test provides a better test in terms of its coverage.

The most interesting feature of Figure 2 is that the critical values for the ratio jump tests are quite stable through time, reflecting the advantage that its asymptotic standard errors involve self-normalising ratios

$$\frac{\widetilde{q}_i^{[1,1,1,1]}}{\left\{\widetilde{v}_i^{[1,1]}\right\}^2} \xrightarrow{p} \frac{\int_{\hbar(i-1)}^{\hbar i} \sigma_s^4 \mathrm{d}s}{\left\{\int_{\hbar(i-1)}^{\hbar i} \sigma_s^2 \mathrm{d}s\right\}^2} \geq \frac{1}{\hbar}, \tag{18}$$

of realised terms. Hence they are not overly effected by changing volatility patterns between days. This feature can be very helpfully exploited to another important degree. In small samples the estimator (18) can fall below $1/\hbar$, which suggests that we replace the standard ratio jump test statistic by the feasible *adjusted ratio jump test*

$$\frac{\delta^{-1/2}}{\sqrt{\max\left[\frac{1}{\hbar}, \widetilde{q}_i^{[1,1,1,1]} \Big/ \left\{\widetilde{v}_i^{[1,1]}\right\}^2\right]}} \left(\frac{\mu_1^{-2}\widetilde{v}_i^{[1,1]}}{\widehat{v}_i} - 1\right) \xrightarrow{L} N(0, \vartheta), \tag{19}$$

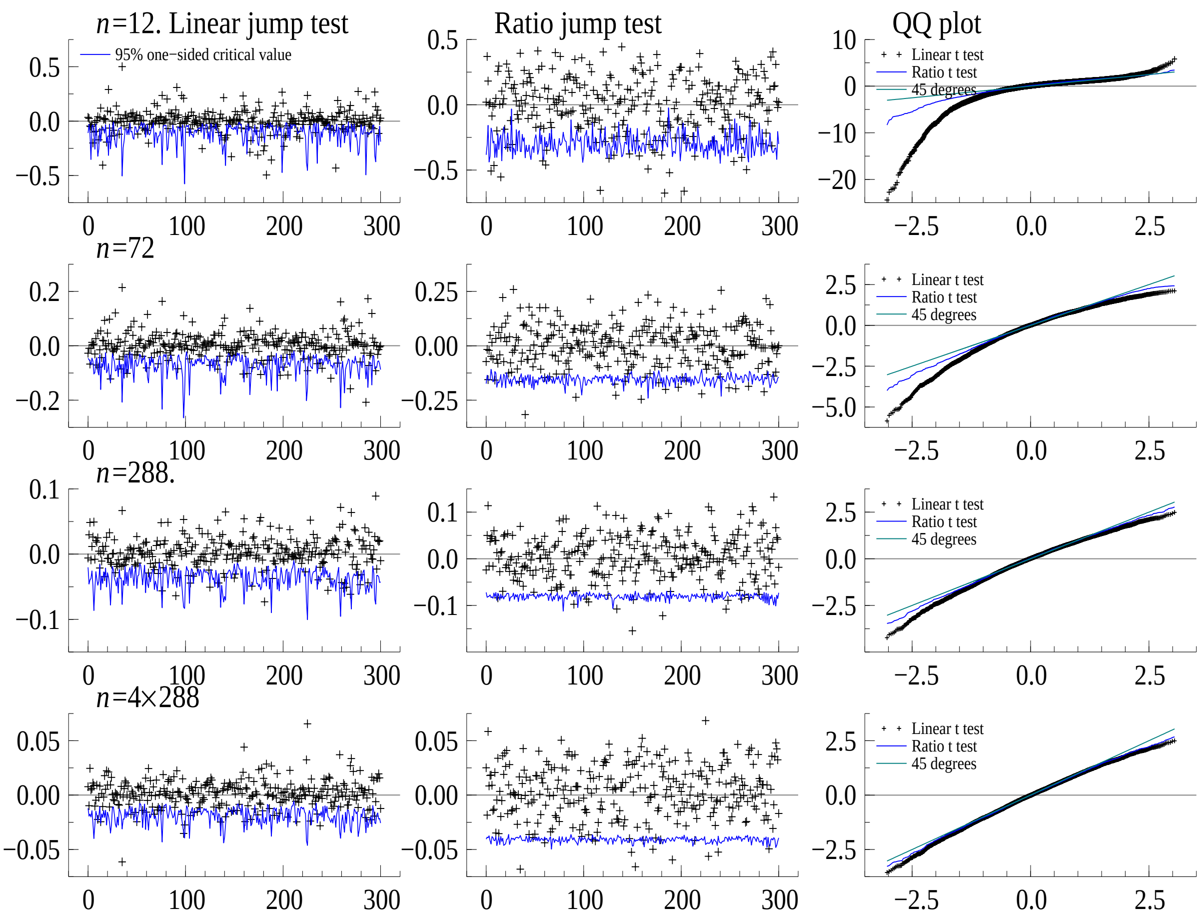

Figure 2: *Simulation from the null distribution of the feasible limit theory for the linear and rato jump tests for a variety of values of n. Right hand side gives the QQ plots of the t-statistics.* Code: `jump_RV_daily.ox`.

which we again use to reject the null of no jump if we observe large negative values. Our hope is that, in practice, it may produce better finite sample behaviour. This type of adjustment was used by Barndorff-Nielsen and Shephard (2004c) in the context of the asymptotic distribution of the log of realised variances. The simulation results for this adjusted ratio test statistic are given in Table 4. The results seem very close to those obtained by the infeasible theory and show only very modest 95% coverage errors even in the case where there is a fast reverting volatility component. This result holds even when n is small. Further, at the more extreme 99% level the coverage rates are still quite good when n is large. From now on when we refer to the ratio test we will implement this adjusted version.

We now move on to see how the tests are effected by the presence of jumps. Table 5 repeats the experiment from the previous subsection which led to the results given in Table 2 but now with the feasible linear and adjusted ratio jump statistics. We can see that in the linear jump

	Standard setup				Robustness check $\lambda_2 = 5 \times 3.74$			
n	bias	S.D.	Cove(95%)	Cove(99%)	bias	S.D.	Cove(95%)	Cove(99%)
12	-.017	0.992	.929	.980	-.056	0.992	.926	.978
72	-.033	1.01	.933	.983	-.108	1.03	.922	.975
288	-.025	1.01	.938	.985	-.073	1.02	.929	.979
4×288	-.035	0.992	.944	.986	-.073	0.997	.939	.984

Table 4: *Finite sample behaviour of the adjusted feasible ratio test. Cove denotes coverage, designed level is* 0.95 *and* 0.99*. Based on* 5,000 *seperate days. Left is the simulation based on the standard setup with* $\lambda_2 = 3.74$*. Right changes* λ_2 *to* 5×3.74 *to check for robustness to very fast decaying components.* Code: `jump_RV_daily.ox`.

tests there are many substantial changes compared to the infeasible case, with a great number of rejections of the null hypothesis. However, these changes are mostly caused by the substantial size distortion under the null. When we look at the adjusted ratio jump statistic, which has very good coverage properties, the results are very similar to the infeasible theory for the ratio test. This is a rather encouraging result. Overall when we use this statistic we can see that we have very little power when n is small unless there are a lot of jumps or the jumps are very large. However, with large n the performance of the test improves a great deal.

	$N(1) = 1$						$N(1) = 2$					
20%	linear test			adjusted ratio test			linear test			adjusted ratio test		
n	mean	S.D.	Cove	mean	S.D.	Cove	bias	S.D.	Cove	mean	S.D.	Cove
12	-1.05	3.93	.760	-.156	1.07	.894	-1.31	4.54	.730	-.251	1.10	.881
72	-1.61	3.53	.676	-.891	1.64	.735	-2.84	4.33	.521	-1.55	1.88	.573
288	-3.63	6.60	.526	-2.27	3.06	.546	-6.76	8.58	.292	-4.00	3.55	.310
10%												
12	-0.772	3.22	.790	-.073	1.02	.916	-.893	3.41	.774	-.110	1.04	.911
72	-0.797	2.07	.781	-.465	1.30	.837	-1.35	2.45	.673	-.835	1.46	.739
288	-1.73	3.43	.654	-1.25	2.14	.679	-3.21	4.43	.457	-2.29	2.55	.484
5%												
12	-0.642	2.84	.802	-.035	0.99	.926	-.723	2.97	.797	-.0566	1.01	.920
72	-0.423	1.46	.842	-.226	1.11	.895	-.657	1.62	.789	-.405	1.19	.847
288	-0.820	1.94	.776	-.640	1.52	.799	-1.49	2.39	.646	-1.18	1.76	.677

Table 5: *Feasible case. Effect of jumps on the linear and adjusted ratio tests. On the right hand side we show results for the case where there are 2 jumps per day. On the left hand side, there is a single jump per day. The variance of the jumps are* 20%*,* 10% *and* 5% *respectively of the expectation of the variance process* σ_t^2*, with the results for the* 20% *case given at the top of the Table.* Code: `jump_RV_daily.ox`

6 Testing for jumps empirically

6.1 Dataset

We now turn our attention to using our adjusted ratio jump test (19) on economic data. We follow Barndorff-Nielsen and Shephard (2002) in using the bivariate United States Dollar/ German Deutsche Mark and Dollar/ Japanese Yen exchange rate series, which covers the ten year period from 1st December 1986 until 30th November 1996. The original dataset records every 5 minutes the most recent mid-quote to appear on the Reuters screen. We have multiplied all returns by 100 in order to make them easier to present. The database has been kindly supplied to us by Olsen and Associates in Zurich, who document their pathbreaking work in this area in Dacorogna, Gencay, Müller, Olsen, and Pictet (2001).

6.2 Adjusted ratio jump test

Figure 3 plots the ratio test

$$\frac{\mu_1^{-2}\widetilde{v}_i^{[1,1]}}{\widehat{v}_i},$$

and its corresponding 99% critical values, computed under the assumption of no jump using the theory given in (19), for each of the first 250 working days in the sample for $n = 12$ and $n = 72$. We reject the null if the ratio is significantly below one. The values of n are quite small, corresponding to 2 hour and 20 minute returns, respectively. They were chosen to try to ensure our results were not overly sensitive to market microstructure errors. Results for larger values of n will be reported in a moment. Importantly the critical values do not change very much between different days. When n is very small this is mainly due to the use of the maximum function in the calculation of the standard error for the asymptotic distribution.

Figure 3 shows quite a lot of rejections of the null of no jumps, although the times when the rejections happen change quite a lot as n changes. Further, the average ratio is below one. When n is 12 the percentage of ratios below 1 is 70% and 73%, while when n increases to 72 these percentages become 71% in both cases.

Table 6 reports the corresponding results for the whole 10 year sample. This Table provides a warning of the use of too high a value of n for it shows the sum of the first to fifth serial correlation coefficients of the high frequency data. This is denoted by $r_{\cdot}$. We see that in the Dollar/DM series as n increases this correlation builds up, probably due to bid/ask bounce effects. By the time n has reached 288 the summed correlation had reached nearly -0.1 which means the realised variance overestimates the variability of prices by around 20%. Of course this effect could be removed by using a further level of pre-filtering before we analyse the data. The situation is worse for the Dollar/Yen series which has a moderate amount of negative correlation amongst the high frequency returns even when n is quite small. We will ignore these market microstructure effects.

Table 6 holds the average value of $\mu_1^{-2}\widetilde{v}_i^{[1,1]}$ and $\widehat{v}_i$ as well as the proportion of times the null is rejected using 95% and 99% tests. They are given for a variety of values of n and for both

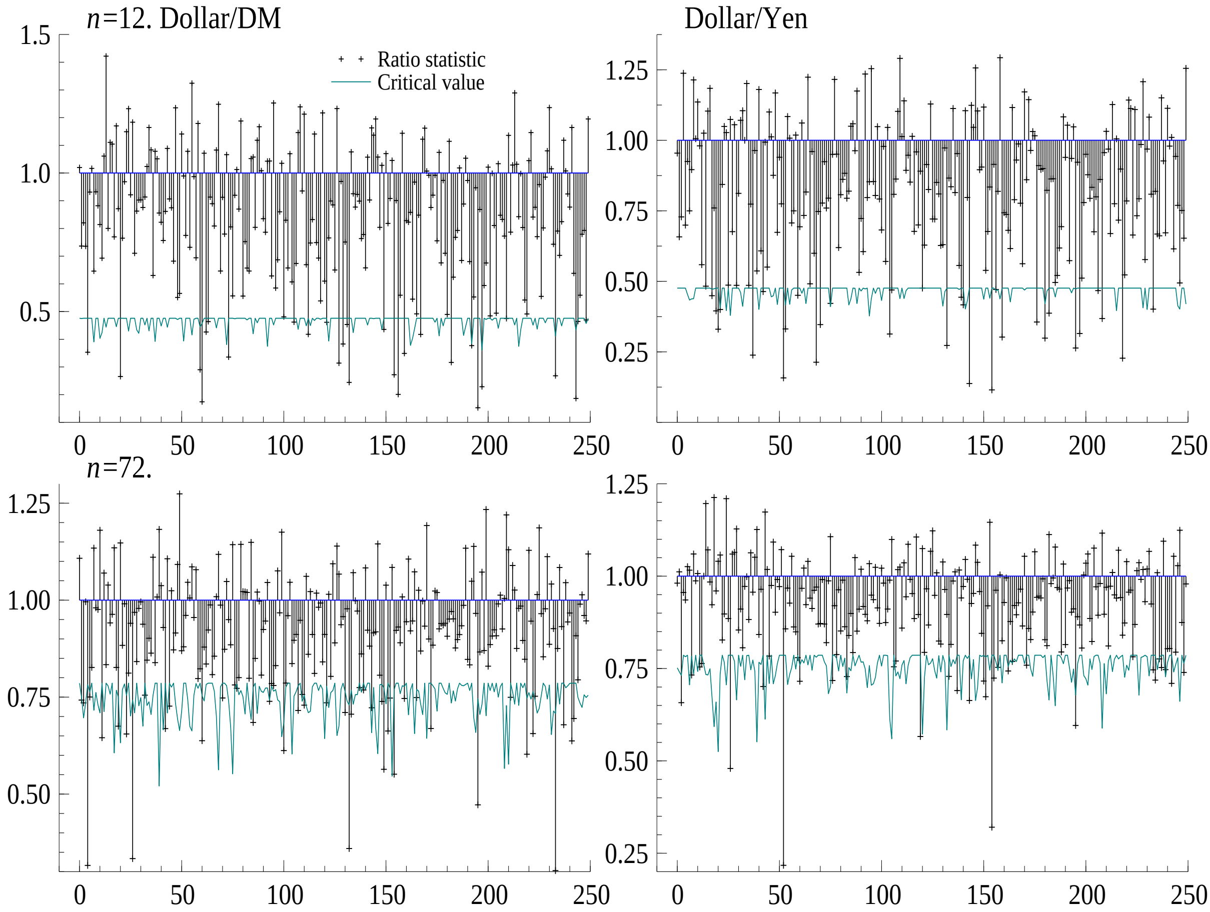

Figure 3: *Data based on the first year of the sample for the Dollar/DM (left hand side) and Dollar/Yen (right hand side) using $n = 12$ and $n = 72$. An index plot shows the ratio statistic computed each day, which should be around 1 if the null of no jumps is true. The corresponding 99% adjusted asymptotic critical value is also shown for each day.* Code: `jump_RV_daily.ox`.

exchange rates. They show that the results are reasonably stable with respect to n, although the percentage due to jumps do drift as n changes.

The Table shows that for the Dollar/DM series the variation of the jumps is estimated to contribute between around 3% and 12% of the QV. On many days there is enough evidence to reject the null hypothesis of no jumps, with around 20% rejections at the 5% level and 10% at the 1% level.

The results for the Dollar/Yen are rather similar, with the jumps contributing between 2% and 12% of the variation in the prices. A similar rate of rejection of the null is obtained for the Yen series as we saw for the DM rate.

Overall this analysis suggests that there is quite a lot of statistical evidence that there are jumps in the exchange rate series. Interestingly the percentage of rejections and proportions

	Dollar/DM						Dollar/Yen					
n	$r_{\cdot}$	cont	QV	jump %	5% rej	1% rej	$r_{\cdot}$	cont	QV	jump %	5% rej	1% rej
12	.001	.355	.452	12.0	.202	.090	-.041	.328	.420	12.2	.201	.086
48	.012	.408	.467	6.80	.219	.114	-.032	.409	.458	5.72	.209	.101
72	-.001	.437	.487	5.41	.225	.120	-.032	.429	.471	4.68	.195	.095
144	-.056	.471	.510	3.91	.220	.116	-.077	.473	.506	3.35	.223	.107
288	-.092	.502	.531	2.83	.181	.092	-.100	.512	.539	2.56	.187	.095

Table 6: $r_{\cdot}$ *denotes the sum of the first five serial correlation coefficients of the high frequency data. "cont" denotes the average value of* $\mu_1^{-2}\{Y_\delta\}_i^{[1,1]}$ *over the sample.* QV *gives the corresponding result for the average realised variance over the sample. jump % denotes the estimate of the percentage of the quadratic variation due to jumps in the sample. 5% rej shows the proportion of rejections at the 5% level, while 1% indicates the same thing but at the 1% level.*

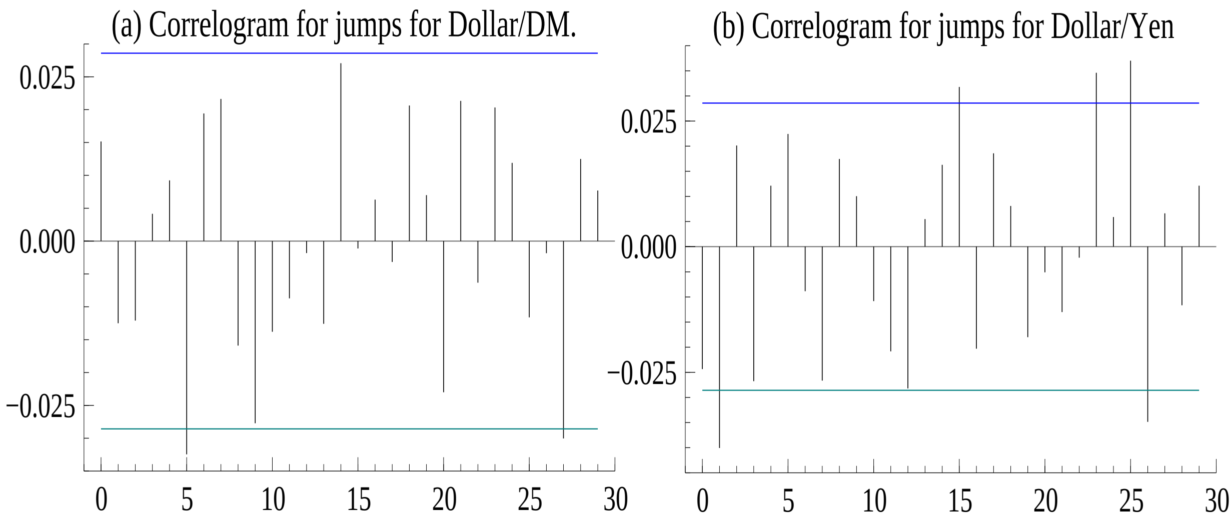

Figure 4: *Correlogram for a time series of indicators which note if the jump test was failed. This was based on* $n = 72$ *and using a 5% size test. Straightline are standard Bartlett 95% confidence intervals for these statistics.* Code: `jump_RV_daily.ox`.

due to jumps seems rather stable as we move between the two exchange rates.

Figure 4 shows the correlogram of a daily time series which records a one if the adjusted ratio jump test rejects the null of no jump and a zero if the null is accepted. Hence this attempts to measure if there is any serial dependence in the occurrence of jumps between days. For both currencies there is little evidence for clustering of jumps between days. Of course fully parametric models may be more successful at detecting this type of subtle dependence.

6.3 Case studies

6.3.1 Two contrasting days

In this subsection we will look at some specific days in the sample which have large realised variances[3] to see if we can link together the outcomes from the formal statistical analysis with more informal discussions. Throughout we focus on the Dollar/DM rate. To start we will give a detailed discussion of two extreme days, which we will put in context by analysing them together with a few days each side of the extreme events. We plot Y_δ for a variety of values of n using dots, rather than the more standard time series lines, as well as giving the adjusted ratio jump statistics with its corresponding 99% critical values.

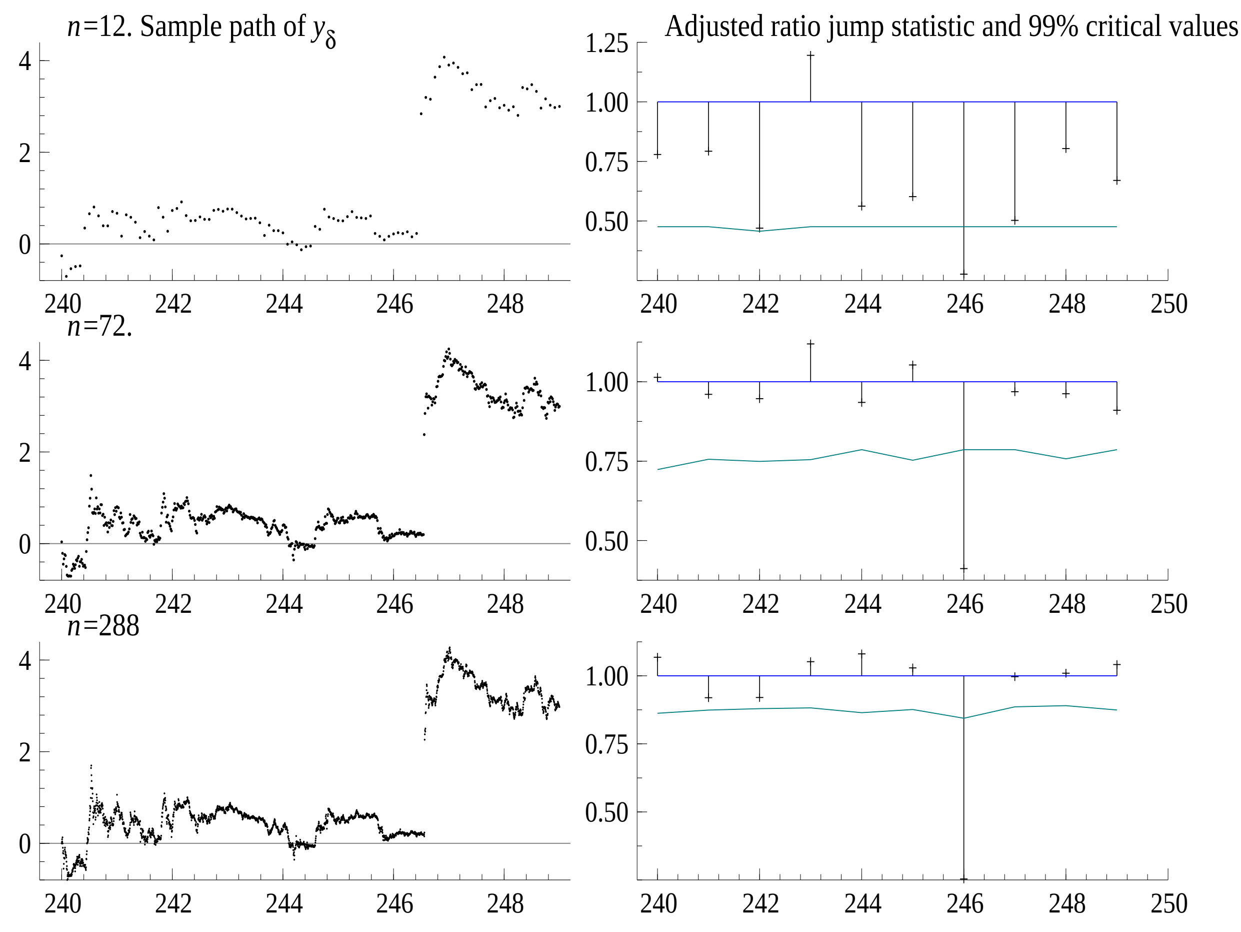

Figure 5: *Example of small stretches of data with large realised variances. Left hand side show Y_δ for a variety of values of n. Right shows the adjusted ratio jump statistic, together with 99% critical values. The large step change in the prices, occurred on 15th January 1988 when surprising U.S. balance of payment figures were announced. Code is available at:* `jump_RV.ox`.

In Figure 5 there is a large uptick in the Dollar against the D-mark, with a movement of

[3] The result does not change whether one uses the Andersen, Bollerslev, Diebold, and Labys (2001) version of this data, or the dataset which includes the Barndorff-Nielsen and Shephard (2002) stochastic adjustments.

nearly two percent in a five minute period. This occurred on January 15th 1988. The Financial Times reported on its front page the next day

> "The dollar and share prices soared in hectic trading on world financial markets yesterday after the release of official figures showing that the US trade deficit had fallen to \$13.22 bn in November from October's record level of \$17.63 bn. The US currency surged 4 pfennigs and 4 yen within 10 minutes of the release of the figures and maintained the day's highest levels in late New York business ... ."

The data for January 15th had a large realised variance but a much smaller estimate of the integrated variance. Hence the statistics are attributing a large component of the realised variance to the jump with the adjusted ratio statistic being larger than the corresponding 99% critical value. When n is small the statistic is on the borderline of being significant, while the situation becomes much clearer as n becomes large.

This jump effect contrasts with Figure 6 where on the 1134th day, August 19th 1991, there is a three percentage strengthening in the Dollar, but this happens over an hour long period with many positive returns. In the early hours of that day the Russian President Mikhail Gorbachev was overthrown after a coup by Communist hardliners. The Financial Times reported on its front page the next day:

> "Share prices around the world plummeted ... and the dollar climbed by more than 5 pfennigs against the D-Mark yesterday, as financial markets experienced their most turbulent trading conditions since the crash of October 1987. ... President Mikhail Gorbachev's removal from power led to intense investor nervousness about the effects of a Soviet political crisis spilling over into the rest of Europe and disturbing the outlook for the world economy....The repercussions were especially marked in Germany....On currency markets, dealers sold the D-Mark for dollars....Central bank intervention by Germany, Britain, Italy, France and Japan damped the dollar's rise. After touching a high of DM1.832, it closed last night in London at DM1.8165, up more than 5 pfennigs. In New York if finished at DM1.8235."

The corresponding realised variance is very high, but so is the estimated integrated variance. Hence in this case the statistics have not flagged up a jump in the price, even though prices were moving rapidly.

More surprisingly Figure 6 flags up a possible jump on the next day, August 20th. This happened around 10.20 GMT, where we had consecutive percentage returns -0.577, -0.999 and 1.027, showing a sharp sell-off in the Dollar followed by a recovery. The London Times reported the next day

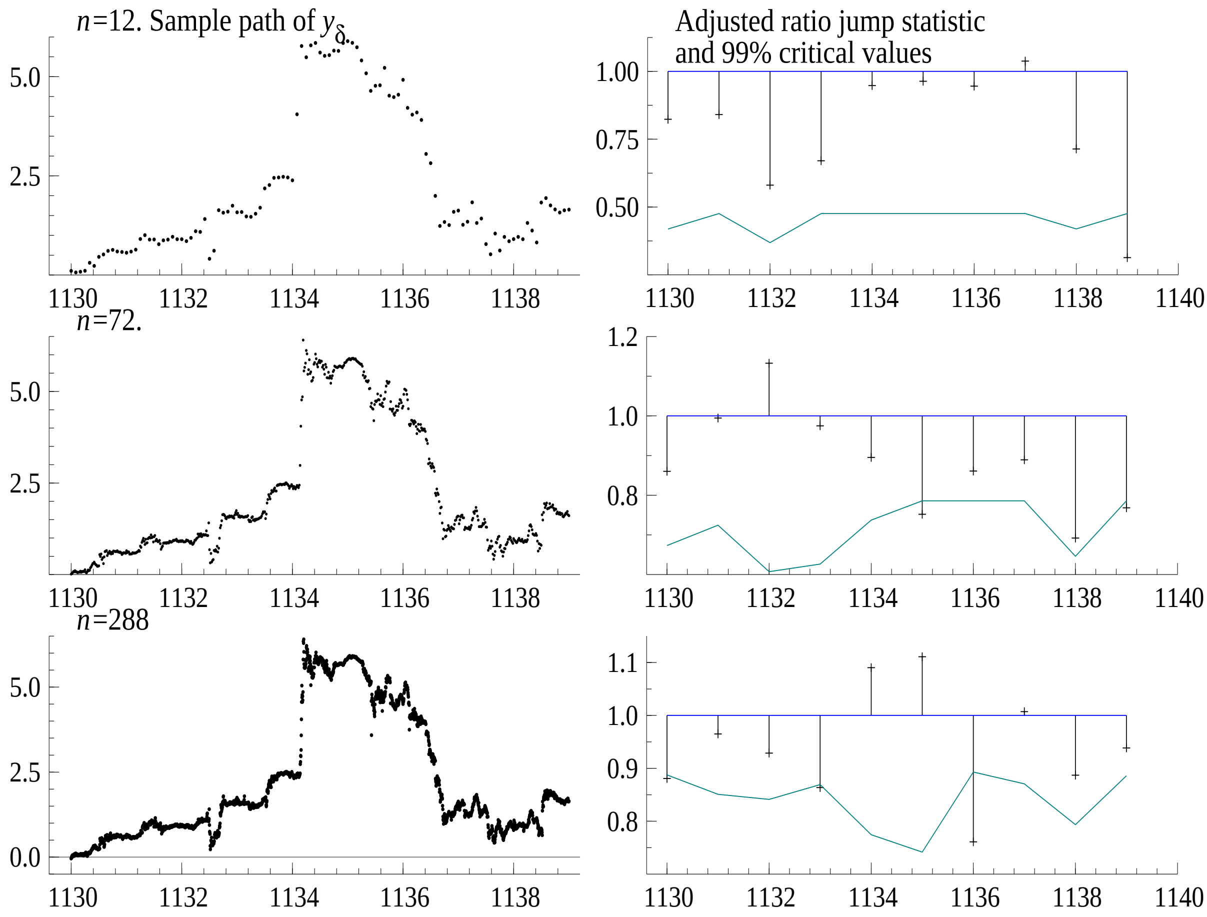

Figure 6: *Example of small stretches of data with large realised variances. Left hand side show* Y_δ *for a variety of values of* n*. Right shows the adjusted ratio jump statistic, together with 99% critical values. Large change in prices attributed to high volatility, not a jum. The change occured on 19th August 1991 when Gorbachev was removed by a coup. Code is available at:* `jump_RV.ox`.

> "During the European trading morning, a report that Mikhail Gorbachev was back in the Kremlin sent the dollar into an immediate three-pfennig plunge against the mark. However, a further report - that Mr. Gorbachev was still in the Crimea - wiped out the fall just as quickly."

6.3.2 High volatility days

An important question is whether these two days were typical of extreme days on the foreign exchange market? Here we focus will be on two sets of days: all those days where the ratio statistic is small or large and the realised variance is quite large. Throughout $n = 288$ is used.

Figure 7 plots results for all the 8 days when the ratio statistic is less than 0.6, suggesting a jump, and where the realised variance is above 1.2. On each day the Figure shows a single big

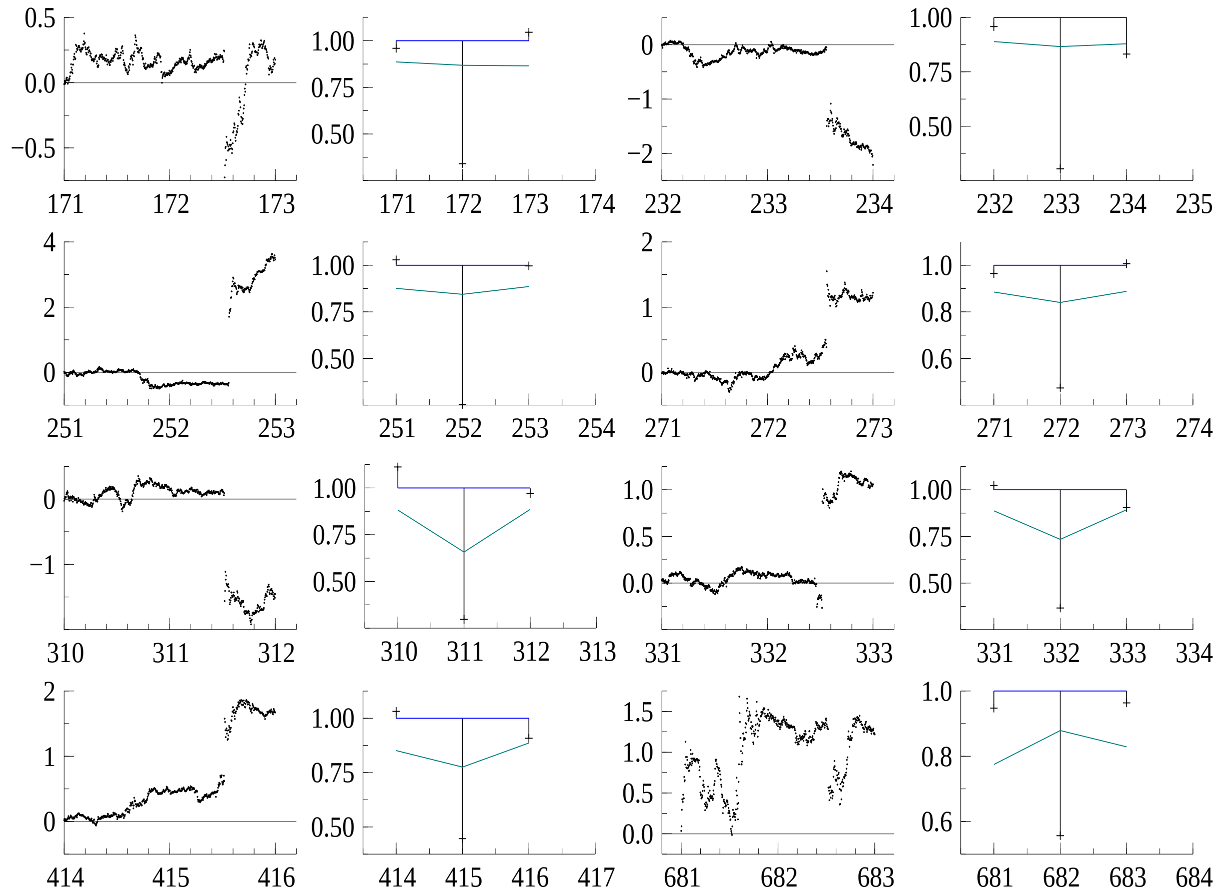

Figure 7: *Days on which there is a high realised variance and the adjusted ratio jump test found a jump using* $n = 288$*. Depicted is* Y_δ *and the corresponding jump test, with 99% critical values. Code is available at:* `jump_RV.ox`.

movement which is much larger in magnitude than the others on that day. These big changes are listed below.

Sequence	Day	GMT	move
173th	Friday 11th September, 1987	12.35	$-.967$
234th	Thursday, 10th December, 1987	13.35	-1.44
253th	Friday, 15th January, 1988	13.35	2.03
273th	Friday, 12th February, 1988	13.35	1.16
312th	Thursday, 14th April, 1988	12.35	-1.65
333th	Tuesday, 17th May, 1988	12.35	1.14
416th	Wednesday, 14th September, 1988	12.35	0.955
683th	Tuesday, 17th October, 1989	12.35	-0.714

Most U.S. macroeconomic announcements are made at 8.30 EST, which is 12.30 GMT from early April to late October and 13.30 otherwise (the precise dates of daylight saving times vary from year to year). This means that all the jumps we have seen in this Figure correspond to macroeconomic announcements. There is a substantial economic literature trying to match up movements in exchange rates to macroeconomic announcements (e.g. Ederington and Lee

(1993), Andersen and Bollerslev (1998b) and Andersen, Bollerslev, Diebold, and Vega (2003)). Generally this concludes that such news is quickly absorbed into the market, moving the rates vigourously, but with little long term impact on the subsequent volatility of the rates.

Finally in Figure 8 we plots results for days where the realised volatility is greater than 3.79. On each of these days the jump statistic indicates no jump in the process.

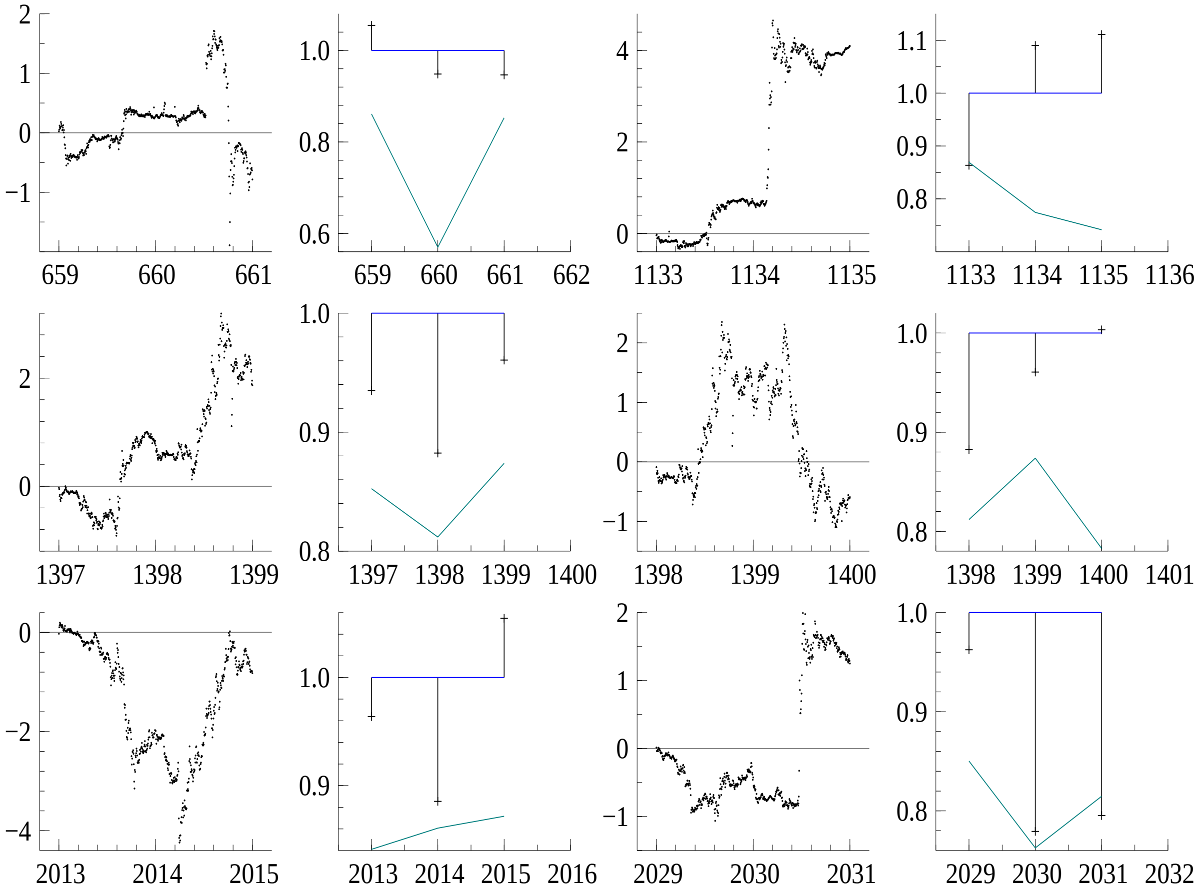

Figure 8: *Days on which there is a high realised variance and the adjusted ratio jump test did not find a jump using $n = 288$. Depicted is Y_δ and the corresponding jump test, with 99% critical values. Code is available at:* `jump_RV.ox`.

The non-jump days with very high realised variances are listed below.

Sequence	Day
661st	Friday, 15th September 1989
1135th	Monday, 19th August 1991
1399th	Wednesday, 16th September 1992
1400th	Thursday, 17th September 1992
2015th	Wednesday, 8th March 1995
2031th	Thursday, 30th March 1995

The most interesting period in this table is around 16th September 1992, which is when the UK suspended its ERM membership. This did cause a small immediate weakening in the Dollar on this day but generally this seems to have been largely anticipated. Instead the dominant feature is the very high level of volatility during this period.

7 Extensions and discussion

7.1 Leverage

A significant limitation of our analysis has been that we have assumed that (A, σ) are independent from W. This no leverage assumption (e.g. Black (1976), Nelson (1991) and Ghysels, Harvey, and Renault (1996)) is empirically reasonable with exchange rates but clashes with what we observe for equity data. This is very important. It is not clear how our results change in the case where we have leverage. Huang and Tauchen (2003) have some simulation results on this topic. Some discussion of this in the connected realised power variation case is given in Barndorff-Nielsen, Graversen, and Shephard (2004) where the very limited known theoretical results are outlined.

7.2 Multivariate processes

Our discussion of jumps in financial economics has been entirely univariate. How can we think of multivariate versions of the objects we discussed in this paper?

Quadratic covariation plays an essential role in financial econometrics (e.g. Barndorff-Nielsen and Shephard (2004a) and Andersen, Bollerslev, Diebold, and Labys (2003)). We will discuss this in the context of $(X, Y)' \in \mathcal{SM}$. Then the quadratic covariation between X and Y is

$$[X, Y]_t = \text{p}-\lim_{n\to\infty} \sum_{j=1}^{n} \left(X_{t_j} - X_{t_{j-1}}\right)\left(Y_{t_j} - Y_{t_{j-1}}\right).$$

Note that using this notation $[Y, Y] = [Y]$.

An important property of QV is the so called polarisation result that

$$[X, Y] = \frac{1}{2}\left\{[X + Y] - [X] - [Y]\right\}.$$

Further, of course, we can use realised BPV to consistently estimate

$$[X^c, Y^c] = \frac{1}{2}\left\{[X^c + Y^c] - [X^c] - [Y^c]\right\}$$

by estimating each of the individual terms. This allow us to also estimate $[X^d, Y^d]$. We are currently studying this and an alternative estimator of this type based on

$$[X, Y] = \frac{1}{4}\left\{[X + Y] - [X - Y]\right\}.$$

In principle this style of analysis allows us to understand the dependence between assets in standard time and in times of jumps. This topic has received considerable attention in the economics literature which often postulates that the dependence between assets can vary considerably during periods of extreme variability.

7.3 Building reduced form models

Following Barndorff-Nielsen and Shephard (2004d), Andersen, Bollerslev, and Diebold (2003) have used bipower variation as an input into new reduced form forecasting devices for modelling future values of realised variances (which in turn proxy the variability of future prices). This follows the influential line of thinking of Andersen, Bollerslev, Diebold, and Labys (2003) who modelled realised variances in terms of lags of previous realised variances. Following initial versions of the work reported in this paper, they used the result given in Theorem 1 to truncate the BPV based estimator of the QV of the jump component if the jumps are not significant. This shrinkage style estimator seems successful in empirical work, yielding fresh insights and improved forecast accuracy.

7.4 Market microstructure effects

In some recent stimulating work a number of researchers have been trying to formally measure and mitigate the impact of market microstructure noise (e.g. irregularly spaced data, rounding, volume effects, etc.) on the estimation of integrated variance using realised QV. Andersen, Bollerslev, Diebold, and Labys (2000) studied the biases induced by noise using the so-called signature plot. Leading recent papers which study the sampling properties of realised QV in the presence of noise include Bandi and Russell (2003a), Bandi and Russell (2003b) and Zhang, Mykland, and Aït-Sahalia (2003). Related work in the probability literature on the impact of noise on discretely observed diffusions can be found in Gloter and Jacod (2000a) and Gloter and Jacod (2000b), while Delattre and Jacod (1997) report results on the impact of rounding on sums of functions of discretely observed diffusions. Another stream of authors either suggest various adjustments to the return sequences or the definition of realised QV to overcome the effect of market microstructure noise. Elegant papers on this include Hansen and Lunde (2003), Martens (2003) and Curci and Corsi (2003). Also, of course, special mention should be made to Dacorogna, Gencay, Müller, Olsen, and Pictet (2001) who have carried out seminal work on the careful construction of reliable price data for the study of volatility. Finally, Zhang, Mykland, and Aït-Sahalia (2003) propose a subsampling scheme to attempt to overcome biases due to market microstructure effects.

Here we very briefly discuss the impact of noise on realised BPV. We draw our inspiration from the papers of Bandi and Russell (2003a), Bandi and Russell (2003b) and Zhang, Mykland, and Aït-Sahalia (2003), although our model for noise will be different. We suppose we make observations at equally space time intervals 0, δ, 2δ, ... and we inherit some measurement noise u each time we do this. In particular we observe $X_{\delta j} = Y_{\delta j} + u_j$, the true equilibrium price plus an error. In all the above three papers the noise is assumed to be i.i.d. with a distribution which does not vary with δ. It is the second of those assumptions which is crucial to their results. We depart from this by instead assuming a component model for u_j with $u_j = u_{j,1} + u_{j,2}$, where $u_{j,1} = o_p(\delta^{1/2})$ while

$$u_{2,j} = \sum_{i=N_{\delta(j-1)}+1}^{N_{\delta j}} v_i, \quad v_i = O_p(1),$$

where N is a finite activity process (that is $N_t < \infty$ for all t). This means that most pieces of noise will be small, but once in a while there will be a wild measurement. If the time gap is large then the probability of seeing a wild observation is large, while for small time gaps the errors will be modest. This has some relation to the work of Gloter and Jacod (2000a) and Gloter and Jacod (2000b) who studied the impact of noise on the estimation of the parameters in a fully parametric diffusion observed with errors. In those papers they allowed the properties of noise to change with δ.

The model for $X_{\delta j}$ means that the recorded returns are

$$x_j = X_{\delta j} - X_{\delta(j-1)} = y_j + u_j - u_{j-1}.$$

The QV process for X is

$$[X]_t = \int_0^t \sigma_s^2 \mathrm{d}s + 2\sum_{j=1}^{N_t} v_j^2,$$

so the jumps due to $u_{j,1}$ become asymptotically negligible. An important feature of this result is that this QV is finite, with probability one. Thus, this differs from the result which would have been obtained if $u_{j,1} = O_p(1)$ and $u_{2,j} = 0$, which was the case studied by Bandi and Russell (2003a), Bandi and Russell (2003b) and Zhang, Mykland, and Aït-Sahalia (2003).

If the 2nd lag realised bipower variation measure of the error free returns has

$$\frac{1}{\mu_1^2}\sum_{j=3}^{n} |y_{j-2}|\,|y_j| \xrightarrow{p} \int_0^t \sigma_s^2 \mathrm{d}s, \quad \text{then} \quad \frac{1}{\mu_1^2}\sum_{j=3}^{n} |x_{j-2}|\,|x_j| \xrightarrow{p} \int_0^t \sigma_s^2 \mathrm{d}s,$$

due to the finite activity of the counting process and the fact that the non-jump induced errors are of a smaller order of magnitude than the returns. It is necessary to use a 2nd lag bipower variation measure as the misrecording of prices induces a first order moving average structure in the returns, which means there is a non-negligible probability of contiguous jumps in the counting process even asymptotically.

An interesting feature of the analysis is that the intensity of the counting process can be allowed to be dynamic without effecting the line of argument made here. The sole argument which matters is that the arrival process is of finite activity. Serial dependence between the errors is, in principle, irrelevant.

The above analysis is only instructive. It does not mean that market microstructure effects have no impact on BPV. Rather, we take this as meaning that it should be more robust to market microstructure effects than realised variance.

In practice when we compute skipped versions of realised BPV, then it makes sense to use the finite sample correction

$$\widetilde{v}(X)_{i,k}^{[1,1]} = \frac{n}{n-k} \sum_{j=k+1}^{n} |x_{j-k}| \, |x_j| \, ,$$

for this delivers an unbiased estimator of the variance in the Brownian motion case.

Table 7 shows the average value of the daily realised quantities for a variety of values of n for the entire United States Dollar/German Deutsche-Mark Olsen series from 1986 to 1996 we previous studied in Section 6. We can see that the average values of the realised BPVs falls as n reduces, however the results are rather stable across different realised BPVs so long as the lag is larger than one. The Table shows a sharp fall as we move from the average value of the realised variance to the average value of $\mu_1^{-2}\widetilde{v}(X)_{i,k}^{[1,1]}$, but another sharp decrease for long lagged realised BPVs. This suggests the presence of both jumps and market microstructure effects.

	$(X_i - X_{i-1})^2$	$\widehat{v}_i$	$\mu_1^{-2}\widetilde{v}(X)_{i,1}^{[1,1]}$	$\mu_1^{-2}\widetilde{v}(X)_{i,2}^{[1,1]}$	$\mu_1^{-2}\widetilde{v}(X)_{i,3}^{[1,1]}$
$n = 288$	0.50679	0.53085	0.49964	0.46804	0.45442
$n = 144$	0.50679	0.51148	0.47496	0.44401	0.43474
$n = 72$	0.50679	0.49064	0.44526	0.41814	0.41298
$n = 24$	0.50679	0.46469	0.40653	0.38361	0.36369

Table 7: *Average value of the whole sample of the statistics per day. Returns scaled by 100.*

We can also study the autocorrelation between days of these volatility measures. Figure 9 shows the results for a variety of values of n. The results are clear. All realised BPV measures have more serial dependence than the RV, but the measures using lags longer than one period have even higher serial correlations. The increase in dependence seems to be of the same order of magnitude we have seen going from RV to one lagged realised BPV.

8 Conclusions

In this paper we have provided detailed results on the asymptotic distribution of tests for the presence of jumps. Monte Carlo results suggest an adjusted ratio jump statistic can be reliably used to test for jumps even if the sample size is small. We applied this test to some exchange rate data and found many rejections of the null of no jumps. In some case studies we related the

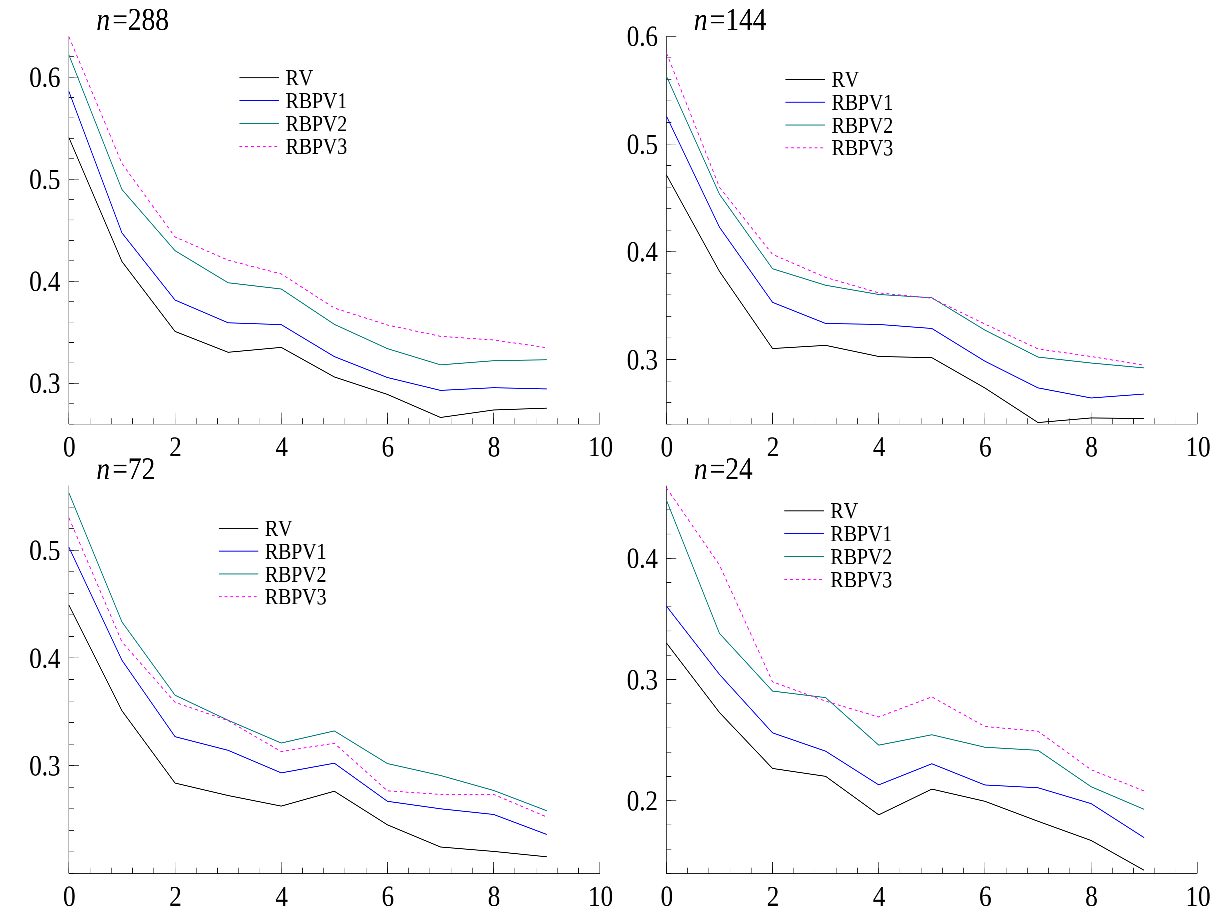

Figure 9: *Correlograms of the RV and realised BPVs at various lags. Code:* `jump_RV_daily.ox`.

rejections to economic news. We see that the large jumps in our dataset are mainly caused by macroeconomic news announcements. Our results contrast with previous results, such as that reported by Eraker, Johannes, and Polson (2003), which tend to find a small number of jumps associated with large daily moves in the asset prices.

The data we have used comes from one of the most thickly traded financial markets. The overwhelming evidence of jumps in this market suggests that the commonly used assumption that prices have continuous sample paths seems at odds with the empirical evidence.

9 Acknowledgments

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made in this paper are based on software written by the authors using the Ox language of Doornik (2001). We have benefited from correspondence with Tim Bollerslev and Xin Huang on this topic, while the comments of Clive Bowsher, Jean Jacod and Bent Nielsen on an earlier draft were also very helpful.

A Proof of theorem 1

A.1 Assumptions and statement of two theorems

Recall the three assumptions we use in Theorem 1. We will carry out the limit result for a fixed value of t, allowing $t = \delta n$. So as $\delta \downarrow 0$ then necessarily $n \to \infty$.

(a) The volatility quantity

$$\psi_j = \sqrt{\delta^{-1} \int_{\delta(j-1)}^{\delta j} \sigma_s^2 \mathrm{d}s}$$

is bounded away from 0 and infinity, uniformly in j and δ.

(b) The mean process A satisfies, (pathwise) as $\delta \downarrow 0$,

$$\delta^{-1} \max_{1 \leq j \leq \lfloor t/\delta \rfloor} \left| A_{j\delta} - A_{(j-1)\delta} \right| = O(1). \tag{20}$$

(c) The joint process (A, σ) is independent of the Brownian motion W.

In this Appendix we prove two results we state in this subsection: (i) Theorem 2 which shows consistency of realised BPV when $A \neq 0$, (ii) Theorem 3 which gives a joint central limit theory for realised BPV and QV under $\mathcal{SVSM}^c$. These two results then delivers Theorem 1 immediately.

Theorem 2 *Let $Y \in \mathcal{SVFASM}$ and suppose conditions **(a)**, **(b)** and **(c)** hold, then*

$$\{Y\}_t^{[1,1]} = \mu_1^2 \int_0^t \sigma_s^2 \mathrm{d}s. \tag{21}$$

Theorem 3 *Let $Y \in \mathcal{SVSM}^c$ and suppose conditions **(a)**, **(b)** and **(c)** hold. Then conditionally on (A, σ), the realised BPV and QV processes*

$$[Y_\delta]_t \quad \textit{and} \quad \mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \tag{22}$$

follow asymptotically, as $\delta \downarrow 0$, a bivariate normal law with common mean $\int_0^t \sigma_s^2 \mathrm{d}s$. The asymptotic covariance of

$$\delta^{-1/2} \left\{ \begin{pmatrix} [Y_\delta]_t \\ \mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \end{pmatrix} - \begin{pmatrix} \int_0^t \sigma_s^2 \mathrm{d}s \\ \int_0^t \sigma_s^2 \mathrm{d}s \end{pmatrix} \right\}$$

is

$$\Pi \int_0^t \sigma_s^4 \mathrm{d}s \tag{23}$$

where

$$\begin{aligned} \Pi &= \begin{pmatrix} \mathrm{Var}(u^2) & 2\mu_1^{-2}\mathrm{Cov}\left(u^2, |u||u'|\right) \\ 2\mu_1^{-2}\mathrm{Cov}\left(u^2, |u||u'|\right) & \mu_1^{-4}\left\{\mathrm{Var}\left(|u||u'|\right) + 2\mathrm{Cov}\left(|u||u'|, |u'|\,|u''|\right)\right\} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ 2 & \left(\pi^2/4\right) + \pi - 3 \end{pmatrix} \simeq \begin{pmatrix} 2 & 2 \\ 2 & 2.6090 \end{pmatrix} \end{aligned}$$

with u, u', u'' being independent standard normals.

□

A.2 Consistency of realised bipower variation: Theorem 2

This proof goes in four stages. We first set the jump process to zero. We provide some preliminary results on discretisation of integrated variance. Then we prove consistency of bipower variation when $A = 0$. We then prove Theorem 2 in the case of no jumps by showing that allowing $A \neq 0$ has negligible impact. This is by far the hardest part of our proof as we prove the effect of the drift is $o_p(\delta^{-1/2})$ which will be needed for Theorem 3. Finally, we note here that the theorem extends to the jump case trivially using the argument given in Barndorff-Nielsen and Shephard (2004d). Hence the focus of this subsection is entirely on detailing the first three steps of this argument.

We first recall a result.

Proposition 1 ***(Barndorff-Nielsen and Shephard (2004d)).*** *Under* **(a)** *we have for $r > 0$ and*

$$\sigma_j^2 = \int_{(j-1)\delta}^{j\delta} \sigma_s^2 \mathrm{d}s$$

that

$$\delta^{1-r}\left\{\sum_{j=2}^n \sigma_{j-1}^r \sigma_j^r - \sum_{j=1}^n \sigma_j^{2r}\right\} = O_p(\delta).$$

□

This result is obtained in the course of the proof of Theorem 2 of Barndorff-Nielsen and Shephard (2004d), cf. equation (13) of that paper.

Corollary 1 *Under* ***(a)*** *we have that*

$$\sum_{j=2}^n \sigma_{j-1}\sigma_j - \int_0^t \sigma_s^2 \mathrm{d}s = O_p(\delta).$$

□

This Corollary is a considerable strengthening (in the special case of $r = 1$) of the result in Barndorff-Nielsen and Shephard (2004d) that for $r > 0$ then $\delta^{1-r} \sum_{j=2}^{n} \sigma_{j-1}^{r} \sigma_{j}^{r}$ minus $\int_0^t \sigma_s^{2r} \mathrm{d}s$ is $o_p(1)$. This strengthening is vital later as it will allow us to derive a central limit theorem without imposing strong conditions on the volatility process (e.g. finite variation). It does not hold for more general values of r.

Proof. Trivially from Proposition 1 in the special case where $r = 1$, using the fact that

$$\sum_{j=1}^{n} \sigma_j^2 = \int_0^t \sigma_s^2 \mathrm{d}s.$$

□

This allows us to prove the existence of bipower variation in the case $r = s = 1$ under weaker assumptions than used by Barndorff-Nielsen and Shephard (2004d).

Theorem 4 *Suppose $Y \in \mathcal{SVSM}^c$ and additionally* (**a**)*,* (**c**) *and $A = 0$, then*

$$\left(\sum_{j=2}^{\lfloor t/\delta \rfloor} |y_{j-1}||y_j| \right) - \mu_1^2 \int_0^t \sigma_s^2 \mathrm{d}s = o_p(1).$$

□

Proof. Using (**c**) and the fact that $A = 0$ we have that

$$\sum_{j=2}^{n} |y_{j-1}|\,|y_j| \overset{L}{=} \sum_{j=2}^{n} \sigma_{j-1}\sigma_j |u_{j-1}||u_j|,$$

where the u_j are i.i.d. standard normal. This means that

$$\sum_{j=2}^{n} |y_{j-1}|\,|y_j| - \mu_1^2 \sum_{j=2}^{n} \sigma_{j-1}\sigma_j \overset{L}{=} \sum_{j=2}^{n} \sigma_{j-1}\sigma_j (|u_{j-1}||u_j| - \mu_1^2).$$

In our proof it suffices to show two things:

1. As $\delta \downarrow 0$
$$R = \sum_{j=2}^{n} \sigma_{j-1}\sigma_j (|u_{j-1}||u_j| - \mu_1^2) = o_p(1). \qquad (24)$$

2. As $\delta \downarrow 0$
$$\sum_{j=2}^{n} \sigma_{j-1}\sigma_j - \int_0^t \sigma_s^2 \mathrm{d}s = o_p(1). \qquad (25)$$

But Corollary 1 is a stronger result than (25). Hence we are left to prove part 1. To establish (24), it is sufficient to show that

$$S = \sum_{j=2}^{n} c_{nj} x_j \xrightarrow{p} 0, \quad \text{where} \quad c_{nj} = \sigma_{j-1}\sigma_j \quad \text{and} \quad x_j = (|u_j||u_j'| - \mu_1^2),$$

where u_j and u'_j are independent standard normal sequences[4]. Barndorff-Nielsen and Shephard (2003a, Corollary 4.3) applies to such a sum. Specifically to see that $S \xrightarrow{p} 0$ let $\sigma_j = \delta^{1/2}\psi_j$ then we write $c_{nj} = \delta\psi_{j-1}\psi_j$. The important issue is that assumption **(a)** bounds $\psi_{j-1}\psi_j$. Then, as is simple to verify, the following three conditions are satisfied as $\delta \downarrow 0$ and with x distributed as $|u|\,|u'| - \mu_1^2$

$$c_n = \max_{1\le j<n} c_{nj} = \delta \max_{1\le j<n} \psi_{j-1}\psi_j \to 0,$$

$$nP\{|x| \ge c_n^{-1}\varepsilon\} = nP\left\{|x| \ge \frac{1}{\delta \max_j \psi_{j-1}\psi_j}\varepsilon\right\} \to 0,$$

$$\sup_n \sum_{j=2}^{n} c_{nj} = \sup_n \delta \sum_{j=2}^{n} \psi_{j-1}\psi_j < \infty,$$

and thus the conditions of Corollary 4.3 in Barndorff-Nielsen and Shephard (2003a) are satisfied. □

To complete the Proof of Theorem 3 we need to show that the impact of the drift is negligible. To do this we recall that in the $\mathcal{SVSM}^c$ model

$$M_t = \int_0^t \sigma_s \mathrm{d}W_s,$$

which implies

$$Y = A + M.$$

The remaining task is to show that, to the order concerned, A does not affect the asymptotic limit behaviour, provided conditions **(a)**, **(b)** and **(c)** hold. For this it suffices to show that

$$\delta^{-1/2}\left\{[Y_\delta]_t^{[1,1]} - [M_\delta]_t^{[1,1]}\right\} = o_p(1).$$

We shall in fact prove the following stronger result, which covers a variety of versions of realised bipower variation.

---

[4]This is a consequence of the following argument. Define

$$R = \sum_{j=1}^{n} v_{n,j}, \quad \text{where} \quad v_{n,j} = \sigma_{j-1}\sigma_j(|u_{j-1}||u_j| - \mu_1^2),$$

and construct $R' = \sum_{j=1}^{n} z_{n,j}$, where R' is an independent copy of R. We observe that $v_{n,j}$ is independent from $v_{n,j-s}$ for all $|s| > 1$. Then we can rewrite

$$\begin{aligned} R + R' &= (v_{n,1} + v_{n,2} + ... + v_{n,n}) + (z_{n,1} + z_{n,2} + ... + z_{n,n}) \\ &= (v_{n,1} + z_{n,2} + v_{n,3} + ...) + (z_{n,1} + v_{n,2} + z_{n,3} + ...) \\ &= S + S', \end{aligned}$$

so that S and S' are each sums of independent terms. S and S' are not independent but are identically distributed. However, if as $n \to \infty$ so $S \xrightarrow{p} 0$, then, consquently, $R + R' \xrightarrow{p} 0$. As R is independent from R' this means that $R \xrightarrow{p} 0$.

Proposition 2 *Under conditions* **(a)**, **(b)** *and* **(c)** *for* $r, s > 0$

$$\delta^{-(r+s)/2}\left\{[Y_\delta]^{[r,s]} - [M_\delta]^{[r,s]}\right\} = O_p(\delta^{-1/2+\varepsilon})$$

for every $\varepsilon \in \left(0, \frac{1}{4}\right)$. $\square$

Proof. Let

$$\underline{\sigma^2} = \inf_{0\le u\le t} \sigma_u^2 \quad \text{and} \quad \overline{\sigma^2} = \sup_{0\le u\le t} \sigma_u^2,$$

$$m_j = M_{j\delta} - M_{(j-1)\delta},$$

and

$$\sigma_j^2 = \int_{\delta(j-1)}^{\delta j} \sigma_u^2 \mathrm{d}u,$$

$$\gamma_j = \delta^{-1} a_j, \quad a_j = A_{j\delta} - A_{(j-1)\delta}, \quad j = 1, 2, ..., n-1.$$

and note that (pathwise for (A, σ^2)), by assumption **(a)**,

$$0 < \underline{\sigma^2} \le \overline{\sigma^2} < \infty,$$

implying if

$$\theta_j \delta = \sigma_j^2$$

that

$$0 < \min_j \theta_j \le \max_j \theta_j < \infty,$$

while, due to assumption **(b)**, there exists (pathwise) a constant c for which

$$\max_j |\gamma_j| \le c\delta,$$

whatever the value of n.

We have using **(c)** and writing now $m_j \overset{L}{=} \sigma_j |u_j|$, then

$$\begin{aligned}
[Y_\delta]_t^{[r,s]} - [M_\delta]_t^{[r,s]} &= \sum_{j=2}^{n} (|a_{j-1} + m_{j-1}|^s |a_j + m_j|^r - |m_{j-1}|^s |m_j|^r) \\
&= \sum_{j=2}^{n} \{|\delta\gamma_{j-1} + \delta^{1/2}\theta_{j-1}^{1/2} u_{j-1}|^s |\delta\gamma_j + \delta^{1/2}\theta_j^{1/2} u_j|^r \\
&\quad - |\delta^{1/2}\theta_{j-1}^{1/2} u_{j-1}|^s |\delta^{1/2}\theta_j^{1/2} u_j|^r\} \\
&= \delta^{r/2}\delta^{s/2} \sum_{j=2}^{n} \theta_{j-1}^{s/2}\theta_j^{r/2} \{|(\gamma_{j-1}/\theta_{j-1}^{1/2})\delta^{1/2} + u_{j-1}|^s \\
&\quad \cdot |(\gamma_j/\theta_j^{1/2})\delta^{1/2} + u_j|^r - |u_{j-1}|^s |u_j|^r\}
\end{aligned}$$

and hence

$$\delta^{-(r+s)/2}\left\{[Y_\delta]_t^{[r,s]}-[M_\delta]_t^{[r,s]}\right\}=\sum_{j=2}^{n}\theta_{j-1}^{s/2}\theta_j^{r/2}h_{r,s}\left(u_{j-1},u_j;\gamma_{j-1}/\theta_{j-1}^{1/2},\gamma_j/\theta_j^{1/2}\right)$$

where

$$h_{r,s}(u,v;\rho_1,\rho_2)=|\rho_1\delta^{1/2}+u|^r|\rho_2\delta^{1/2}+v|^s-|u|^r|v|^s.$$

As $\left|\gamma_j/\theta_j^{1/2}\right|$ is bounded for all j, the conclusion of Proposition 2 now follows from Corollary 2, which is given below.

□

To obtain that Corollary we establish three Lemmas, 1, 2 and 3. Lemma 1 collates several results from Barndorff-Nielsen and Shephard (2003b) which are used to prove Lemmas 2 and 3.

Let u be a standard normal random variable and define

$$h_r(u;\rho)=|\rho\delta^{1/2}+u|^r-|u|^r.$$

Lemma 1 *(Barndorff-Nielsen and Shephard (2003b)) For any $r>0$ and $\rho\geq 0$, we have*

$$\mathrm{E}\{h_r(u;\rho)\}=O(\delta),$$

$$\mathrm{E}\{|u|^r h_r(u;\rho)\}=O(\delta^{(1+1\wedge r)/2}).$$

$$\mathrm{E}\{h_r^2(u;\rho)\}=O(\delta^{(1+1\wedge r)/2}).$$

and

$$\mathrm{Var}\{h_r(u;\rho\}=O(\delta^{(1+1\wedge r)/2}).$$

□

The results given in Lemma 1 are derived in the course of the proof of Proposition 3.3 in Barndorff-Nielsen and Shephard (2003b), so a separate proof will not be given here.

We proceed to state and prove Lemmas 2 and 3. Let u and v be independent $N(0,1)$ variables.

Lemma 2 *For any $r,s>0$ and ρ_1 and ρ_2 nonnegative constants, we have*

$$\mathrm{E}\{h_{r,s}(u,v;\rho_1,\rho_2)\}=O(\delta).$$

Proof. The independence of u, v together with the first equation in Lemma 1 implies

$$\begin{aligned}
\mathrm{E}\left\{h_{r,s}(u,v;\rho)\right\} &= \mathrm{E}\left\{|\rho_1\delta^{1/2}+u|^r\right\}\mathrm{E}\left\{|\rho_2\delta^{1/2}+v|^s\right\} - \mathrm{E}\left\{|u|^r\right\}\mathrm{E}\left\{|v|^s\right\} \\
&= \mathrm{E}\{h_r(u;\rho_1)\}\mathrm{E}\{h_r(v;\rho_2)\} + \mathrm{E}\{h_r(u;\rho_1)\}\mathrm{E}\left\{|v|^s\right\} + \mathrm{E}\left\{|u|^s\right\}\mathrm{E}\{h_s(v;\rho_2)\} \\
&= O(\delta).
\end{aligned}$$

□

Lemma 3 *For u, v independent standard normal random variables and ρ_1 and ρ_2 nonnegative constants, we have*

$$\mathrm{E}\{h^2_{r,s}(u,v;\rho_1,\rho_2)\} = O\left(\delta^{(1+1\wedge r\wedge s)/2}\right).$$

□

Proof. Clearly

$$\begin{aligned}
h^2_{r,s}(u,v;\rho) &= |\rho_1\delta^{1/2}+u|^{2r}|\rho_2\delta^{1/2}+v|^{2s} + |u|^{2r}|v|^{2s} - 2|\rho_1\delta^{1/2}+u|^r|\rho_2\delta^{1/2}+v|^s|u|^r|v|^s \\
&= h_{2r,2s}(u,v;\rho) + 2|u|^{2r}|v|^{2s} - 2|\rho_1\delta^{1/2}+u|^r|\rho_2\delta^{1/2}+v|^s|u|^r|v|^s,
\end{aligned}$$

so, by Lemma 2 and the independence of u and v,

$$\begin{aligned}
\mathrm{E}\{h^2_{r,s}(u,v;\rho)\} &= \mathrm{E}\{h_{2r,2s}(u,v;\rho)\} + 2\mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\left\{|u|^{2s}\right\} \\
&\quad -2\mathrm{E}\left\{|u|^r|\rho_1\delta^{1/2}+u|^r\right\}\mathrm{E}\left\{|u|^s|\rho_2\delta^{1/2}+u|^s\right\} \\
&= O(\delta) - 2\left(\mathrm{E}\left\{|u|^r|\rho_1\delta^{1/2}+u|^r\right\}\mathrm{E}\left\{|u|^s|\rho_2\delta^{1/2}+u|^s\right\} - \mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\left\{|u|^{2s}\right\}\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathrm{E}\left\{|u|^r|\rho_1\delta^{1/2}+u|^r\right\}\mathrm{E}\left\{|u|^s|\rho_2\delta^{1/2}+u|^s\right\} - \mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\left\{|u|^{2s}\right\} \\
=\ &\mathrm{E}\left\{|u|^r|\rho_1\delta^{1/2}+u|^r - |u|^{2r}\right\}\mathrm{E}\left\{|u|^s|\rho_2\delta^{1/2}+u|^s\right\} \\
&+\mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\left\{|u|^s|\rho_2\delta^{1/2}+u|^s\right\} - \mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\left\{|u|^{2s}\right\} \\
=\ &\mathrm{E}\left\{|u|^r|\rho_1\delta^{1/2}+u|^r - |u|^{2r}\right\}\mathrm{E}\left\{|u|^s|\rho_2\delta^{1/2}+u|^s - |u|^{2s}\right\} \\
&+\mathrm{E}\left\{|u|^{2s}\right\}\mathrm{E}\left\{|u|^r|\rho_1\delta^{1/2}+u|^r - |u|^{2r}\right\} \\
&+\mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\left\{|u|^s|\rho_2\delta^{1/2}+u|^s\right\} - \mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\left\{|u|^{2s}\right\} \\
=\ &\mathrm{E}\{|u|^r h_r(u;\rho_1)\}\mathrm{E}\{|u|^s h_s(u;\rho_2)\} \\
&+\mathrm{E}\left\{|u|^{2s}\right\}\mathrm{E}\{|u|^r h_r(u;\rho_1)\} + \mathrm{E}\left\{|u|^{2r}\right\}\mathrm{E}\{|u|^s h_s(u;\rho_2)\}.
\end{aligned}$$

All in all, on account of Lemma 1, this means that

$$\begin{aligned}\mathrm{E}\{h_{r,s}^2(u,v;\rho_1,\rho_2)\} &= O(\delta)+O(\delta^{(1+1\wedge r)/2})O(\delta^{(1+1\wedge s)/2})+O(\delta^{(1+1\wedge r)/2})+O(\delta^{(1+1\wedge s)/2}) \\ &= O(\delta^{(1+1\wedge r\wedge s)/2}).\end{aligned}$$

□

Lemmas 2 and 3 and the Cauchy-Schwarz inequality together imply

Corollary 2 *For u, v, u', v' independent standard normal random variables and ρ_1, ρ_2, ρ_1', ρ_2' nonnegative constants, we have*

$$\mathrm{Var}\{h_{r,s}(u,v;\rho_1,\rho_2))\} = O\left(\delta^{(1+1\wedge r\wedge s)/2}\right)$$

and

$$\mathrm{Cov}\{h_{r,s}(u,v;\rho_1,\rho_2))h_{r,s}(u',v';\rho_1',\rho_2'))\} = O\left(\delta^{(1+1\wedge r\wedge s)/2}\right).$$

□

As already mentioned, the conclusion of Proposition 2 follows from Corollary 2.

Remark 2 *From the final equation in the proof of Lemma 3 one sees that in the special case when $r=s=1$ then*

$$\mathrm{Var}\{h_r(u;\rho)\} = O(\delta)$$

and hence the conclusion of Proposition 2 may be sharpened to

$$[Y_\delta]^{[1,1]} - [M_\delta]^{[1,1]} = O_p(\delta).$$

□

The result in Theorem 2 now follows from the combination of Theorem 4 and Proposition 2.

A.3 Asymptotic distribution of bipower variation: Theorem 3

Given Proposition 2, what remains is to prove Theorem 3 when $Y \in \mathcal{SVSM}^c$ and the additional conditions (**a**), (**c**) and $A=0$ hold. The key feature is that, ignoring the asymptotically negligible y_1^2 and conditioning on the σ process, we can write the process as

$$\begin{pmatrix} \sum_{j=2}^n y_j^2 \\ \sum_{j=2}^n |y_{j-1}|\,|y_j| \end{pmatrix} - \begin{pmatrix} \int_0^t \sigma_s^2 \mathrm{d}s \\ \mu_1^2 \int_0^t \sigma_s^2 \mathrm{d}s \end{pmatrix} \overset{L}{=} \sum_{j=2}^n \begin{pmatrix} \sigma_j^2 v_j \\ \sigma_{j-1}\sigma_j w_j \end{pmatrix}$$

where

$$v_j = u_j^2 - 1 \quad \text{and} \quad w_j = |u_{j-1}||u_j| - \mu_1^2.$$

The sequences $\{v_j\}$ and $\{w_j\}$ have zero means, with the former being i.i.d., while the latter satisfy $w_j \perp\!\!\!\perp w_{j+s}$ for $|s| > 1$. Then the Theorem follows if we can show that

$$\delta^{-1/2} \sum_{j=2}^{n} \begin{pmatrix} \sigma_j^2 v_j \\ \sigma_{j-1}\sigma_j w_j \end{pmatrix} \xrightarrow{L} N\left(0, \int_0^t \sigma_s^4 \mathrm{d}s \begin{pmatrix} \mathrm{Var}(v_1) & 2\mathrm{Cov}\,(v_1, w_1) \\ 2\mathrm{Cov}\,(v_1, w_1) & \mathrm{Var}\,(w_1) + 2\mathrm{Cov}\,(w_1, w_2) \end{pmatrix}\right). \tag{26}$$

Our strategy for proving this is to show[5] the limiting Gaussian result that using any real constants c_1 and c_2,

$$\begin{aligned} &\delta^{-1/2} \sum_{j=2}^{n} \left(c_1 \sigma_j^2 v_j + c_2 \sigma_{j-1}\sigma_j w_j\right) \\ &\xrightarrow{L} N\left(0, \int_0^t \sigma_s^4 \mathrm{d}s \left[c_1^2 \mathrm{Var}(v_1) + 4c_1c_2 \mathrm{Cov}\,(v_1, w_1) + c_2^2 \left\{\mathrm{Var}\,(w_1) + 2\mathrm{Cov}\,(w_1, w_2)\right\}\right]\right). \end{aligned}$$

The asymptotic Gaussianity follows from standard calculations from the classical central limit theorem for martingale sequences due to Lipster and Shiryaev (see, for example, Shiryayev (1981, p. 216)).

What remains is to derive the asymptotic variance of this sum. Clearly

$$\delta^{-1/2} \sum_{j=2}^{n} \left(c_1 \sigma_j^2 v_j + c_2 \sigma_{j-1}\sigma_j w_j\right) = \delta^{-1/2} \left\{\delta \sum_{j=2}^{n} \left(c_1 \psi_j^2 v_j + c_2 \psi_{j-1}\psi_j w_j\right)\right\}$$

has the variance

$$\delta \sum_{j=2}^{n} \mathrm{Var}\left(c_1 \psi_j^2 v_j + c_2 \psi_{j-1}\psi_j w_j\right) + 2\delta \sum_{j=2}^{n} \mathrm{Cov}\left(c_2 \psi_{j-1}\psi_j w_j, c_2 \psi_{j-2}\psi_{j-1} w_{j-1}\right).$$

Now using Riemann integrability

$$\begin{aligned} &\delta \sum_{j=2}^{n} \mathrm{Var}\left(c_1 \psi_j^2 v_j + c_2 \psi_{j-1}\psi_j w_j\right) \\ = \;& \mathrm{Var}\,(v_1)\, c_1^2 \delta \sum_{j=2}^{n} \psi_j^4 + \mathrm{Var}\,(w_1)\, c_2^2 \delta \sum_{j=2}^{n} \psi_{j-1}^2 \psi_j^2 \\ & + 2c_1c_2 \left\{\mathrm{Cov}(v_1, w_1) + \mathrm{Cov}(v_2, w_1)\right\} \delta \sum_{j=2}^{n} \psi_{j-1}\psi_j^3 \\ \to \;& \int_0^t \sigma_s^4 \mathrm{d}s \left\{c_1^2 \mathrm{Var}\,(v_1) + c_2^2 \mathrm{Var}\,(w_1) + 4c_1c_2 \mathrm{Cov}(v_1, w_1)\right\}. \end{aligned}$$

[5]Recall that if $z_n = (z_{n1}, ..., z_{nq})$ is a sequence of random vectors having mean 0 then to prove that $z_n \xrightarrow{L} N_q(0, \Psi)$ for some nonnegative definite matrix Ψ it suffices to show that for arbitrary real constants $c_1, ..., c_q$ we have $c' z_n \xrightarrow{L} N_q(0, c'\Psi c)$, where $c = (c_1, ..., c_q)'$. (This follows directly from the characterisation of convergence in law in terms of convergence of the characteristic functions.)

Likewise

$$
\begin{aligned}
& \delta \sum_{j=3}^{n} \operatorname{Cov}\left(c_2 \psi_{j-1} \psi_j w_j, c_2 \psi_{j-1} \psi_{j-2} w_{j-1}\right) \\
= \quad & c_2^2 \operatorname{Cov}\left(w_1, w_2\right) \delta \sum_{j=3}^{n} \psi_{j-2} \psi_{j-1}^2 \psi_j \\
\rightarrow \quad & c_2^2 \operatorname{Cov}\left(w_1, w_2\right) \int_0^t \sigma_s^4 \mathrm{d}s.
\end{aligned}
$$

This confirms the required covariance pattern stated in (26).

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