On the Power of the Conditional Likelihood Ratio and Related Tests for Weak-Instrument Robust Inference

Nicolas Van de Sijpe* Frank Windmeijer†

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Abstract

Power curves of the Conditional Likelihood Ratio (CLR) and related tests for testing $\mathrm{H}_0{:}\,\beta = \beta_0$ in linear models with a single endogenous variable, $y = x\beta + u$, estimated using potentially weak instrumental variables have been presented for two different designs. One design keeps the variance matrix of the structural and first-stage errors, Σ, constant, the other instead keeps the variance matrix of the reduced-form and first-stage errors, Ω, constant. The values of Σ govern the endogeneity features of the model. The fixed-Ω design changes these endogeneity features with changing values of β in a way that makes it less suitable for an analysis of the behaviour of the tests in low to moderate endogeneity settings, or when β and the correlation of the structural and first-stage errors, ρ_{uv}, have the same sign. At larger values of $|\beta|$, the fixed-Ω design implicitly selects values for Σ where the power of the CLR test is high. We show that the Likelihood Ratio statistic is identical to the $t_0(\widehat{\beta}_L)^2$ statistic as proposed by Mills, Moreira, and Vilela (2014), where $\widehat{\beta}_L$ is the LIML estimator. In fixed-Σ design Monte Carlo simulations, we find that LIML- and Fuller-based conditional Wald tests and the Fuller-based conditional t_0^2 test are more powerful than the CLR test when the degree of endogeneity is low to moderate. The conditional Wald tests are further the most powerful of these tests when β and ρ_{uv} have the same sign. We show that in the fixed-Ω design, setting $\beta_0 = 0$ and the diagonal elements of Ω equal to 1 is not without loss of generality, unlike in the fixed-Σ design.

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*Dept. of Economics, University of Sheffield, UK, n.vandesijpe@sheffield.ac.uk.

†Dept. of Statistics and Nuffield College, University of Oxford, UK, frank.windmeijer@stats.ox.ac.uk, corresponding author.

1 Introduction

For the linear model with one endogenous explanatory variable,

$$y_i = x_i\beta + u_i, \tag{1}$$

for $i = 1, \ldots, n$, estimated using instrumental variables, the Conditional Likelihood Ratio (CLR) test of Moreira (2003) and related tests, like the AR (Anderson and Rubin, 1949), LM (Kleibergen, 2002, Moreira, 2002), and conditional Wald (CW) tests are tests for the hypothesis H_0: $\beta = \beta_0$. They are robust to weak instruments in the sense that they have correct size when instruments are weak, with CLR, AR, and LM unbiased, similar tests, whereas the CW tests are not unbiased.

For the evaluation of the power of these tests, two different designs have been used in the literature. Let z_i be the k_z vector of instruments and let the first-stage model be given by

$$x_i = z_i'\pi + v_i. \tag{2}$$

Then the fixed-Σ design specification is given by (1) and (2), with the variance matrix of the structural and first-stage errors, Σ, fixed in the sense that it is not a function of β,

$$\Sigma = Var\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{bmatrix}.$$

The reduced form for y_i is given by $y_i = z_i'\pi\beta + u_i + \beta v_i = z_i'\pi_y + r_i(\beta)$, with $r_i(\beta) = u_i + \beta v_i$ and so the variance of the reduced-form errors is a function of β in this fixed-Σ design. The values of Σ govern the endogeneity features of the model.

The fixed-Ω design specifies a constant reduced-form variance matrix Ω. This design has the same first-stage specification (2), but specifies the linear model for y_i as

$$y_i = x_i\beta + r_i - \beta v_i, \tag{3}$$

as then the reduced form for y_i is given by $y_i = z_i'\pi\beta + r_i$ and the reduced-form error variance is fixed

$$\Omega = Var\begin{pmatrix} r_i \\ v_i \end{pmatrix} = \begin{bmatrix} \sigma_r^2 & \sigma_{rv} \\ \sigma_{rv} & \sigma_v^2 \end{bmatrix}.$$

In this case, the structural error is $u_i(\beta) = r_i - \beta v_i$ and hence the variance of the structural errors is a function of β in this fixed-Ω design, and so here the values of $\Sigma(\beta)$ govern the endogeneity features of the model.

The simulations in Kleibergen (2002), Moreira (2003, 2009), and Stock, Wright, and Yogo (2002), amongst others, are based on the fixed-Σ design, whereas Andrews, Moreira, and Stock (2006, 2007), Hillier (2009), Mills, Moreira, and Vilela (2014) and Moreira and Moreira (2019) are based on the fixed-Ω design. These fixed-Ω design examples all start with specifying the model of interest as the structural and first-stage equations as in (1) and (2), but then subsequently fix Ω, without explicitly specifying the model of interest as in (3). An argument used for fixing Ω is that it can be consistently estimated and hence treated as known, see e.g. the discussion in Andrews, Marmer, and Yu (2019).

Poskitt and Skeels (2008) discuss these two designs and show that simulation results can differ substantially between them, but do not provide an explanation for these differences. Davidson and MacKinnon (2008) highlight that a design with Ω fixed changes Σ when changing the value of β and conclude that Ω is "not a sensible quantity to keep fixed" (Davidson and MacKinnon 2008, p 455). Andrews, Marmer, and Yu (2019) propose a design where the value of β is fixed, but the value of β_0 in H_0: $\beta = \beta_0$ is varied instead, arguing that this keeps Ω fixed. Their motivation for this design is its direct link to the formation of confidence intervals based on inverting test statistics. As we show in Section 3.1, this design where β is kept fixed but β_0 is varied is essentially the same as the fixed-Σ design. This follows as, ceteris paribus, for $\delta \in \mathbb{R}$, the test statistics for H_0: $\beta = \beta_* - \delta$ when $\beta = \beta_*$ in the Andrews et al. (2019) design are identical to the test statistics for H_0: $\beta = \beta_*$ when $\beta = \beta_* + \delta$ in the fixed-Σ design.

The main contribution of our paper is that we examine in detail the relationship between power analyses conducted using the fixed-Σ and fixed-Ω designs. The standard fixed-Σ design power curve varies the value of β but keeps the structural endogeneity features constant, in particular the degree of endogeneity $\rho_{uv} = \frac{\sigma_{uv}}{\sigma_u \sigma_v}$. This is not the case for the fixed-Ω design, where σ_v^2 is kept fixed, but $\sigma_u^2(\beta)$ and $\rho_{uv}(\beta)$ change with the value of β in the DGP in a very specific way, as shown in Figure 2 in Section 4. In particular, for the usual setting of $\sigma_r^2 = \sigma_v^2 = 1$, the structural correlations $\rho_{uv}(\beta)$ are predominantly negative for positive values of β and vice versa. Further, $\rho_{uv}(\beta)$ approaches 1 for large negative values of β, and -1 for large positive values of β, with accompanied levels of the variance $\sigma_u^2(\beta)$ such that the power of the AR and CLR tests approaches 1, even with very weak instruments.

Therefore, significant parts of the fixed-Ω power curve implicitly consider variance and endogeneity structures that favour the AR and CLR tests. As we further discuss in Section 4 and also highlight in Figure 5 in Section 5, only a very small part of a fixed-Ω power curve relates to settings of low to moderate endogeneity, or settings where β and ρ_{uv} have the same sign. The fixed-Σ design therefore is better suited to evaluate the power of the tests for these cases. Low endogeneity settings seem important, as e.g. Chernozhukov and Hansen (2008) report a median estimated value of $\rho_{uv} = 0.3$ for a survey of applied instrumental variables papers.

Our paper makes a number of further contributions. We first document in Section 2, which introduces the model and test statistics, a standalone finding that the AR test statistic is identical to a Hausman (1978) type test statistic comparing the LIML estimator of π under the null, H_0: $\beta = \beta_0$, to the OLS estimator of π. We then show that the LR test statistic is identical to the $t_0(\widehat{\beta}_L)^2 \equiv W_0(\widehat{\beta}_L)$ statistic proposed by Mills et al. (2014). The only difference between the $W_0(\widehat{\beta}_L)$ statistic and the standard LIML-based Wald statistic is the estimator for σ_u^2 in the denominator of the test. For $W_0(\widehat{\beta}_L)$ this variance is estimated under the null. From this equivalence result it follows that a conditional $CW_0\left(\widehat{\beta}_{Full}\right)$ test, where $\widehat{\beta}_{Full}$ is a Fuller (1977) estimator, and which is not an unbiased test, is a direct alternative to the LIML-based CLR test.

In Section 3, we follow the analysis of Andrews et al. (2006) and evaluate the noncentrality parameters of the weak-instrument limiting noncentral Wishart distribution of the maximal invariant. We find that for the fixed-Σ design the noncentrality parameters depend on the values of β and β_0 only through the difference $\delta = \beta - \beta_0$, and hence one can set $\beta_0 = 0$ without loss of generality (wlog) when evaluating power. We also find that one can set $\sigma_u^2 = \sigma_v^2 = 1$ wlog for evaluating power in the fixed-Σ design. In contrast, these findings do not hold for the fixed-Ω design, contradicting the statements in footnotes 7 and 8 in Andrews et al. (2006), i.e. one cannot set $\beta_0 = 0$ wlog.[1] One also cannot set the diagonal elements of Ω equal to 1 wlog, unless β_0 is set equal to 0. The latter combination is standard practice in the literature, but the special nature of the fixed-Ω, $\beta_0 = 0$ case is highlighted by our finding for the AR test in Section 3.2, where we find that the limiting weak-instrument power of the AR test is identical to the local-to-zero

[1] Footnote 8 in Andrews et al. (2006) refers to setting $\beta_0 = 0$ wlog, but in the structural equation (1), not equation (3), and hence applies to the fixed-Σ design.

power of the standard OLS-based Wald test for testing H_0: $\pi_y = 0$ in the reduced form specification $y_i = z_i'\pi_y + r_i$, which hence does not depend on σ_{rv} and σ_v^2.

Through an analysis of the fixed-Σ design weak-instrument noncentrality parameters we can highlight the behaviour of the tests when $|\rho_{uv}| \to 1$. Section 3.1.1 shows that the LR test, using the critical value of the strong-instruments χ_1^2 limiting distribution, is not size distorted when $|\rho_{uv}| \to 1$, see also Andrews et al. (2019), and has a maximum size distortion at $\rho_{uv} = 0$. Section 3.1.2 then shows that for each value of β there is a value of σ_u^2, such that the power of the AR and CLR tests approaches 1 when $\beta < \beta_0$ and $\rho_{uv} \to 1$, or $\beta > \beta_0$ and $\rho_{uv} \to -1$. We show in Section 4 and illustrate in Figure 3 that the fixed-Ω design maps onto those particular combinations of β, ρ_{uv} and σ_u^2 for large values of $|\beta|$, confirming that significant parts of the fixed-Ω power curve implicitly considers parameter configurations with high power of the AR and CLR tests. From the noncentrality parameters, we can show in Section 3.1.3 that the fixed-Σ power curve of the CLR test is asymmetric around β_0 as a function of $\beta - \beta_0$.

As discussed above, conclusions based on fixed-Ω designs about which test has superior power may be based on only very partial information. We illustrate this in Section 5 by comparing the behaviours of the LIML- and Fuller-based conditional Wald (CW) tests and the CW_0-Fuller test to that of the CLR ($= CW_0$-Liml) test in fixed-Σ design Monte Carlo simulations for different degrees of endogeneity ρ_{uv}. As far as we are aware these tests have only been compared in the fixed-Ω design (see Andrews et al., 2007; Mills et al., 2014), where the CLR test was found to dominate for most values of β. In contrast, we find in the fixed-Σ design that, for low to moderate degrees of endogeneity, the conditional Wald tests are more powerful than the CLR test. Even for medium to high values of ρ_{uv} the CW_0-Fuller test is well behaved, with higher power than the CLR test for part of the parameter space. Also, of these tests, the CW tests have the most power when β and ρ_{uv} have the same sign, including for the highest value of $\rho_{uv} = 0.75$ considered in the simulations. In a fixed-Ω analysis, these findings are easy to miss, as only a small segment of the power curve encapsulates low to moderate degrees of endogeneity or situations where ρ_{uv} and β have the same sign.

Our main focus is on the CLR test. As the LR test statistic is based on the AR test statistic, we introduce and discuss issues related to the CLR and AR tests in the next sections, but refrain from a general discussion of the LM test.

2 Model and Tests

We start with the standard structural and first-stage linear model specifications for a sample $\{y_i, x_i, z_i'\}_{i=1}^n$, given by

$$\begin{aligned} y_i &= x_i\beta + u_i \\ x_i &= z_i'\pi + v_i, \end{aligned} \tag{4}$$

where z_i is the k_z vector of instrumental variables. The instruments satisfy $E(z_i u_i) = 0$. Standard assumptions on the data, see e.g. Assumption M in Stock and Yogo (2005), needed for limiting normal distributions and consistent estimation of variance matrices are assumed to hold. The explanatory variable x_i is endogenous as $E(x_i u_i) = E(u_i v_i) \neq 0$. Other exogenous explanatory variables, including the constant, have been partialled out. The errors are assumed to be conditionally homoskedastic, with

$$\Sigma = Var\left(\left(\begin{array}{c} u_i \\ v_i \end{array}\right) | z_i\right) = \left[\begin{array}{cc} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{array}\right], \tag{5}$$

and correlation $\rho_{uv} = \frac{\sigma_{uv}}{\sigma_u \sigma_v}$.

The reduced form for y_i is given by

$$\begin{aligned} y_i &= z_i'\pi\beta + u_i + v_i\beta \\ &= z_i'\pi_y + r_i(\beta), \end{aligned} \tag{6}$$

with $r_i(\beta) = u_i + v_i\beta$, and the reduced-form error variance of $(r_i(\beta) \;\; v_i)'$ is given by

$$\Omega(\beta) = \left[\begin{array}{cc} \sigma_u^2 + 2\beta\sigma_{uv} + \beta^2\sigma_v^2 & \sigma_{uv} + \beta\sigma_v^2 \\ \sigma_{uv} + \beta\sigma_v^2 & \sigma_v^2 \end{array}\right]. \tag{7}$$

Let y and x be the n-vectors (y_i) and (x_i) and Z the $n \times k_z$ matrix of instruments. The standard 2SLS estimator for β is given by

$$\widehat{\beta}_{2sls} = \frac{x'P_Z y}{x'P_Z x},$$

where $P_Z = Z(Z'Z)^{-1} Z'$. The 2SLS estimator is based on the OLS estimator for π, given by $\widehat{\pi} = (Z'Z)^{-1} Z'x$. Let $\widehat{x} = Z\widehat{\pi}$, then $\widehat{\beta}_{2sls} = \widehat{x}'y/(\widehat{x}'x) = \widehat{x}'y/(\widehat{x}'\widehat{x})$.

Dropping notationally the dependence of Ω on β for ease of exposition, an estimator for

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix}$$

is $\widehat{\Omega} = W'M_Z W/n$, where $W = [y \;\; x]$ and $M_Z = I_n - P_Z$. The LIML estimator for β is then given by

$$\widehat{\beta}_L = \frac{x'P_Z y - n\widehat{\kappa}\widehat{\omega}_{12}}{x'P_Z x - n\widehat{\kappa}\widehat{\omega}_{22}},$$

where $\widehat{\kappa}$ is the minimum eigenvalue,

$$\widehat{\kappa} = \min \operatorname{eval}\left(\left(n^{-1}W'P_Z W\right)\widehat{\Omega}^{-1}\right). \tag{8}$$

Let $a_L = (\widehat{\beta}_L \;\; 1)'$. The definition of the LIML estimator for π as used in Moreira (2003) is given by

$$\widehat{\pi}_L = \left(Z'Z\right)^{-1} Z'W\widehat{\Omega}^{-1}a_L \left(a_L'\widehat{\Omega}^{-1}a_L\right)^{-1}. \tag{9}$$

Let $\widehat{x}_L = Z\widehat{\pi}_L$, then $\widehat{\beta}_L = \widehat{x}_L'y/\left(\widehat{x}_L'x\right) = \widehat{x}_L'y/\left(\widehat{x}_L'\widehat{x}_L\right)$, see Windmeijer (2018) for the latter equality.

Consider testing the null $\text{H}_0\colon \beta = \beta_0$ against the two-sided alternative $\text{H}_1\colon \beta \neq \beta_0$. The distributional properties of the AR and LR tests as described below are exact under fixed instruments, known Ω and normally distributed errors. Instrument strength is determined by the concentration parameter λ_n/σ_v^2, where $\lambda_n = \pi'Z'Z\pi$. The limiting distributions of the tests under the null are the same when relaxing these assumptions and using $\widehat{\Omega}$ as an estimator for Ω, see Moreira (2003) and Kleibergen (2002). Weak instrument asymptotics imply $\pi = \pi_n = c/\sqrt{n}$, where c is a vector of constants, with instrument strength then determined by λ/σ_v^2, with $\lambda = \operatorname{plim}\left(\pi_n'Z'Z\pi_n\right) = c'A_{zz}c$, where $A_{zz} = \operatorname{plim}\left(Z'Z/n\right)$.

Let $u_0 = y - x\beta_0$. The Anderson-Rubin test statistic is given by

$$AR = \frac{u_0'P_Z u_0}{\widehat{\sigma}_0^2}, \tag{10}$$

where $\widehat{\sigma}_0^2 = b_0'\widehat{\Omega}b_0 = u_0'M_Z u_0/n$, with $b_0 = (1 \;\; -\beta_0)'$. AR has a limiting $\chi^2_{k_z}$ distribution under the null, independent of the strength of the instruments. The AR test is a test for overidentifying restrictions in model (4), imposing the null.

Let $\widehat{\pi}_{L0}$ be the LIML estimator of π under the null, given by

$$\widehat{\pi}_{L0} = (Z'Z)^{-1} Z'W\widehat{\Omega}^{-1}a_0 / \left(a_0'\widehat{\Omega}^{-1}a_0\right),$$

with $a_0 = (\beta_0 \;\; 1)'$. As $\widehat{\pi}_{L0}$ is a consistent and efficient estimator of π under the null, H_0: $\beta = \beta_0$, but inconsistent under the alternative, H_1: $\beta \neq \beta_0$, whereas $\widehat{\pi}$ is consistent in both cases, we can use the Hausman (1978) specification test principle to construct the test statistic

$$H_\pi = (\widehat{\pi} - \widehat{\pi}_{L0})' \left(V\widehat{a}r(\widehat{\pi}) - V\widehat{a}r(\widehat{\pi}_{L0})\right)^{-1} (\widehat{\pi} - \widehat{\pi}_{L0}). \tag{11}$$

Under the null, H_π has a limiting $\chi^2_{k_z}$ distribution. An interesting, and standalone, observation is that the H_π statistic is identical to the AR statistic, as stated in the following proposition.

Proposition 1. *Let the Anderson-Rubin test statistic AR be as defined in (10) and let the Hausman test statistic H_π be defined as in (11). Then $H_\pi = AR$.*

Proof. See Appendix A.1 □

The Likelihood Ratio test we consider here is the test denoted LR_1 in Moreira (2003), which is a criterion difference test. This LR statistic is given by

$$\begin{aligned} LR &= \frac{u_0'P_Z u_0}{\widehat{\sigma}_0^2} - \frac{\widehat{u}_L' P_Z \widehat{u}_L}{\widehat{\sigma}_L^2} \\ &= AR - B\left(\widehat{\beta}_L\right), \end{aligned} \tag{12}$$

where $\widehat{\sigma}_L^2 = \widehat{u}_L' M_Z \widehat{u}_L / n = b_L'\widehat{\Omega} b_L$, with $\widehat{u}_L = y - x\widehat{\beta}_L$ and $b_L = (1 \;\; -\widehat{\beta}_L)'$, and where $B\left(\widehat{\beta}_L\right)$ is the Basmann (1960) test for overidentifying restrictions in model (4), with $B\left(\widehat{\beta}_L\right) = n\widehat{\kappa}$. Under standard strong instrument asymptotics, LR has a limiting χ^2_1 distribution. However, under weak instruments, its distribution is not invariant with respect to the value of $\pi_n = c/\sqrt{n}$, unlike AR. As Moreira (2003) showed, the asymptotic conditional distribution of LR under the null, conditional on the value of the $\widehat{\pi}_{L0}$-based Wald test statistic for testing H_0: $\pi = 0$,

$$\tau_0 = \widehat{\pi}_{L0}' \left(V\widehat{a}r(\widehat{\pi}_{L0})\right)^{-1} \widehat{\pi}_{L0} = \frac{a_0'\widehat{\Omega}^{-1}W'P_Z W\widehat{\Omega}^{-1}a_0}{a_0'\widehat{\Omega}^{-1}a_0}, \tag{13}$$

is given by

$$f\left(LR|\tau_0\right) = \frac{1}{2}\left(\xi_1 + \xi_{k_z-1} - \tau_0 + \sqrt{\left(\xi_1 + \xi_{k_z-1} + \tau_0\right)^2 - 4\xi_{k_z-1}\tau_0}\right),$$

where ξ_1 and ξ_{k-1} are independent χ^2_1 and $\chi^2_{k_z-1}$ distributed random variables. Conditional critical values for the LR test can then be simulated, or the conditional p-values calculated by numerical integration (Moreira, 2003; Mikusheva and Poi, 2006; Andrews et al., 2007; Hillier, 2009), resulting in correct size for this conditional LR (CLR) test, also when instruments are weak or uninformative. In the following we refer to the LR test when using critical values from the strong-instruments limiting χ^2_1 distribution, and the CLR test when using the conditional on τ_0 critical values.

Conditional tests with correct size under the null in weakly identified models can also be obtained for standard Wald tests, for example based on 2SLS, LIML, Fuller and bias-corrected 2SLS estimators, see Andrews, Moreira, and Stock (2007) and Mills, Moreira, and Vilela (2014). Mills et al. (2014) provide the details for obtaining the distributions of these test statistics conditional on τ_0, and they also considered one-sided conditional t-tests. A LIML-based test considered by Mills et al. (2014) is given by

$$W_0\left(\widehat{\beta}_L\right) = t_0(\widehat{\beta}_L)^2 = \frac{\left(\widehat{\beta}_L - \beta_0\right)^2\left(x'P_Z x - n\widehat{\kappa}\widehat{\omega}_{22}\right)}{\widehat{\sigma}^2_0}. \tag{14}$$

The difference with the standard LIML-based Wald test is the use of the restricted estimator $\widehat{\sigma}^2_0$ instead of the unrestricted $\widehat{\sigma}^2_L$. We find that the $W_0\left(\widehat{\beta}_L\right)$ statistic is identical to the LR statistic, as stated in the following proposition.

Proposition 2. *Let LR be as defined in (12) and let $W_0(\widehat{\beta}_L)$ be as defined in (14). Then $W_0\left(\widehat{\beta}_L\right) = LR$.*

Proof. See Appendix A.2. □

It follows from Proposition 2 and the results in Mills et al. (2014) that the conditional $W_0(\widehat{\beta}_L)$ ($CW_0(\widehat{\beta}_L)$) and CLR tests are also equivalent. This implies that a conditional $CW_0\left(\widehat{\beta}_{Full}\right)$ test, where $\widehat{\beta}_{Full}$ is a Fuller (1977) estimator of β, and which is not an unbiased test, is a direct alternative to the LIML-based CLR test, and this will therefore be one of the tests whose power we evaluate in section 5.

For the just-identified case, $k_z = 1$, it follows from Proposition 2 that $W_0\left(\widehat{\beta}_{IV}\right) = LR = AR$, where $\widehat{\beta}_{IV} = (z'x)^{-1} z'y$. The equivalence of $W_0\left(\widehat{\beta}_{IV}\right)$ and AR was derived by Feir, Lemieux, and Marmer (2016), see also Lee, McCrary, Moreira, and Porter (2020).

3 Properties of Tests in the Two Designs

In this section, we first show that, in the fixed-Σ design, we can set $\beta_0 = 0$ without loss of generality. We also establish that the Andrews et al. (2019) approach of keeping the value of β fixed, but varying the values of β_0 when testing H_0: $\beta = \beta_0$, is equivalent to the fixed-Σ design. We then analyse the noncentrality parameters of the fixed-Σ design weak-instrument noncentral Wishart distribution of the maximal invariant in order to examine properties of the AR and $(C)LR$ tests that will help us understand the link between the fixed-Ω and fixed-Σ designs. We show that the power curve of the CLR test is asymmetric, and establish the conditions in which the weak-instrument power of the AR and CLR tests goes to 1. Later on in Section 4, we show that the fixed-Ω design tends to implicitly select those values of Σ where the CLR test reaches a power of 1 as the value of $|\beta|$ increases. We end the current section by analysing the noncentrality parameters of the fixed-Ω design and showing that, unlike in the fixed-Σ case, in the fixed-Ω case we cannot set $\beta_0 = 0$ and the diagonal elements of Ω equal to 1 without loss of generality.

3.1 Fixed-Σ design

It is clear from the model specification (4), that for given values of $\{z_i', u_i, v_i\}_{i=1}^n$ and π, and hence a given value of $\lambda_n = \pi' Z' Z \pi$, all distributional properties of the 2SLS estimator remain unchanged but for the location of the estimator, when β is changed from $\beta = \beta_*$ to $\beta = \beta_* + \gamma$. For the first case we have $\widehat{\beta}_{2sls,1} = \beta_* + \frac{x' P_Z u}{x' P_Z x}$, and for the second case, $\widehat{\beta}_{2sls,2} = \beta_* + \gamma + \frac{x' P_Z u}{x' P_Z x} = \widehat{\beta}_{2sls,1} + \gamma$, as the ratio $\frac{x' P_Z u}{x' P_Z x}$ is unaffected. As we show in Appendix A.3, the same holds for the LIML estimator, leading to the following result. We focus here on the CLR test, but results here and below hold equivalently for the AR and conditional Wald tests.

Result 1. *Given values $\{z_i', u_i, v_i\}_{i=1}^n$ and π, and hence given values $x_i = z_i'\pi + v_i$ for $i = 1, \ldots, n$, and $\lambda_n = \pi' Z' Z \pi$, denote the LR test statistic for testing* H_0: $\beta = \beta_0$ *when $\beta = \beta_*$, and so $y_i = x_i \beta_* + u_i$, by $LR\left(\beta_0\right)_{\beta=\beta_*}$, and τ_0 is denoted $\tau_0\left(\beta_0\right)_{\beta=\beta_*}$. Then keeping* *everything constant, but only changing the value of* β *to* $\beta = \beta_* + \gamma$*, with* $\gamma \in \mathbb{R}$*, and so only changing the values* y_i *to* $y_{\gamma,i} = x_i(\beta_* + \gamma) + u_i = y_i + x_i\gamma$*, we have the result that, for testing* H_0: $\beta = \beta_0 + \gamma$,

$$\begin{aligned} LR(\beta_0 + \gamma)_{\beta=\beta_*+\gamma} &= LR(\beta_0)_{\beta=\beta_*} \\ \tau_0(\beta_0 + \gamma)_{\beta=\beta_*+\gamma} &= \tau_0(\beta_0)_{\beta=\beta_*} . \end{aligned}$$

Proof. See Appendix A.3 □

It follows directly from Result 1 that the power of the CLR test in the fixed-Σ design, only depends on $\delta = \beta - \beta_0$, and hence there is no loss of generality in taking $\beta_0 = 0$ when generating power curves using Monte Carlo simulation methods.

Corollary 1. *Let* $\delta_* \in \mathbb{R}$*. Under the conditions of Result 1, it follows that*

$$\begin{aligned} LR(\beta_*)_{\beta=\beta_*+\delta_*} &= LR(\beta_* - \delta_*)_{\beta=\beta_*} ; \\ \tau_0(\beta_*)_{\beta=\beta_*+\delta_*} &= \tau_0(\beta_* - \delta_*)_{\beta=\beta_*} . \end{aligned}$$

Proof. Follows directly from Result 1, with $\gamma = \delta_* = \beta_* - \beta_0$. □

It follows from Corollary 1 that the Andrews et al. (2019) approach of keeping the value of β fixed at β_*, but varying the values of β_0 for testing H_0:$\beta = \beta_0$, results in a power curve which is the mirror image around β_* of the standard fixed-Σ power curve when varying the value of β and testing H_0: $\beta = \beta_*$. This follows as the values of $\{z_i', u_i, v_i\}_{i=1}^n$ and π are kept constant when varying the value for β_0 in the Andrews et al. (2019) approach.

Next, consider the weak instruments limiting distribution results in Andrews et al. (2006, Lemma 4, p 736). Defining

$$\begin{aligned} \widehat{S} &= (Z'Z)^{-1/2} Z'Wb_0 \left(b_0'\widehat{\Omega}b_0\right)^{-1/2} \\ \widehat{T} &= (Z'Z)^{-1/2} Z'W\widehat{\Omega}^{-1}a_0 \left(a_0'\widehat{\Omega}^{-1}a_0\right)^{-1/2} , \end{aligned}$$

then the AR and LR test statistics are given by

$$AR = \widehat{S}'\widehat{S}$$
$$LR = \frac{1}{2}\left(\widehat{S}'\widehat{S} - \widehat{T}'\widehat{T} + \sqrt{\left(\widehat{S}'\widehat{S} - \widehat{T}'\widehat{T}\right)^2 + 4\left(\widehat{S}'\widehat{T}\right)^2}\right),$$

and $\tau_0 = \widehat{T}'\widehat{T}$.

Andrews et al. (2006) show that $\left(\widehat{S}, \widehat{T}\right) \xrightarrow{d} (S, T)$, with $S \sim N\left(c_{\beta,\beta_0}\mu, I_{k_z}\right)$ and $T \sim N\left(d_{\beta,\beta_0}\mu, I_{k_z}\right)$, where $\mu = A_{zz}^{1/2}c$ and

$$\begin{aligned} c_{\beta,\beta_0} &= (\beta - \beta_0)\left(b_0'\Omega b_0\right)^{-1/2}, \\ d_{\beta,\beta_0} &= a'\Omega^{-1}a_0\left(a_0'\Omega^{-1}a_0\right)^{-1/2} \\ &= b'\Omega b_0\left(b_0'\Omega b_0\right)^{-1/2}|\Omega|^{-1/2}, \end{aligned}$$

where $a = (\beta\ \ 1)'$, $b = (1\ \ -\beta)'$, $|\Omega|$ is the determinant of Ω, and where here $\Omega = \Omega(\beta)$.

The power properties of the tests when $n \to \infty$ are determined by the properties of the noncentral Wishart distribution of the maximal invariant, given by

$$Q = \begin{bmatrix} S'S & S'T \\ T'S & T'T \end{bmatrix}. \tag{15}$$

As above, let $\delta = \beta - \beta_0$, $\lambda = c'A_{zz}c = \mu'\mu$ and note that

$$\sigma_0^2(\beta) := b_0'\Omega(\beta)b_0 = (1\ \ \delta)\Sigma(1\ \ \delta)' = \sigma_u^2 + 2\sigma_{uv}\delta + \sigma_v^2\delta^2.$$

Then we get for the noncentrality parameters, see Appendix A.4.1 for details,

$$c_{\beta,\beta_0}^2\lambda = \frac{\lambda\delta^2}{\sigma_0^2(\beta)} = \frac{\lambda/\sigma_v^2}{\left(\frac{\sigma_u}{\delta\sigma_v}\right)^2 + 2\rho_{uv}\frac{\sigma_u}{\delta\sigma_v} + 1}, \tag{16}$$

$$d_{\beta,\beta_0}^2\lambda = \frac{\left(\lambda/\sigma_v^2\right)\left(\frac{\sigma_u}{\delta\sigma_v} + \rho_{uv}\right)^2}{\left(\left(\frac{\sigma_u}{\delta\sigma_v}\right)^2 + 2\rho_{uv}\frac{\sigma_u}{\delta\sigma_v} + 1\right)\left(1 - \rho_{uv}^2\right)}, \tag{17}$$

$$c_{\beta,\beta_0}d_{\beta,\beta_0}\lambda = \frac{\left(\lambda/\sigma_v^2\right)\left(\frac{\sigma_u}{\delta\sigma_v} + \rho_{uv}\right)}{\left(\left(\frac{\sigma_u}{\delta\sigma_v}\right)^2 + 2\rho_{uv}\frac{\sigma_u}{\delta\sigma_v} + 1\right)\sqrt{\left(1 - \rho_{uv}^2\right)}}.$$

It follows that in order to investigate the weak-instrument power properties of the tests for $n \to \infty$, there is no loss of generality in setting $\sigma_u^2 = \sigma_v^2 = 1$, because, given A_{zz}, the distribution of Q under $(\delta^*, c^*, \sigma_u^2, \sigma_v^2, \rho_{uv})$ equals its distribution under $(\delta, c, 1, 1, \rho_{uv})$, where $\delta = \delta^* \sigma_v / \sigma_u$ and $c = c^*/\sigma_v$. The results also confirm that the power properties of the tests in the fixed-Σ design only depend on β and β_0 via their difference $\delta = \beta - \beta_0$.

Notice further that the noncentrality parameters $c_{\beta,\beta_0}^2 \lambda$, $d_{\beta,\beta_0}^2 \lambda$ and $|c_{\beta,\beta_0} d_{\beta,\beta_0} \lambda|$ are symmetric in δ and ρ_{uv}, in the sense that their values, and hence the asymptotic power of the tests, are the same for any values $\{\delta, \rho_{uv}\}$ and $\{-\delta, -\rho_{uv}\}$.

3.1.1 Size properties of LR test

The noncentrality parameter of the weak-instrument limiting noncentral chi-squared distribution of τ_0 is given by $d_{\beta,\beta_0}^2 \lambda$. Under the null, H$_0$: $\beta = \beta_0$, we have that $\delta = 0$, and so

$$d_{\beta_0,\beta_0}^2 \lambda = \frac{\lambda / \sigma_v^2}{1 - \rho_{uv}^2}.$$

Therefore, given values $\lambda > 0$ and $\sigma_v^2 > 0$, $d_{\beta_0,\beta_0}^2 \lambda \to \infty$ and hence $\tau_0 \to \infty$, if $|\rho_{uv}| \to 1$. For the CLR test, the critical values depend on the value of τ_0. They range from the critical values of the χ_1^2 distribution as $\tau_0 \to \infty$ to the critical values of the $\chi_{k_z}^2$ distribution as $\tau_0 \to 0$, see Moreira (2003). Whilst the standard 2SLS-based Wald tests have their largest weak instrument size distortions at $|\rho_{uv}| = 1$, see Stock and Yogo (2005), it follows that this is not the case for the LR test. For $\lambda > 0$, the LR test has *no* size distortion when $|\rho_{uv}| \to 1$, as then $\tau_0 \to \infty$ and hence the critical values of the χ_1^2 distribution apply, see also Andrews et al. (2019). The noncentrality parameter $d_{\beta_0,\beta_0}^2 \lambda$ is minimised at $\rho_{uv} = 0$, and hence the weak-instrument size distortion of the LR test is maximised at $\rho_{uv} = 0$, see Figure B.1 in Appendix B for an illustration.

3.1.2 Conditions for the Power of AR and CLR Tests to Approach 1

The noncentrality parameter for the weak-instrument limiting distribution of the AR test statistic is given by $c_{\beta,\beta_0}^2 \lambda$. Given values $\sigma_u^2 > 0$, $\sigma_v^2 > 0$ and $\lambda > 0$, $c_{\beta,\beta_0}^2 \lambda$ is maximised at $\delta^+ = -\frac{1}{\rho_{uv}} \frac{\sigma_u}{\sigma_v}$, or, given β_0, $\beta^+ = \beta_0 - \frac{1}{\rho_{uv}} \frac{\sigma_u}{\sigma_v}$. It follows that the power of the AR test in the fixed-Σ design is asymmetric, with the maximum power to the left of β_0 if $\rho_{uv} > 0$ and to the right of β_0 if $\rho_{uv} < 0$. The asymmetry of the power of the AR test follows directly from the asymmetry of the function $c_{\beta,\beta_0}^2 \lambda$ as illustrated in Figure 1 below.

The noncentrality parameter at δ^+ is given by

$$c^2_{\beta^+,\beta_0}\lambda = \frac{\lambda/\sigma_v^2}{1-\rho_{uv}^2}.$$

It follows that $c^2_{\beta^+,\beta_0}\lambda \to \infty$, and so $AR\left(\beta_0\right)_{\beta=\beta^+} \to \infty$, if $|\rho_{uv}| \to 1$, with $\beta^+ \to \beta_0 - \frac{\sigma_u}{\sigma_v}$ for $\rho_{uv} \to 1$, and $\beta^+ \to \beta_0 + \frac{\sigma_u}{\sigma_v}$ for $\rho_{uv} \to -1$. As Davidson and MacKinnon (2015, pp 831-832) show, for $\lambda > 0$, $n\widehat{\kappa} = B\left(\widehat{\beta}_L\right) \xrightarrow{d} \chi^2_{k_z-1}$ when $|\rho_{uv}| \to 1$. It therefore follows that $LR\left(\beta_0\right)_{\beta=\beta^+} = AR\left(\beta_0\right)_{\beta=\beta^+} - B\left(\widehat{\beta}_L\right) \to \infty$ when $|\rho_{uv}| \to 1$. Thus the power of the AR and CLR tests approaches 1 at $\beta = \beta^+$ when $|\rho_{uv}| \to 1$. Alternatively, and for later reference, this can be rephrased in the following way. If we standardise $\sigma_v^2 = 1$, then for each value of β there is a value of σ_u^2, namely $\sigma_u^{2+} = \left(\delta^+\right)^2$, such that the power of the AR and CLR tests approaches 1 when $\beta < \beta_0$ and $\rho_{uv} \to 1$, or $\beta > \beta_0$ and $\rho_{uv} \to -1$.

3.1.3 Asymmetry of the Power of the CLR Test

As the distribution of $B\left(\widehat{\beta}_L\right) = n\widehat{\kappa}$ is invariant to the value of β in the fixed-Σ design, see the proof of Result 1 in Appendix A.3, it follows that the power of the LR test is asymmetric due to the asymmetry in power of the AR test, and because the LR test uses the constant critical values of the χ^2_1 distribution. As the critical values of the CLR test depend on the observed value of τ_0, it does not immediately follow that the power of the CLR test is also asymmetric, but we show here that it is.

From the expressions of the noncentrality parameters $c^2_{\beta,\beta_0}\lambda$ and $d^2_{\beta,\beta_0}\lambda$ in (16) and (17) respectively, it follows that

$$c^2_{\beta,\beta_0}\lambda + d^2_{\beta,\beta_0}\lambda = \frac{\lambda/\sigma_v^2}{1-\rho_{uv}^2}. \tag{18}$$

As this is not a function of $\delta = \beta - \beta_0$, it implies that the sum of the two noncentrality parameters is constant, given values $\lambda > 0$, $\sigma_v^2 > 0$ and $-1 < \rho_{uv} < 1$. This is illustrated in the left panel of Figure 1, which graphs the values of $c^2_{\beta,\beta_0}\lambda$ and $d^2_{\beta,\beta_0}\lambda$ as a function of δ, for $\lambda = 1$, $\sigma_u^2 = \sigma_v^2 = 1$ and $\rho_{uv} = 0.5$. The symmetry in the values of $c^2_{\beta,\beta_0}\lambda$ and $d^2_{\beta,\beta_0}\lambda$ is clear, with larger values of $c^2_{\beta,\beta_0}\lambda$ accompanied by smaller values of $d^2_{\beta,\beta_0}\lambda$ and vice versa, with their sum being constant. A higher value of $c^2_{\beta,\beta_0}\lambda$ leads to a higher expected value of the AR statistic and hence, ceteris paribus, a higher expected value of the LR statistic. However, the accompanied lower value of $d^2_{\beta,\beta_0}\lambda$ leads to a lower expected value

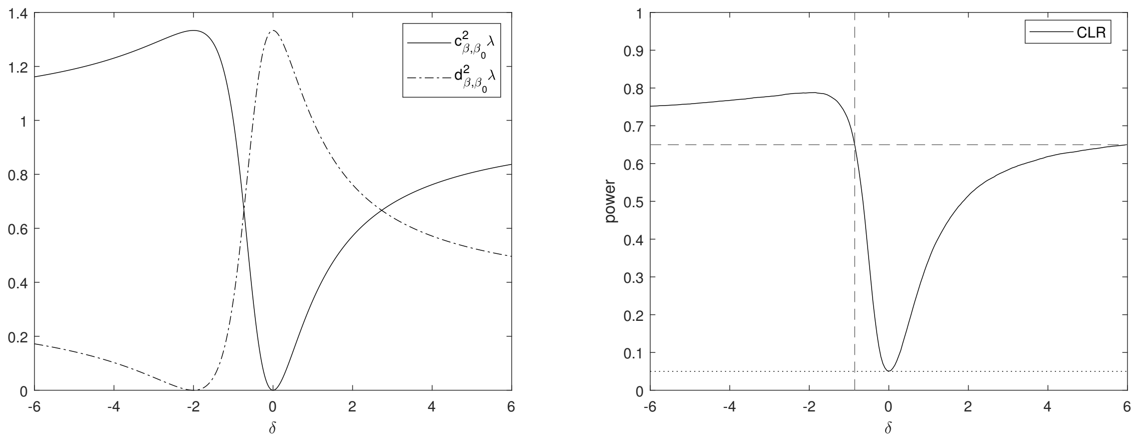

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Figure 1: Left panel, values of $c^2_{\beta,\beta_0}\lambda$ and $d^2_{\beta,\beta_0}\lambda$, $\lambda = 1$, $\sigma_u^2 = \sigma_v^2 = 1$, $\rho_{uv} = 0.5$. Right panel, asymptotic power of the CLR test, $\lambda = 10$, $\sigma_u^2 = \sigma_v^2 = 1$, $\rho_{uv} = 0.5$, horizontal dashed line is power at $\delta = 6$, the vertical dashed line is at $\delta = -0.8571$, see text for explanation.

of τ_0, and hence a larger expected value of the conditional critical value for the CLR test. We can therefore not directly assess the properties of the power curve of the CLR test without investigating the distribution of the conditional critical values for different values of $d^2_{\beta,\beta_0}\lambda$.

However, we are able to make a statement about the asymmetry of the power of the CLR test. This is due to the fact that S and T are independently distributed, see Lemma 2 in Andrews et al. (2006). Therefore for a given value of β_0, and a value β_* of β, with value for $c^2_{\beta,\beta_0}\lambda$ equal to $c^2_{\beta_*,\beta_0}\lambda$, if there is a value $\tilde{\beta} \neq \beta_*$ of β with $c^2_{\tilde{\beta},\beta_0}\lambda = c^2_{\beta_*,\beta_0}\lambda$, then the power of the CLR test for testing H$_0$: $\beta = \beta_0$ when $\beta = \beta_*$ is the same as the power of the test when $\beta = \tilde{\beta}$. This follows as, ceteris paribus, the distributions of the AR test statistics and hence the LR test statistics are the same due to the equal values of the noncentrality parameters $c^2_{\beta^*,\beta_0}\lambda = c^2_{\tilde{\beta},\beta_0}\lambda$. But it follows from (18) that then also $d^2_{\beta^*,\beta_0}\lambda = d^2_{\tilde{\beta},\beta_0}\lambda$, and hence, ceteris paribus, the distributions of the τ_0 statistics are the same. Because S and T are independently distributed, it follows that the distributions of the LR statistics and the conditional critical values for the CLR test are the same at $\beta = \beta_*$ and $\beta = \tilde{\beta}$ and hence the rejection probabilities are the same.

From this result, the asymmetry of the power function of the CLR test follows from the asymmetry of $c^2_{\beta,\beta_0}\lambda$ and $d^2_{\beta,\beta_0}\lambda$ as a function of δ, as displayed in the left panel of Figure 1. We have $c^2_{\beta_0,\beta_0}\lambda = 0$ and $\lim_{\delta\to\infty} c^2_{\beta,\beta_0}\lambda = \lambda/\sigma_v^2$. It is further easily derived that for $\rho_{uv} > 0$, $\partial c^2_{\beta,\beta_0}\lambda/\partial\delta > 0$ for $\delta > 0$, and $\partial c^2_{\beta,\beta_0}\lambda/\partial\delta < 0$ for $-\frac{\sigma_u/\sigma_v}{\rho_{uv}} < \delta < 0$. Further at the value $\delta^m = \beta^m - \beta_0 = -\frac{\sigma_u/\sigma_v}{2\rho_{uv}}$ we have that $c^2_{\beta^m,\beta_0}\lambda = \lambda/\sigma^2_v$, and so for any value $\delta_* \in (0,\infty)$ there is a unique value $\delta^m < \tilde{\delta} < 0$ with the power of the CLR test at $\tilde{\delta}$ the same as that at δ_*, and for each $\delta_{*,1} > \delta_{*,2}$ we have that $\tilde{\delta}_1 < \tilde{\delta}_2$.[2]

This is illustrated in the right panel of Figure 1, which shows the weak instruments asymptotic power of the CLR test from simulations, for 20,000 replications at values $\delta = -6, -5.95, \ldots, 6$, with $\lambda = 10$, $\sigma^2_u = \sigma^2_v = 1$, $\rho_{uv} = 0.5$, and so $\delta^m = -1$ here. The horizontal dashed line is the rejection frequency at $\delta = \delta_* = 6$. We have here that $c^2_{\beta_*,\beta_0}\lambda = c_{\tilde{\beta},\beta_0}\lambda$ at $\tilde{\delta} = -0.8571$. The vertical dashed line is at $\delta = -0.8571$ and the two dashed lines cross exactly on the power curve of the CLR test, confirming the results.

3.2 Fixed-Ω Design

We now contrast the results found above for the fixed-Σ design with those for the fixed-Ω design. We can write the data generating process for the fixed-Ω design as

$$\begin{aligned} y_i &= x_i\beta + r_i - \beta v_i \\ x_i &= z_i'\pi + v_i, \end{aligned} \tag{19}$$

as then the reduced form is given by $y_i = z_i'\pi\beta + r_i$, and Ω is constant for all values of β and given by

$$\Omega = \begin{bmatrix} \sigma^2_r & \sigma_{rv} \\ \sigma_{rv} & \sigma^2_v \end{bmatrix},$$

and correlation $\rho_\Omega := \rho_{rv} = \frac{\sigma_{rv}}{\sigma_r\sigma_v}$.

Changing in this design the value of β from $\beta = \beta_*$ to $\beta = \beta_* + \gamma$ whilst keeping the values of $\{z_i', r_i, v_i\}_{i=1}^n$ and π constant does not lead to a location shift only, but changes the distributions of the estimators. For example, for the 2SLS estimator, we have for the first case $\widehat{\beta}_{2sls,1} = \beta_* + \frac{x'P_Z(r-\beta_*v)}{x'P_Zx}$, whereas for the second case, $\widehat{\beta}_{2sls,2} = \beta_* + \gamma + \frac{x'P_Z(r-\beta_*v)-\gamma x'P_Zv}{x'P_Zx} = \widehat{\beta}_{2sls,1} + \gamma\left(1 - \frac{x'P_Zv}{x'P_Zx}\right)$. The same applies to the LIML estimator and we therefore have in general in this fixed-Ω design that $LR\left(\beta_0+\gamma\right)_{\beta=\beta_*+\gamma} \neq LR\left(\beta_0\right)_{\beta=\beta_*}$, everything else constant. Hence, choosing $\beta_0 = 0$ is now not without loss of generality.

This can further be seen by investigating the noncentrality parameters of the maximal

[2] The equivalence of power at these points δ_* and $\tilde{\delta}$ clearly also holds for the AR test.

invariant Q as defined in (15) for the fixed-Ω design. We get in this case, see Appendix A.4.2 for details,

$$c^2_{\beta,\beta_0}\lambda = \frac{(\lambda/\sigma_v^2)}{\left(\frac{\sigma_r}{\delta\sigma_v}\right)^2 - 2\frac{\beta_0}{\delta}\rho_\Omega\left(\frac{\sigma_r}{\delta\sigma_v}\right) + \left(\frac{\beta_0}{\delta}\right)^2},$$

$$d^2_{\beta,\beta_0}\lambda = \frac{(\lambda/\sigma_v^2)\left(\frac{\sigma_r}{\delta\sigma_v} - \rho_\Omega + \frac{\beta_0}{\delta}\left(\beta\frac{\sigma_v}{\sigma_r} - 2\rho_\Omega\right)\right)^2}{\left(\left(\frac{\sigma_r}{\delta\sigma_v}\right)^2 - 2\frac{\beta_0}{\delta}\rho_\Omega\frac{\sigma_r}{\delta\sigma_v} + \left(\frac{\beta_0}{\delta}\right)^2\right)(1-\rho_\Omega^2)},$$

$$c_{\beta,\beta_0}d_{\beta,\beta_0}\lambda = \frac{(\lambda/\sigma_v^2)\left(\frac{\sigma_r}{\delta\sigma_v} - \rho_\Omega + \frac{\beta_0}{\delta}\left(\beta\frac{\sigma_v}{\sigma_r} - 2\rho_\Omega\right)\right)}{\left(\left(\frac{\sigma_r}{\delta\sigma_v}\right)^2 - 2\frac{\beta_0}{\delta}\rho_\Omega\frac{\sigma_r}{\delta\sigma_v} + \left(\frac{\beta_0}{\delta}\right)^2\right)\sqrt{(1-\rho_\Omega^2)}}.$$

where, as before, $\delta = \beta - \beta_0$. These noncentrality parameters, and hence the asymptotic power of the tests under weak instrument asymptotics, depend on the value β_0/δ so that one cannot set $\beta_0 = 0$ or $\sigma_r^2 = \sigma_v^2 = 1$ wlog in this design. The exception is that one can set $\sigma_r^2 = \sigma_v^2 = 1$ wlog when one sets $\beta_0 = 0$, as then

$$c^2_{\beta,0}\lambda = \frac{\beta^2\lambda}{\sigma_r^2} = \frac{(\lambda/\sigma_v^2)}{\left(\frac{\sigma_r}{\beta\sigma_v}\right)^2},$$

$$d^2_{\beta,0}\lambda = \frac{(\lambda/\sigma_v^2)\left(\frac{\sigma_r}{\beta\sigma_v} - \rho_\Omega\right)^2}{\left(\frac{\sigma_r}{\beta\sigma_v}\right)^2(1-\rho_\Omega^2)},$$

$$c_{\beta,0}d_{\beta,0}\lambda = \frac{(\lambda/\sigma_v^2)\left(\frac{\sigma_r}{\beta\sigma_v} - \rho_\Omega\right)}{\left(\frac{\sigma_r}{\beta\sigma_v}\right)^2\sqrt{(1-\rho_\Omega^2)}}.$$

The case $\beta_0 = 0$, $\sigma_r^2 = \sigma_v^2 = 1$ is often the one considered in the literature, but to reiterate this is clearly not without loss of generality in the fixed-Ω design, in contrast to the findings above for the fixed-Σ design.

Notice that when $\beta_0 = 0$, the noncentrality parameter for the AR test, given by $c^2_{\beta,0}\lambda$, and hence its asymptotic power, does not depend on ρ_Ω or σ_v^2, only on β, λ and σ_r^2. The reason for this is that the AR test statistic for testing H_0: $\beta = 0$, is given by

$$AR_{\beta_0=0} = \frac{y'P_Zy}{y'M_Zy/n},$$

as $u_0 = y - x\beta_0 = y$. Consider the reduced form

$$y = Z\pi_y + r, \tag{20}$$

with the OLS estimator for π_y given by $\widehat{\pi}_y = (Z'Z)^{-1} Z'y$. Then the $\widehat{\pi}_y$-based standard Wald test statistic for testing H$_0$: $\pi_y = 0$ is given by

$$W_{\pi_y} = \widehat{\pi}_y' \left(V\widehat{a}r\left(\widehat{\pi}_y\right)\right)^{-1} \widehat{\pi}_y = \frac{y'P_Z y}{y'M_Z y/n} = AR_{\beta_0=0},$$

using as an estimator for the variance $V\widehat{a}r\left(\widehat{\pi}_y\right) = \widehat{\sigma}_r^2 \left(Z'Z\right)^{-1}$ and $\widehat{\sigma}_r^2 = y'M_Z y/n$.

The fixed-Ω design sets $\pi_y = \pi\beta$ and $Var(r_i) = \sigma_r^2$. Weak-instrument asymptotics sets $\pi = \pi_n = c/\sqrt{n}$, and hence $\pi_y = \pi_{y,n} = c\beta/\sqrt{n}$. Therefore the weak-instrument fixed-Ω power curve for the AR test as a function of β, testing H$_0$: $\beta = 0$, is simply the local-to-zero power curve of the OLS-based Wald test for testing H$_0$: $\pi_y = 0$ in the standard linear model (20). Because σ_r^2 is constant, the features of the endogenous explanatory variable x do not enter this specification and hence neither σ_v^2 nor ρ_Ω enter the noncentrality parameter $c_{\beta,0}^2\lambda$.

4 Power of the CLR Test in the Two Designs with $\beta_0 = 0$

In the fixed-Ω design (19) the variance matrix of the structural errors varies with β and $\Sigma(\beta)$ is given by

$$\Sigma(\beta) = \begin{bmatrix} \sigma_r^2 - 2\beta\sigma_{rv} + \beta^2\sigma_v^2 & \sigma_{rv} - \beta\sigma_v^2 \\ \sigma_{rv} - \beta\sigma_v^2 & \sigma_v^2 \end{bmatrix}, \tag{21}$$

see also Andrews et al. (2019, p 466). It is common for simulations based on the fixed-Ω design to set $\sigma_r^2 = \sigma_v^2 = 1$, from which it follows that

$$\sigma_u^2(\beta) = 1 - 2\beta\rho_\Omega + \beta^2, \tag{22}$$

$$\sigma_{uv}(\beta) = \rho_\Omega - \beta. \tag{23}$$

For testing H$_0$: $\beta = 0$, it follows that under the null, $\Sigma(0) = \Omega$, and so ρ_Ω is then an indicator of the degree of endogeneity in the null model only.

Figure 2 displays these values of $\rho_{uv}(\beta) = \sigma_{uv}(\beta)/\sigma_u(\beta)$ and $\sigma_u^2(\beta)$ as a function of β for values of $\rho_\Omega = 0$, 0.5 and 0.95. The latter two values have often been used in simulations. As is clear from the formulae (22) and (23), and highlighted by Figure 2, for every value of β the endogeneity and variance properties of the structural model change. For the correlations $\rho_{uv}(\beta)$ we have that $\rho_{uv}(\beta) < \rho_\Omega$ for $\beta > 0$, and $\rho_{uv}(\beta) > \rho_\Omega$ for $\beta < 0$. Further, $\rho_{uv}(\beta) > 0$ for $\beta < \rho_\Omega$ and $\rho_{uv}(\beta) < 0$ for $\beta > \rho_\Omega$, approaching 1 and -1 quite rapidly, especially for $\rho_\Omega = 0.95$.

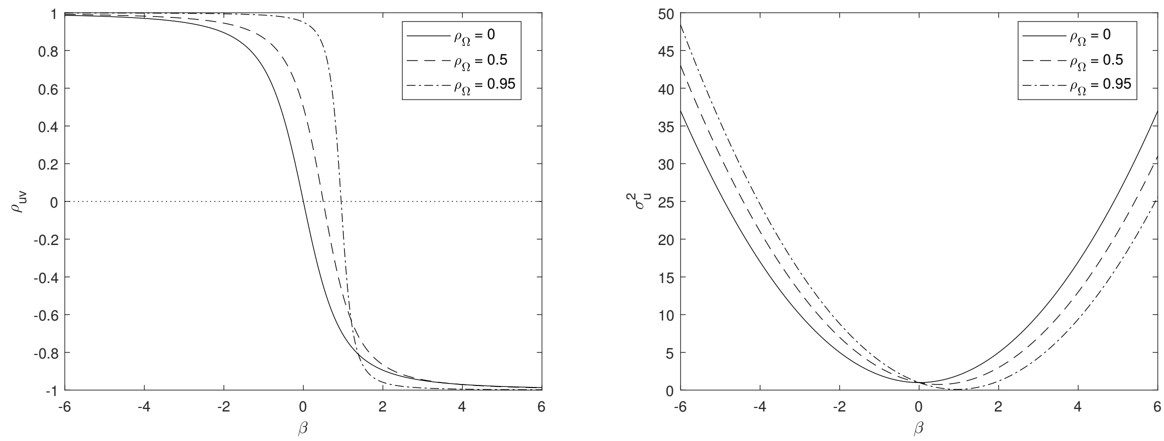

Figure 2: Values of ρ_{uv} and σ_u^2 as a function of β when holding $\Omega = \begin{bmatrix} 1 & \rho_\Omega \\ \rho_\Omega & 1 \end{bmatrix}$ constant.

For any value of ρ_Ω we have that

$$\frac{\sigma_u^2(\beta)}{\beta^2} \to 1 \text{ if } |\beta| \to \infty,$$
$$\rho_{uv}(\beta) \to 1 \text{ if } \beta \to -\infty,$$
$$\rho_{uv}(\beta) \to -1 \text{ if } \beta \to \infty,$$

which are the values for σ_u^2 and ρ_{uv} where the power of the AR and CLR tests approaches 1, as shown in Section 3.1.2, with here $\delta = \beta$. Therefore, the fixed-Ω design selects particular points in the space of the nuisance parameters Σ, selecting those values that result in the power being equal to one for large values of $|\beta|$. This holds for any value $\lambda > 0$, so also for very weak instruments.

We illustrate this in Figure 3, which plots the weak-instrument asymptotic power of the CLR test for testing H$_0$: $\beta = 0$ as a function of ρ_{uv} for different values of σ_u^2 and

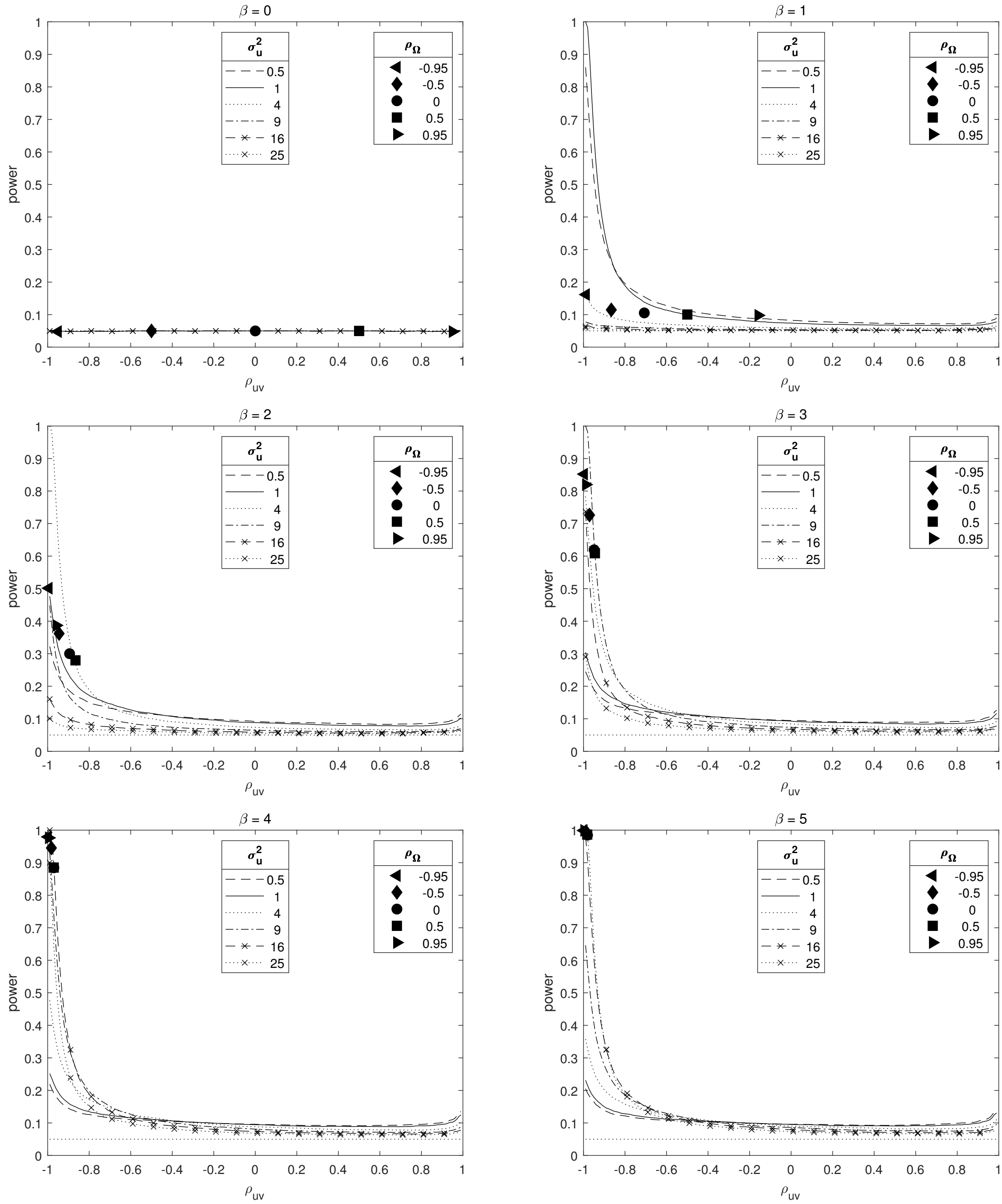

Figure 3: Weak-instrument asymptotic power of CLR test, $k_z = 5$, $\lambda = 1$.

β, normalising $\sigma_v^2 = 1$. The instruments are very weak with $\lambda = 1$ and the number of instruments is $k_z = 5$. Also included are the rejection frequencies of the CLR test for the fixed-Ω design with $\sigma_r^2 = \sigma_v^2 = 1$, for the values of $\rho_\Omega = -0.95, -0.5, 0, 0.5, 0.95$, these points indicated by the solid left-triangle, diamond, circle, square and right-triangle shapes respectively. These graphs confirm our findings as described above. For $\beta = 0$, the CLR test has correct size for all values of ρ_{uv} and σ_u^2, and $\rho_\Omega = \rho_{uv}$. Then for the values of $\beta = 1, 2, 3, 4, 5$, the power of the test is quite low over a wide range of values of ρ_{uv} and σ_u^2, as the value of λ is small, but the power approaches 1 when ρ_{uv} approaches -1 and σ_u^2 is equal to β^2. The path of the fixed-Ω design clearly swings that way for increasing values of β, with the associated $\rho_{uv}(\beta)$ and $\sigma_u^2(\beta)$ approaching -1 and β^2 respectively.

This is further illustrated in Figure 4, which displays the standard weak-instrument asymptotic power curves for the fixed-Ω design, with $\sigma_r^2 = \sigma_v^2 = 1$ and $\rho_\Omega = \{0.5, 0.99\}$, and fixed-$\Sigma$ design, with $\sigma_u^2 = \sigma_v^2 = 1$, and $\rho_{uv} = \{0.5, 0.99\}$, again for $k_z = 5$ and $\lambda = 1$. It is clear that these two power curves display very different types of information. For example, with $\rho_{uv} = 0.5$ fixed, the power of the CLR test is low for all values of β. The power curve for $\rho_{uv} = 0.99$ fixed is highly asymmetric, with low power for positive values of β, a power of 1 for $\beta = -1$, as explained above, and then the power diminishing again for $\beta < -1$. The fixed-Ω power curves are much more symmetric with power approaching 1 for large positive and negative values of β as explained above. The differences in power for the values of $\rho_\Omega = 0.5$ and $\rho_\Omega = 0.99$ are also not as pronounced as those of the fixed-Σ design.

For an applied researcher that makes an assumption of positive structural correlation ρ_{uv} ex ante and expects a value of $\beta > 0$, the above fixed-Σ analysis shows that the power of the CLR test to reject H_0: $\beta = 0$ is low when the instruments are very weak, $\lambda = 1$, for all values of $\beta > 0$, $\rho_{uv} > 0$ and $\sigma_u^2 > 0$. This information is less readily obtained from the fixed-Ω design. For the extreme case of setting $\rho_\Omega = 0$, the power in the fixed-Ω design is only evaluated for $\{\beta > 0, \rho_{uv} < 0\}$ and $\{\beta < 0, \rho_{uv} > 0\}$, with the asymptotic power curve then fully symmetric following the results derived above, and hence providing a very partial set of information. For $\rho_\Omega > 0$, we have $\rho_{uv} > 0$ for $\beta < \rho_\Omega$ and so the fixed-Ω power curve displays the power of the test for $\beta > 0$ and $\rho_{uv} > 0$ for the values $0 < \beta < \rho_\Omega$ only, with the values of ρ_{uv} then between ρ_Ω and 0, $\rho_\Omega > \rho_{uv} > 0$. So for the $\rho_\Omega = 0.5$ case, Figure 4 displays power in the fixed-Ω design for positive β and positive

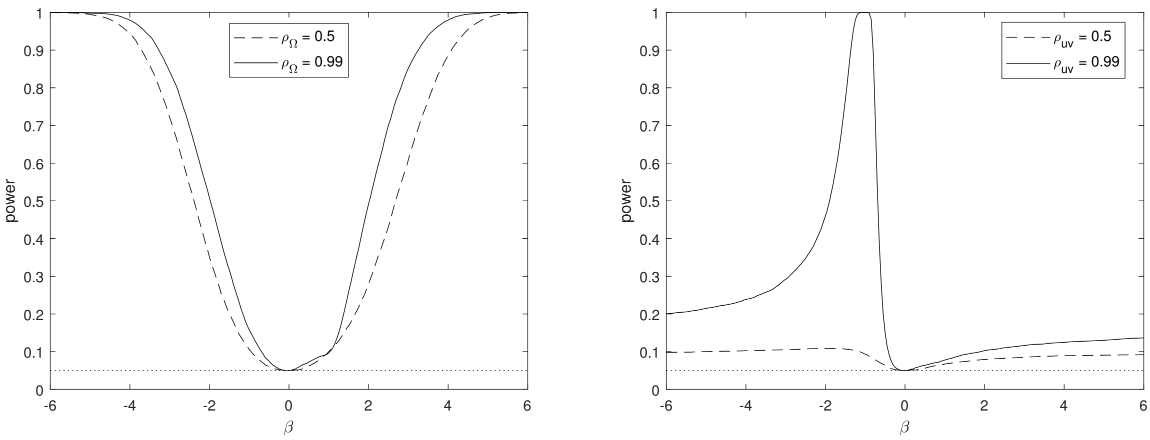

Figure 4: Weak-instrument asymptotic power curves of the CLR test for fixed-Ω design, left panel, and fixed-Σ design, right panel. $k_z = 5$, $\lambda = 1$.

ρ_{uv} only for $\beta \in (0, 0.5)$.

An argument often made for the fixed-Ω design is that Ω can be consistently estimated, unlike Σ when instruments are weak, see e.g. the discussion in Andrews et al. (2019, p 465) who state that "...because ρ_Ω can be consistently estimated, and hence, in large samples can be treated as fixed and known." However, for the structural model (4) of interest, we consistently estimate $\Omega(\beta)$ as defined in (7), and hence $\rho_\Omega(\beta)$, given by

$$\rho_\Omega(\beta) = \frac{\sigma_{uv} + \beta\sigma_v^2}{\sigma_v\sqrt{\sigma_u^2 + 2\beta\sigma_{uv} + \beta^2\sigma_v^2}}.$$

And, once we have estimates of $\Omega(\beta)$ from the data, we can obtain the p-values of the test statistics. Clearly, in large samples, we can treat $\rho_\Omega(\beta)$ as known, but it is not clear why it should be treated as fixed. For the fixed-Ω model specification (19), ρ_Ω can be treated as known and fixed in large samples, but this model has not been posited as the structural model of interest in the literature, and appears a circular argument. Further, knowledge about Ω does not in itself guide a researcher to which test is best to use. For example, we find in the next section that conditional Wald tests have more power than the CLR test for certain combinations of β and values of the nuisance parameters Σ. Knowledge of Ω cannot differentiate between these situations, which is therefore akin to the situation that one cannot estimate Σ in weak-instrument settings.

Further, if a researcher would like to assess power properties of tests or make power calculations ex ante, i.e. before the data are available, she is very likely to be making assumptions about the value of β and ρ_{uv}, but not about the value of ρ_{Ω}, which appears harder to interpret.[3]

5 Power Comparions of LIML- and Fuller-Based CW and CW_0 Tests

Our findings so far suggest that, in the fixed-Ω design, large segments of the power curves gravitate towards particular values for the parameters in Σ that result in high power of the CLR test. As a result, fixed-Ω designs may obscure relevant parameter spaces where the power of the CLR test is weaker, and perhaps lower than that of other tests. We illustrate this by revisiting comparisons between the CLR test and CW-LIML, CW-Fuller and CW_0-Fuller tests. In fixed-Ω designs, CLR power curves tend to dominate the power curves of the conditional Wald tests for most values of β. However, there is also a narrow range of small positive values of β where the conditional Wald tests have higher power. Because in the fixed-Ω design $|\rho_{uv}|$ rises quickly with $|\beta|$, this narrow range of values for β actually encompasses a fairly wide range of values for $|\rho_{uv}|$, covering situations with low and moderate endogeneity, and a section where β and ρ_{uv} have the same sign. We then show that a fixed-Σ design that allows to control ρ_{uv} directly, reveals more clearly that the CLR test can be outperformed in terms of power by CW and CW-Fuller tests in the cases of low to moderate endogeneity or where ρ_{uv} and β have the same sign.

Moreira (2003) compared the behaviour of the conditional 2SLS-based Wald test to that of the CLR test in a fixed-Σ design. Andrews et al. (2007) compared the behaviours of the 2SLS-, LIML- and Fuller(1)-based CW tests to that of the CLR test in the fixed-Ω design. They find that the CW-Fuller test performs best of the three conditional Wald tests, but that its performance is, overall, "..., very poor relative to the CLR test", and that overall "...the CW tests perform worse, often much worse, than the CLR test" (Andrews et al., 2007, p 131). Figure 5 replicates Figure 5, panel (b) in Andrews et al. (2007) for the CLR/CW_0-Liml, CW-Liml and CW-Fuller tests for the fixed-Ω design with $\sigma_r^2 = \sigma_v^2 = 1$, $\rho_{\Omega} = 0.5$, $k_z = 5$ and $\lambda = 5$. We further include the power curve for the CW_0-Fuller test. The W_0-Fuller statistic is equal to $W_0\left(\widehat{\beta}_{Full}\right)$, as per the definition

[3]But note that each point on a fixed-Σ power curve has its equivalent on a fixed-Ω power curve and vice versa. For example, for the setting of Figure 4, the fixed-Σ power at $\beta = 4$, $\rho_{uv} = 0.5$ in the right panel is the same as that of the fixed-Ω power at $\rho_{\Omega} = 0.982$ and $\beta = 0.873$.

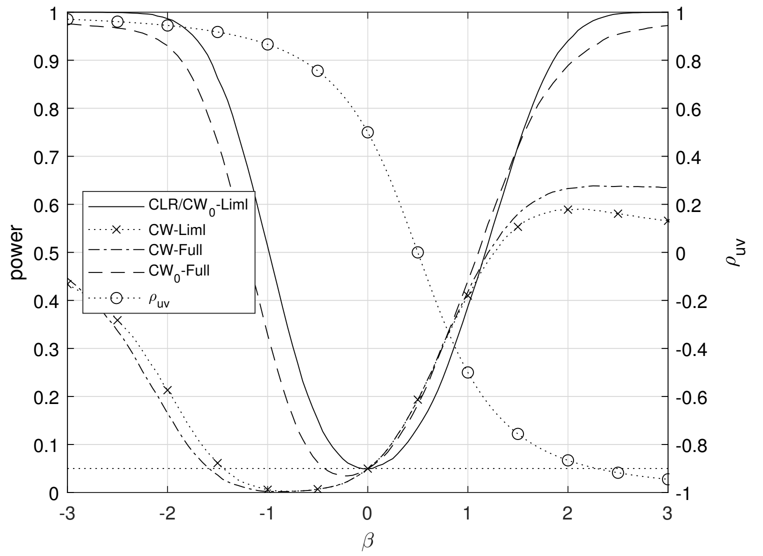

Figure 5: Weak-instrument asymptotic power of tests, fixed-Ω design, $\rho_\Omega = 0.5$, $k_z = 5$, $\lambda = 5$.

in (14), and where $\widehat{\beta}_{Full}$ is the Fuller(1) estimator of β, see also Mills et al. (2014).

From the result of Proposition 2 it follows that the only difference between the W-Liml and LR/W_0-Liml test statistics is the estimator of the variance σ_u^2. For the LR statistic, this is estimated under the null, and for known Ω, or asymptotically, in the fixed-Ω design, $\sigma_0^2 = b_0'\Omega b_0 = \sigma_r^2 = 1$ is constant for all values of β, whereas $\sigma_u^2(\beta) = 1 - 2\beta\rho_\Omega + \beta^2$, and so varies with β as depicted in the right panel of Figure 2, with increasing values of $\sigma_u^2(\beta)$ with increasing values of $|\beta|$. The same observation applies to the difference between the W-Fuller and W_0-Fuller statistics. This is reflected in the power curves in Figure 5. The CW-Liml and CW-Fuller tests have (much) less power than the CW_0-Liml and CW_0-Fuller tests for $\beta < 0$, and for $\beta > 1.2$, and hence the conclusion of a poor performance of the CW tests seems justified.

However, upon closer inspection, it is clear that the CW tests, and also the CW_0-Fuller test, are more powerful than the CLR test for values of β between 0 and 1.05. Figure 5 also displays the amount of endogeneity ρ_{uv} for each value of β. As discussed above, $\rho_{uv} = \rho_\Omega = 0.5$ at $\beta = 0$, decreases to $\rho_{uv} = 0$ at $\beta = 0.5$, and further to $\rho_{uv} = -0.54$ at $\beta = 1.05$. For all negative values of β, $\rho_{uv} > 0.5$, increasing to $\rho_{uv} = 0.97$ at $\beta = -3$, and for all values of $\beta > 1.05$, $\rho_{uv} < -0.54$, decreasing to $\rho_{uv} = -0.97$ at $\beta = 3$.

These results therefore indicate that the CW tests are more powerful for a range of parameter values in low to moderate endogeneity settings. Chernozhukov and Hansen (2008, p 70) report an estimated median value of $\rho_{uv} = 0.3$ for a survey of instrumental variables papers, commenting that this "…suggests that the degree of correlation between structural and first-stage errors is quite modest in many cases". We therefore next compare the performances of the tests in the fixed-Σ design, where we can control the level of endogeneity ρ_{uv} explicitly and can take a closer look at how the tests perform in low to moderate endogeneity settings. This design also enables us to better investigate the behaviour of the tests when ρ_{uv} and β have the same sign. The performances of these tests have not been compared using the fixed-Σ design in the literature before.

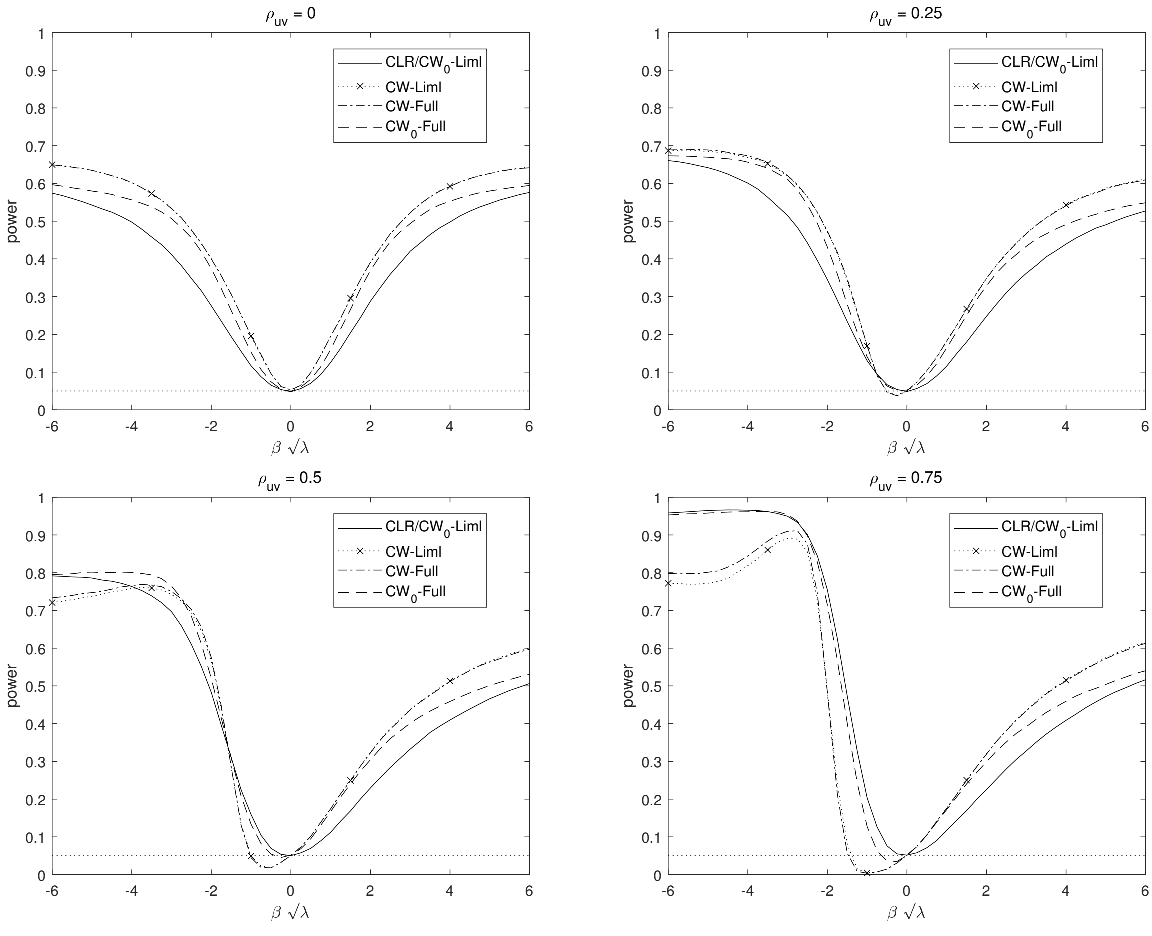

Figure 6: Weak-instrument asymptotic power of tests, fixed-Σ design, $k_z = 5$ and $\lambda = 10$, for different values of ρ_{uv}.

Figure 6 shows the power curves for values of $\rho_{uv} = 0, 0.25, 0.50, 0.75$, with $\sigma_u^2 = \sigma_v^2 = 1$, testing H$_0$: $\beta = 0$. As above, $k_z = 5$, but here the instrument strength is $\lambda = 10$. Appendix C further presents the power curves for $\lambda = 2.5$, 5 and 20. We follow here the practice in the literature to report the rejection frequencies of the tests as a function of $\beta\sqrt{\lambda}$. At low levels of endogeneity, $\rho_{uv} = 0$ and $\rho_{uv} = 0.25$, the behaviour of the CW-Liml and CW-Fuller tests are virtually identical and they are the most powerful across the range of values of $\beta\sqrt{\lambda}$ when $\rho_{uv} = 0$. For $\rho_{uv} = 0.25$ they are also most powerful, but for a small bias of the tests for small negative values of $\beta\sqrt{\lambda}$. The bias of the CW tests increases with increasing values of ρ_{uv}. The CW_0-Fuller test is less biased than the CW tests. Its power dominates that of the CLR test at the lower endogeneity levels $\rho_{uv} = 0$ and $\rho_{uv} = 0.25$. At the higher level of $\rho_{uv} = 0.5$, the power of the CW_0-Fuller test also dominates that of the CLR test except for some negative values of $\beta\sqrt{\lambda}$ close to 0, and where the difference in power between the two tests is small. At the higher level of endogeneity, $\rho_{uv} = 0.75$, the CW-tests and the CW_0-Fuller have more power than the CLR test for positive values of $\beta\sqrt{\lambda}$, whereas the CLR test dominates at negative values of $\beta\sqrt{\lambda}$.

These results show that the biased CW and CW_0-Fuller tests can have more power than the CLR test in low to moderate endogeneity environments, in which case there is also only a small to moderate bias in these tests. This seems an important observation, as this is a situation that may well be encountered in practice as documented by Chernozhukov and Hansen (2008). Also, for all values of $\beta > 0$ and the values of $\rho_{uv} > 0$ considered here, the CW tests are the most powerful. Therefore, for a researcher who believes both β and ρ_{uv} to be both positive, or both negative, these tests are good options in order to detect an effect.

6 Conclusions

We have compared and contrasted the performances of the Conditional Likelihood Ratio and related tests in the fixed-Σ and fixed-Ω designs. Due to the changing endogeneity properties as a function of β in the fixed-Ω design, this design is less suited to show the differences in the properties of the tests in low to moderate endogeneity environments and where the structural correlation ρ_{uv} and β have the same sign, when testing H_0: $\beta = 0$. These characteristics can be controlled directly in the fixed-Σ design, and it is shown more clearly that the LIML- and Fuller-based conditional Wald tests have more power than the CLR test in these circumstances when analysed using the fixed-Σ design. We have also shown that for the the fixed-Ω design, setting $\beta_0 = 0$ in H_0: $\beta = \beta_0$ or the diagonal elements of Ω equal to 1 is not without loss of generality. For the fixed-Σ design, one can set $\beta_0 = 0$ and the diagonal elements of Σ equal to 1 without loss of generality, making the power curves in the latter design more generally applicable.

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Appendix

A Proofs

A.1 Proof of Proposition 1

Alternative expressions for $\widehat{\pi}_L$ are,

$$\widehat{\pi}_L = \left(Z'M_{\widehat{u}_L}Z\right)^{-1} Z'M_{\widehat{u}_L}x \quad \text{(A.1)}$$

$$= \widehat{\pi} - \frac{\left(Z'Z\right)^{-1} Z'\widehat{u}_L\left(\widehat{\omega}_{12} - \widehat{\beta}_L\widehat{\omega}_{22}\right)}{\widehat{\sigma}_L^2}, \quad \text{(A.2)}$$

where $\widehat{u}_L = y - x\widehat{\beta}_L$ and $\widehat{\sigma}_L^2 = \widehat{u}_L'M_Z\widehat{u}_L/n = b_L'\widehat{\Omega}b_L$, with $b_L = (1\ -\widehat{\beta}_L)'$. Expression (A.1) is the standard expression as given in e.g. Bowden and Turkington (1984, p 108), from which (A.2) can be derived, see also Hausman (1983, p 424). The result of Proposition 1 follows as $\widehat{\pi}_{L0}$ can alternatively be expressed as

$$\widehat{\pi}_{L0} = \widehat{\pi} - \frac{\left(Z'Z\right)^{-1} Z'u_0\left(\widehat{\omega}_{12} - \beta_0\widehat{\omega}_{22}\right)}{\widehat{\sigma}_0^2},$$

following (A.2), linking the definitions of Moreira (2003) and Kleibergen (2002). Further,

$$V\widehat{a}r\left(\widehat{\pi}\right) - V\widehat{a}r\left(\widehat{\pi}_{L0}\right) = \left(\widehat{\omega}_{22} - \left(a_0'\widehat{\Omega}^{-1}a_0\right)^{-1}\right)\left(Z'Z\right)^{-1},$$

and $a_0'\widehat{\Omega}^{-1}a_0 = \widehat{\sigma}_0^2/\left|\widehat{\Omega}\right|$. It follows that $\widehat{\omega}_{22} - \left(a_0'\widehat{\Omega}^{-1}a_0\right)^{-1} = \left(\widehat{\omega}_{12} - \beta_0\widehat{\omega}_{22}\right)^2/\widehat{\sigma}_0^2$ and so $H_\pi = AR$.

A.2 Proof of Proposition 2

It follows from (A.2) that

$$\begin{aligned} \widehat{u}_L'P_Zx &= \widehat{u}_L'Z\widehat{\pi} = \widehat{u}_L'Z\widehat{\pi}_L + \frac{\widehat{u}_L'P_Z\widehat{u}_L}{\widehat{\sigma}_L^2}\left(\widehat{\omega}_{12} - \widehat{\beta}_L\widehat{\omega}_{22}\right) \\ &= B\left(\widehat{\beta}_L\right)\left(\widehat{\omega}_{12} - \widehat{\beta}_L\widehat{\omega}_{22}\right), \end{aligned}$$

as $\widehat{u}_L'Z\widehat{\pi}_L = \widehat{u}_L'\widehat{x}_L = 0$.

As $u_0 = \widehat{u}_L + x\left(\widehat{\beta}_L - \beta_0\right)$, it follows that

$$\begin{aligned}\frac{u_0' P_Z u_0}{\widehat{\sigma}_0^2} &= \frac{\widehat{u}_L' P_Z \widehat{u}_L + 2\widehat{u}_L' P_Z x\left(\widehat{\beta}_L - \beta_0\right) + \left(\widehat{\beta}_L - \beta_0\right)^2 x' P_Z x}{\widehat{\sigma}_0^2} \\ &= \frac{\widehat{u}_L' P_Z \widehat{u}_L}{\widehat{\sigma}_0^2} + \frac{2B\left(\widehat{\beta}_L\right)\left(\widehat{\beta}_L - \beta_0\right)}{\widehat{\sigma}_0^2}\left(\widehat{\omega}_{12} - \widehat{\beta}_L\widehat{\omega}_{22}\right) + \frac{\left(\widehat{\beta}_L - \beta_0\right)^2 x' P_Z x}{\widehat{\sigma}_0^2}.\end{aligned}$$

Further

$$\begin{aligned}\frac{\widehat{u}_L' P_Z \widehat{u}_L}{\widehat{\sigma}_0^2} - \frac{\widehat{u}_L' P_Z \widehat{u}_L}{\widehat{\sigma}_L^2} &= \frac{\widehat{u}_L' P_Z \widehat{u}_L}{\widehat{\sigma}_L^2 \widehat{\sigma}_0^2}\left(\widehat{\sigma}_L^2 - \widehat{\sigma}_0^2\right) \\ &= \frac{B\left(\widehat{\beta}_L\right)}{\widehat{\sigma}_0^2}\left(\left(\widehat{\beta}_L^2 - \beta_0^2\right)\widehat{\omega}_{22} - 2\left(\widehat{\beta}_L - \beta_0\right)\widehat{\omega}_{12}\right).\end{aligned}$$

As

$$\left(\widehat{\beta}_L^2 - \beta_0^2\right)\widehat{\omega}_{22} = -\left(\widehat{\beta}_L - \beta_0\right)^2 \widehat{\omega}_{22} + 2\widehat{\beta}_L\left(\widehat{\beta}_L - \beta_0\right)\widehat{\omega}_{22},$$

and $B\left(\widehat{\beta}_L\right) = n\widehat{\kappa}$, it follows that

$$\begin{aligned}\frac{u_0' P_Z u_0}{\widehat{\sigma}_0^2} - \frac{\widehat{u}_L' P_Z \widehat{u}_L}{\widehat{\sigma}_L^2} &= \frac{\left(\widehat{\beta}_L - \beta_0\right)^2\left(x' P_Z x - B\left(\widehat{\beta}_L\right)\widehat{\omega}_{22}\right)}{\widehat{\sigma}_0^2} \\ &= \frac{\left(\widehat{\beta}_L - \beta_0\right)^2\left(x' P_Z x - n\widehat{\kappa}\widehat{\omega}_{22}\right)}{\widehat{\sigma}_0^2} \\ &= W_0\left(\widehat{\beta}_L\right).\end{aligned}$$

A.3 Proof of Result 1

Consider the model specification with $\beta = \beta_*$,

$$\begin{aligned} y &= x\beta_* + u \\ x &= Z\pi + v. \end{aligned} \tag{A.3}$$

Then the LIML estimator is given by

$$\begin{aligned}\widehat{\beta}_L &= \left(x'\left(I_n - (\widehat{\kappa}+1)\, M_Z\right) x\right)^{-1} x'\left(I_n - (\widehat{\kappa}+1)\, M_Z\right) y \\ &= \beta_* + \left(x'\left(I_n - (\widehat{\kappa}+1)\, M_Z\right) x\right)^{-1} x'\left(I_n - (\widehat{\kappa}+1)\, M_Z\right) u.\end{aligned}$$

For testing H$_0$: $\beta = \beta_0$, we have that $u_0 = y - x\beta_0$, and $\widehat{u}_L = y - x\widehat{\beta}_L$.

Next, consider a change in the parameter value only, $\beta = \beta_* + \gamma$, ceteris paribus

$$\begin{aligned}y_\gamma &= x\left(\beta_* + \gamma\right) + u \\ x &= Z\pi + v.\end{aligned} \tag{A.4}$$

It follows that for testing H$_0$: $\beta = \beta_0 + \gamma$, we have that

$$u_{0,\gamma} = y_\gamma - x\left(\beta_0 + \gamma\right) = u_0.$$

Let $W = [y\ x]$ and $W_\gamma = [y_\gamma\ x]$. Let $\widehat{\Omega}_\gamma = W_\gamma' M_Z W_\gamma / n$ and

$$\widehat{\kappa}_\gamma = \min \text{eval}\left(\left(n^{-1} W_\gamma' P_Z W_\gamma\right)\left(\widehat{\Omega}_\gamma\right)^{-1}\right).$$

Then the LIML estimator in model (A.4) is given by

$$\widehat{\beta}_{L,\gamma} = \beta_* + \gamma + \left(x'\left(I_n - (\widehat{\kappa}_\gamma+1)\, M_Z\right) x\right)^{-1} x'\left(I_n - (\widehat{\kappa}_\gamma+1)\, M_Z\right) u.$$

As

$$W_\gamma = W\Gamma;\ \Gamma = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix},$$

it follows that

$$\begin{aligned}\left(n^{-1} W_\gamma' P_Z W_\gamma\right)\left(\widehat{\Omega}_\gamma\right)^{-1} &= \left(n^{-1}\Gamma' W' P_Z W \Gamma\right)\left(n^{-1}\Gamma' W' M_Z W \Gamma\right)^{-1} \\ &= \Gamma'\left(n^{-1} W' P_Z W\right)\widehat{\Omega}^{-1}\left(\Gamma'\right)^{-1}\end{aligned}$$

and so $\widehat{\kappa}_\gamma = \widehat{\kappa}$ and $\widehat{\beta}_{L,\gamma} = \widehat{\beta}_L + \gamma$. It therefore follows that

$$\widehat{u}_{L,\gamma} = y_\gamma - x\widehat{\beta}_{L,\gamma} = y - x\widehat{\beta}_L = \widehat{u}_L.$$

Denote by $AR(\beta_0)_{\beta=\beta_*}$ and $LR(\beta_0)_{\beta=\beta_*}$ the test statistics for testing $H_0 : \beta = \beta_0$ when $\beta = \beta_*$, then it follows, ceteris paribus, that

$$
\begin{aligned}
AR(\beta_0 + \gamma)_{\beta=\beta_*+\gamma} &= AR(\beta_0)_{\beta=\beta_*} \\
LR(\beta_0 + \gamma)_{\beta=\beta_*+\gamma} &= LR(\beta_0)_{\beta=\beta_*}
\end{aligned}
$$

Further, for testing H$_0$: $\beta = \beta_0$ when $\beta = \beta_*$, we have

$$
\tau_0(\beta_0)_{\beta=\beta_*} = \frac{a_0'\widehat{\Omega}^{-1}W'P_Z W\widehat{\Omega}^{-1}a_0}{a_0'\widehat{\Omega}^{-1}a_0},
$$

where $a_0 = (\beta_0 \ \ 1)'$. Then for testing H$_0$: $\beta = \beta_0 + \gamma$ when $\beta = \beta_* + \gamma$, denoting $a_{0,\gamma} = (\beta_0 + \gamma \ \ 1)'$,

$$
\begin{aligned}
\tau_0(\beta_0 + \gamma)_{\beta=\beta_*+\gamma} &= \frac{a_{0,\gamma}'\left(\widehat{\Omega}_\gamma\right)^{-1}W_\gamma'P_Z W_\gamma\left(\widehat{\Omega}_\gamma\right)^{-1}a_{0,\gamma}}{a_{0,\gamma}'\left(\widehat{\Omega}_\gamma\right)^{-1}a_{0,\gamma}} \\
&= \frac{a_{0,\gamma}'\Gamma^{-1}\widehat{\Omega}^{-1}W'P_Z W\widehat{\Omega}^{-1}\left(\Gamma'\right)^{-1}a_{0,\gamma}}{a_{0,\gamma}'\Gamma^{-1}\widehat{\Omega}^{-1}\left(\Gamma'\right)^{-1}a_{0,\gamma}} \\
&= \tau_0(\beta_0)_{\beta=\beta_*},
\end{aligned}
$$

as

$$
\left(\Gamma'\right)^{-1}a_{0,\gamma} = \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix}\begin{pmatrix} \beta_0 + \gamma \\ 1 \end{pmatrix} = a_0.
$$

It therefore follows that, ceteris paribus, the LR test statistic and its conditional p-value for testing H$_0$: $\beta = \beta_0$ when $\beta = \beta_*$ are identical to the test statistic and its conditional p-value for testing H$_0$: $\beta = \beta_0 + \gamma$ when $\beta = \beta_* + \gamma$.

A.4 Derivation of Noncentrality Parameters

A.4.1 Fixed-Σ Design

As

$$\begin{aligned}\sigma_0^2(\beta) &:= b_0'\Omega(\beta)\,b_0 = \begin{pmatrix} 1 & -\beta_0 \end{pmatrix} \begin{bmatrix} \sigma_u^2 + 2\beta\sigma_{uv} + \beta^2\sigma_v^2 & \sigma_{uv} + \beta\sigma_v^2 \\ \sigma_{uv} + \beta\sigma_v^2 & \sigma_v^2 \end{bmatrix} \begin{pmatrix} 1 \\ -\beta_0 \end{pmatrix} \\ &= \sigma_u^2 + 2\sigma_{uv}\delta + \sigma_v^2\delta^2,\end{aligned}$$

with $\delta = \beta - \beta_0$, it follows that

$$\begin{aligned}c_{\beta,\beta_0}^2\lambda &= \frac{\lambda\delta^2}{\sigma_0^2(\beta)} = \frac{\lambda\delta^2}{\sigma_u^2 + 2\rho_{uv}\sigma_u\sigma_v\delta + \sigma_v^2\delta^2} \\ &= \frac{\lambda/\sigma_v^2}{\left(\frac{\sigma_u}{\delta\sigma_v}\right)^2 + 2\rho_{uv}\frac{\sigma_u}{\delta\sigma_v} + 1}.\end{aligned}$$

Further,

$$\begin{aligned}b'\Omega(\beta)\,b_0 &= (1\ -\beta) \begin{bmatrix} \sigma_u^2 + 2\beta\sigma_{uv} + \beta^2\sigma_v^2 & \sigma_{uv} + \beta\sigma_v^2 \\ \sigma_{uv} + \beta\sigma_v^2 & \sigma_v^2 \end{bmatrix} \begin{pmatrix} 1 \\ -\beta_0 \end{pmatrix} \\ &= \sigma_u^2 + \delta\sigma_{uv},\end{aligned}$$

and so

$$\begin{aligned}d_{\beta,\beta_0}^2\lambda &= \frac{\lambda\left(b'\Omega(\beta)\,b_0\right)^2}{\sigma_0^2(\beta)\,|\Omega(\beta)|} = \frac{\lambda\left(\sigma_u^2 + \delta\sigma_{uv}\right)^2}{\left(\sigma_u^2 + 2\sigma_{uv}\delta + \sigma_v^2\delta^2\right)\sigma_u^2\sigma_v^2\left(1 - \rho_{uv}^2\right)} \\ &= \frac{\left(\lambda/\sigma_v^2\right)\left(\frac{\sigma_u}{\delta\sigma_v} + \rho_{uv}\right)^2}{\left(\left(\frac{\sigma_u}{\delta\sigma_v}\right)^2 + 2\rho_{uv}\frac{\sigma_u}{\delta\sigma_v} + 1\right)\left(1 - \rho_{uv}^2\right)}.\end{aligned}$$

Finally,

$$\begin{aligned}c_{\beta,\beta_0}d_{\beta,\beta_0}\lambda &= \frac{\lambda\delta\left(\sigma_u^2 + \delta\sigma_{uv}\right)}{\sigma_0^2(\beta)\,|\Omega(\beta)|^{1/2}} = \frac{\lambda\delta\left(\sigma_u^2 + \delta\sigma_{uv}\right)}{\left(\sigma_u^2 + 2\sigma_{uv}\delta + \sigma_v^2\delta^2\right)\sigma_u\sigma_v\sqrt{\left(1 - \rho_{uv}^2\right)}} \\ &= \frac{\left(\lambda/\sigma_v^2\right)\left(\frac{\sigma_u}{\delta\sigma_v} + \rho_{uv}\right)}{\left(\left(\frac{\sigma_u}{\delta\sigma_v}\right)^2 + 2\rho_{uv}\frac{\sigma_u}{\delta\sigma_v} + 1\right)\sqrt{\left(1 - \rho_{uv}^2\right)}}.\end{aligned}$$

A.4.2 Fixed-Ω Design

We now have

$$\sigma_0^2 := b_0'\Omega b_0 = \begin{pmatrix} 1 & -\beta_0 \end{pmatrix} \begin{bmatrix} \sigma_r^2 & \sigma_{rv} \\ \sigma_{rv} & \sigma_v^2 \end{bmatrix} \begin{pmatrix} 1 \\ -\beta_0 \end{pmatrix}$$
$$= \sigma_r^2 - 2\beta_0\sigma_{rv} + \beta_0^2\sigma_v^2,$$

and so, with $\rho_\Omega := \rho_{rv} = \frac{\sigma_{rv}}{\sigma_r\sigma_v}$,

$$c_{\beta,\beta_0}^2\lambda = \frac{\lambda\delta^2}{\sigma_0^2} = \frac{\lambda\delta^2}{\sigma_r^2 - 2\beta_0\sigma_{rv} + \beta_0^2\sigma_v^2}$$
$$= \frac{\lambda/\sigma_v^2}{\left(\frac{\sigma_r}{\delta\sigma_v}\right)^2 - 2\frac{\beta_0}{\delta}\rho_\Omega\frac{\sigma_r}{\delta\sigma_v} + \left(\frac{\beta_0}{\delta}\right)^2}.$$

Further, $|\Omega| = \sigma_r^2\sigma_v^2 - \sigma_{rv}^2 = \sigma_r^2\sigma_v^2(1-\rho_\Omega^2)$, and $b'\Omega b_0 = \sigma_r^2 - (\beta+\beta_0)\sigma_{rv} + \beta\beta_0\sigma_v^2 = \sigma_r^2 - \delta\sigma_{rv} + \beta_0(\beta\sigma_v^2 - 2\sigma_{rv})$, hence

$$d_{\beta,\beta_0}^2\lambda = \frac{\lambda\left(\sigma_r^2 - \delta\sigma_{rv} + \beta_0\left(\beta\sigma_v^2 - 2\sigma_{rv}\right)\right)^2}{\left(\sigma_r^2 - 2\beta_0\sigma_{rv} + \beta_0^2\sigma_v^2\right)\sigma_r^2\sigma_v^2\left(1-\rho_\Omega^2\right)}$$
$$= \frac{\left(\lambda/\sigma_v^2\right)\left(\frac{\sigma_r}{\delta\sigma_v} - \rho_\Omega + \frac{\beta_0}{\delta}\left(\beta\frac{\sigma_v}{\sigma_r} - 2\rho_\Omega\right)\right)^2}{\left(\left(\frac{\sigma_r}{\delta\sigma_v}\right)^2 - 2\frac{\beta_0}{\delta}\rho_\Omega\frac{\sigma_r}{\delta\sigma_v} + \left(\frac{\beta_0}{\delta}\right)^2\right)\left(1-\rho_\Omega^2\right)},$$

and

$$c_{\beta,\beta_0}d_{\beta,\beta_0}\lambda = \frac{\lambda\delta\left(\sigma_r^2 - \delta\sigma_{rv} + \beta_0\left(\beta\sigma_v^2 - 2\sigma_{rv}\right)\right)}{\left(\sigma_r^2 - 2\beta_0\sigma_{rv} + \beta_0^2\sigma_v^2\right)\sigma_r\sigma_v\sqrt{(1-\rho_\Omega^2)}}$$
$$= \frac{\left(\lambda/\sigma_v^2\right)\left(\frac{\sigma_r}{\delta\sigma_v} - \rho_\Omega + \frac{\beta_0}{\delta}\left(\beta\frac{\sigma_v}{\sigma_r} - 2\rho_\Omega\right)\right)}{\left(\left(\frac{\sigma_r}{\delta\sigma_v}\right)^2 - 2\frac{\beta_0}{\delta}\rho_\Omega\frac{\sigma_r}{\delta\sigma_v} + \left(\frac{\beta_0}{\delta}\right)^2\right)\sqrt{(1-\rho_\Omega^2)}}$$

B Size Distortion of the LR Test

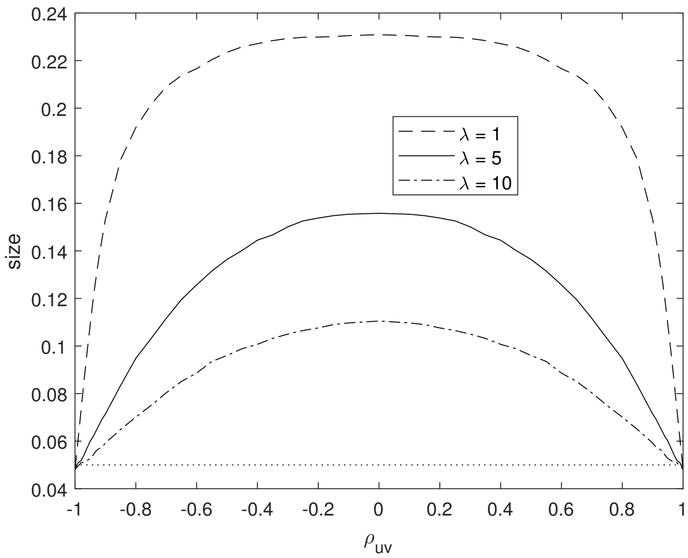

Figure B.1: Size properties of LR test using 5% critical value of the χ^2_1 distribution, for different values of ρ_{uv} and instrument strength λ. $\sigma^2_u = \sigma^2_v = 1$, $k_z = 5$.

C Power Curves of CLR/CW_0-Liml, CW-Liml, CW-Fuller and CW_0-Fuller Tests

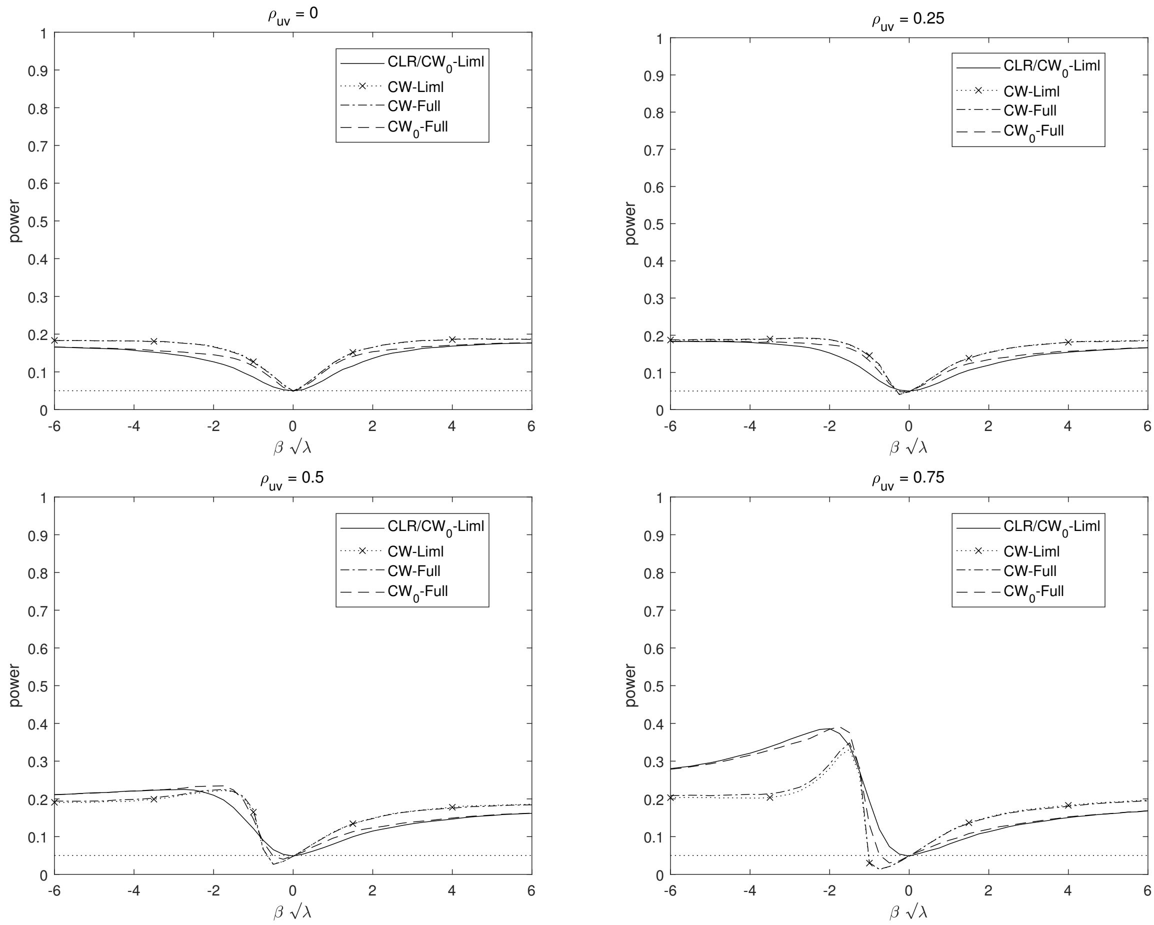

Figure C.1: $\lambda = 2.5$, $k_z = 5$.

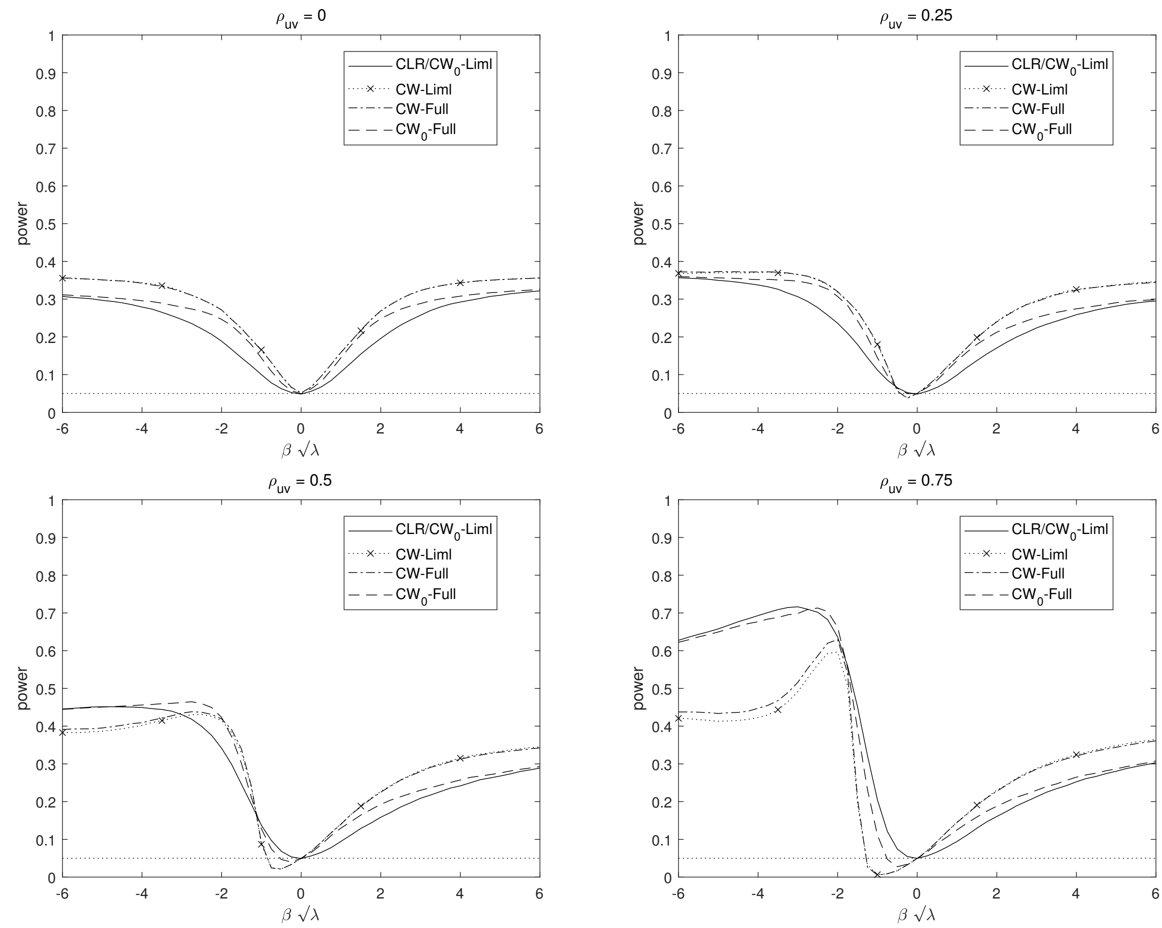

Figure C.2: $\lambda = 5$, $k_z = 5$.

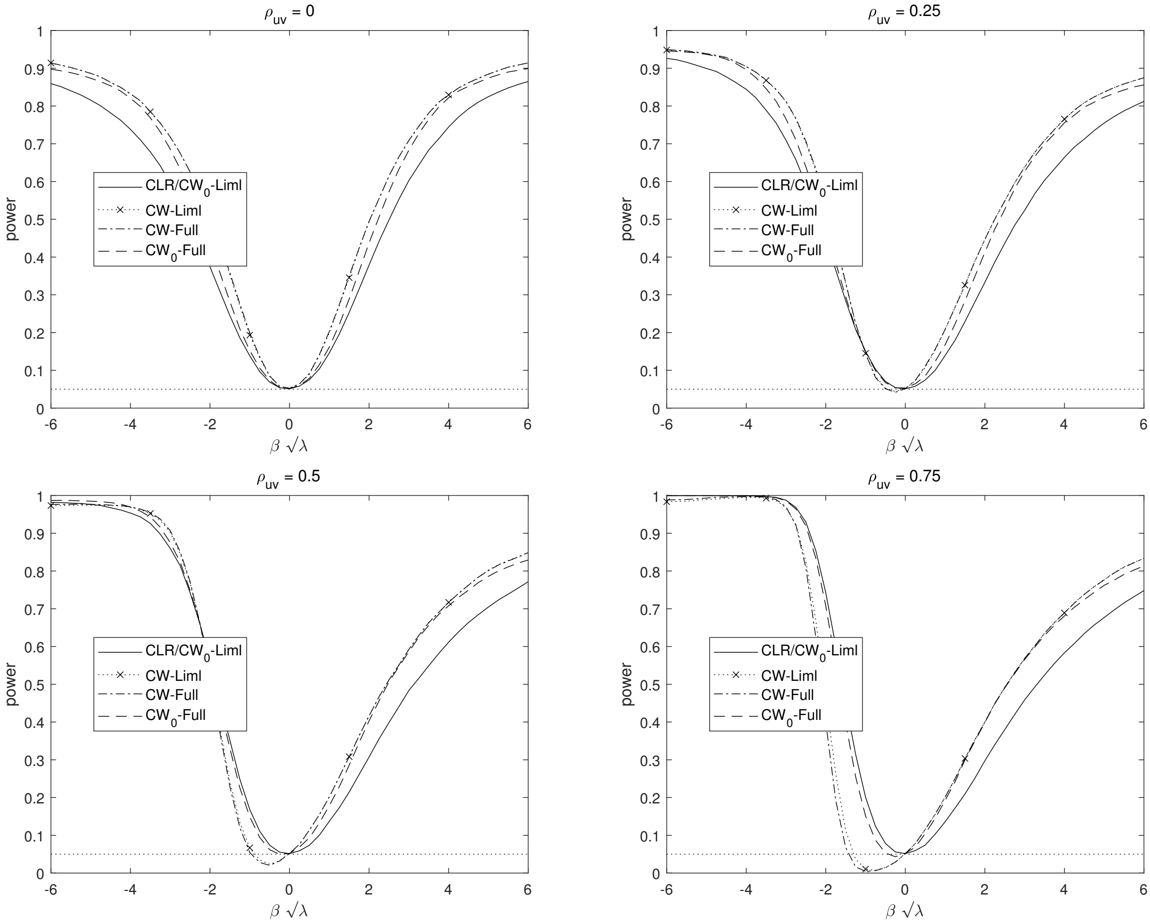

Figure C.3: $\lambda = 20$, $k_z = 5$.