Bargaining

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Bargaining: introduction

- **Bargaining**: exchange situation, or production-and-exchange situation, in which all parties have some control over terms of trade
- *Examples*: Firm + union; debtor nations + creditors; politicians; firms + regulator; plaintiff + defendant; buyers + sellers of houses, cars, antiques, etc.
- These examples involve "thin" markets, hence imperfect competition. Possible reasons for "thinness" of markets:
  - inherently small numbers of participants
  - search costs
  - transaction-specific investments
- With inherently small numbers, search costs, or transaction-specific investments, bargainers have limited ability to turn to alternative trading partners (limited "outside options").
- In the extreme case, none of the parties has any alternative trading partner. With two parties, this extreme case is "bilateral monopoly". This is the case we'll focus on.

Bargaining: introduction

Motivations for studying bargaining

- Characterize "rational" behavior in specific situations
- Explain and predict bargaining behavior
  - e.g. Why do we see strikes? What determines whether lawsuits result in trials or in out-of-court settlements?
- Reduce bargaining inefficiencies, such as delays or failures
  - e.g. by changing procedures or distribution of information
- Provide building blocks, based on strategic behavior, for theories of price formation in specific markets
  - e.g. housing markets, financial markets

Two approaches adopted for analyzing bargaining

- axiomatic (cooperative game theory)
- strategic (noncooperative game theory)

Axiomatic approach

- *"One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely."* (Nash, 1953)
  - uses minimal information about bargaining environment and says nothing about procedure
  - provides map from preferences of players and feasible set to bargaining outcomes
  - intended to apply to a wide range of bargaining situations
- Advantages
  - easy to apply and computationally straightforward
- Disadvantages
  - How to evaluate reasonableness of axioms?
  - Which cooperative solution concept to apply when?
  - How to modify a given solution concept to reflect a change in bargaining situation/procedure?

Strategic approach

- *"One makes the players' steps of negotiation ... moves in the non-cooperative model... The negotiation process must be formalized and restricted, but in such a way that each participant is still able to utilize all the essential strengths of his position."* (Nash, 1953)
  - bargaining environment and procedure represented in an extensive-form game
  - equilibrium concept defined, and equilibria calculated
- Advantages
  - explicitly analyses bargaining *behavior*, not just outcomes
  - can be used to study effects of changes in rules
- Disadvantages
  - for a given game, may be multiple equilibria
  - eqm. may be very sensitive to particular extensive-form game analysed. What regularities are robust to rule changes?

The "Nash Program"

1. *"[t]he two approaches to the problem, via the negotiation model or via the axioms, are complementary; each helps to justify and clarify the other"* (Nash, 1953)

2. Nash Program: relate axiomatic bargaining solutions to equilibria of strategic models

3. We will illustrate the Nash program by showing the close relationship between the *Nash Bargaining Solution* and the unique subgame perfect equilibrium of *Rubinstein's model of alternating-offer bargaining*.

Nash Bargaining Solution (NBS)

- 2 players, $i = 1, 2$
- Given the primitives:
  - set A of possible agreements
  - disagreement event D
  - utility functions $u_i : A \cup \{D\} \to \mathbb{R}$
- Define:
  - $S \equiv$ set of all pairs $(u_1(a), u_2(a))$ for $a \in A$
  - $d \equiv (u_1(D), u_2(D))$
- Definitions
  1. A *bargaining problem* is a pair $\langle S, d \rangle$ where $S \subset \mathbb{R}^2$ is compact and convex, $d \in S$, and there exists $s \in S$ such that $s_i > d_i$ for $i = 1, 2$. The set of all bargaining problems is denoted $\mathcal{B}$.
  2. A *bargaining solution* is a function $f : \mathcal{B} \to \mathbb{R}^2$ that assigns to each bargaining problem $\langle S, d \rangle \in \mathcal{B}$ a unique element of S. $(f_1(S, d), f_2(S, d))$ is the utility pair singled out by the bargaining solution f.

Nash's Axioms

1. **Invariance to Equivalent Utility Representations:** If $d_i' = \alpha_i d_i + \beta_i$, for $i = 1, 2$, and $S' = \left\{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 \ : \ (s_1, s_2) \in S\right\}$, where $\alpha_i > 0$ for $i = 1, 2$, then $f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$, for $i = 1, 2$.
   - *Interpretation:* Changing the numerical representation of the preferences, *without changing the attitudes towards lotteries*, should not change the nature of the solution.

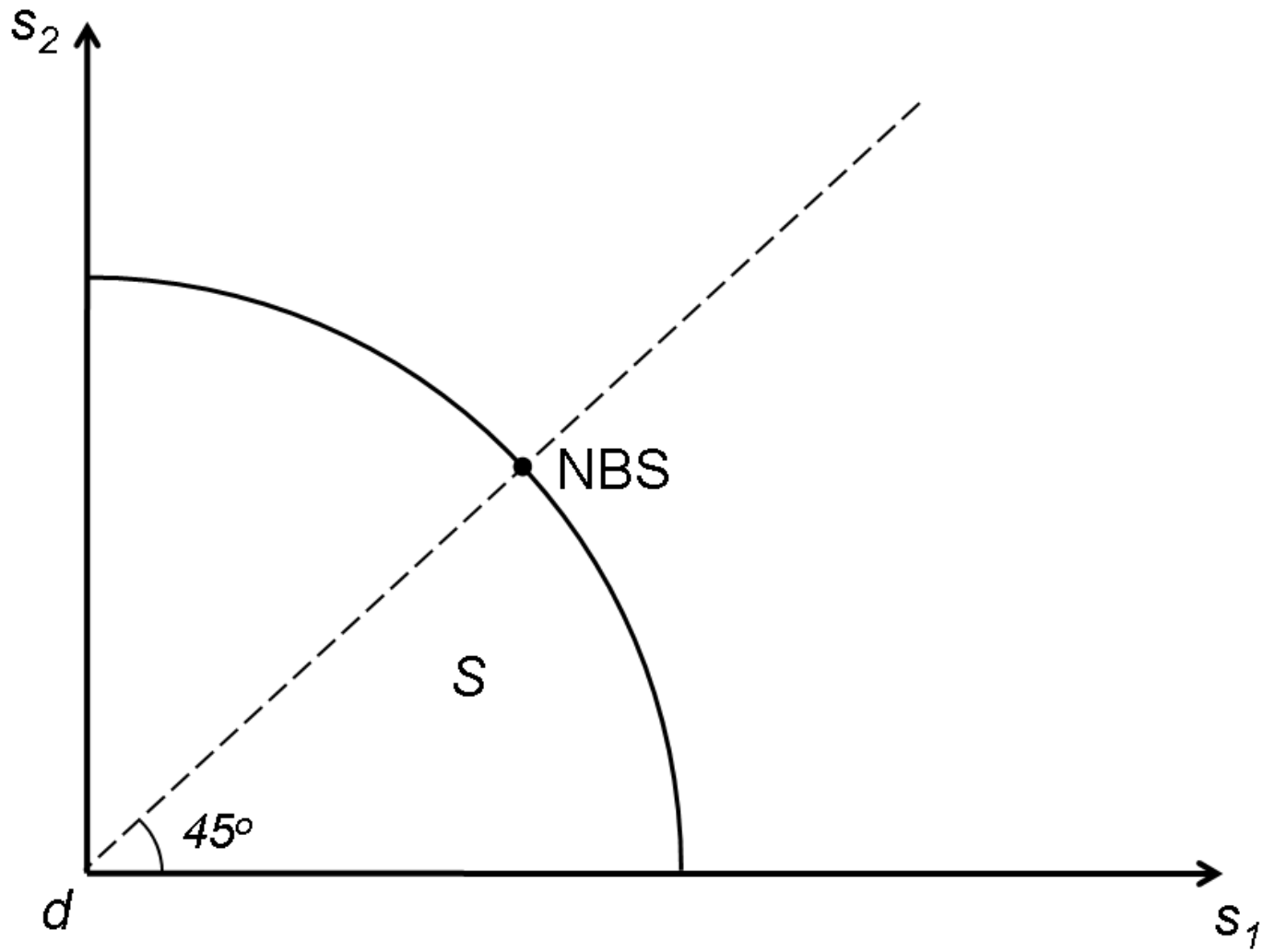
2. **Symmetry:** If S is symmetric ($(s_1, s_2) \in S \Leftrightarrow (s_2, s_1) \in S$) and $d_1 = d_2$, then $f_1(S, d) = f_2(S, d)$.
   - *Interpretation:* If $\langle S, d \rangle$ is symmetric, there is no information distinguishing one player from the other, so the solution should not distinguish between players.

Nash's Axioms

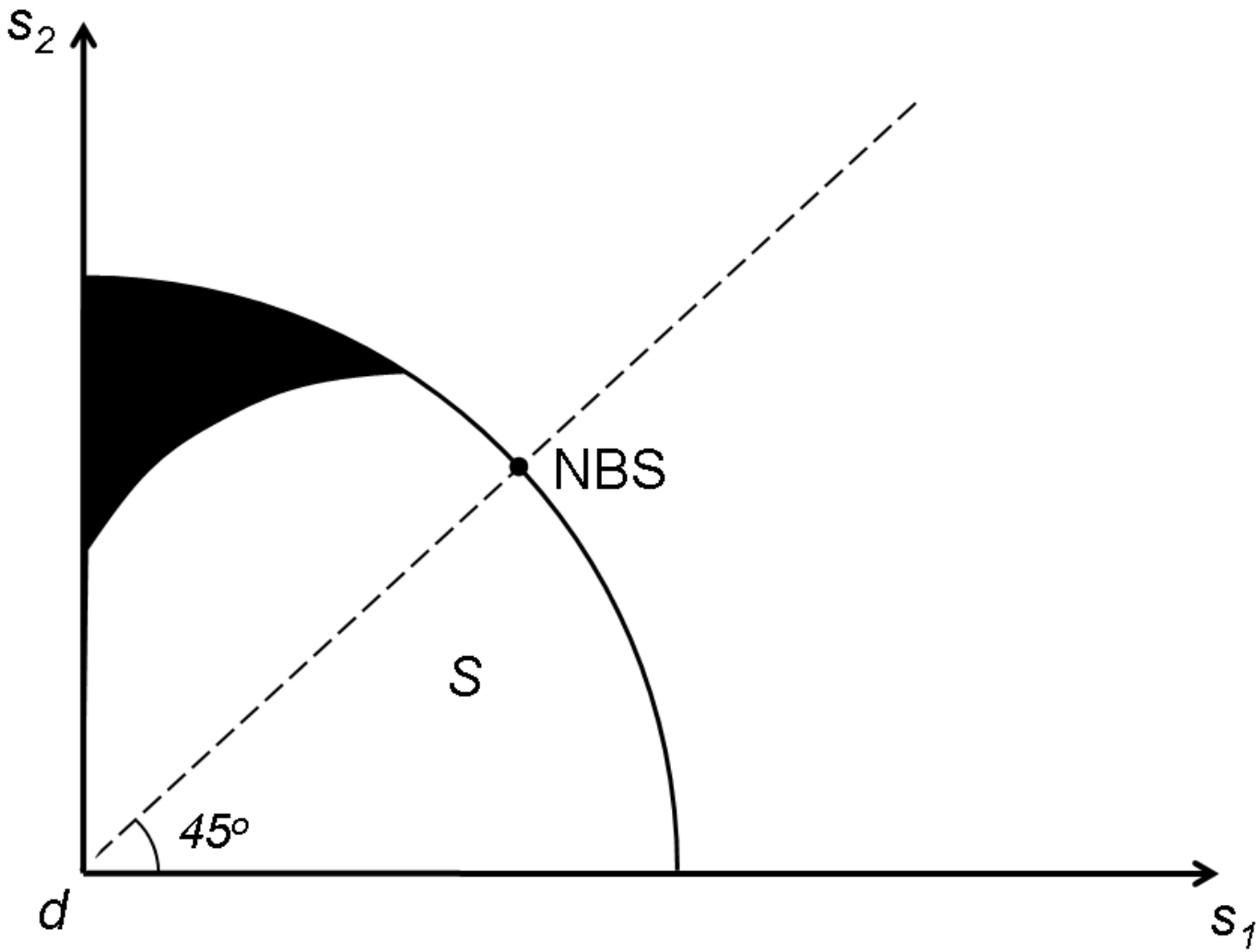
3. **Independence of Irrelevant Alternatives:** If $\langle S, d \rangle$ and $\langle T, d \rangle$ are bargaining problems with $S \subset T$ and $f(T, d) \in S$, then $f(S, d) = f(T, d)$.
   - *Interpretation:* If alternatives in T but not in S were not selected when feasible set was T, then when these alternatives are no longer available (because not in S), the solution should be unchanged.

4. **Pareto Efficiency:** Suppose $\langle S, d \rangle$ is a bargaining problem, $s \in S$, $t \in S$, and $t_i > s_i$ for $i = 1, 2$. Then $f(S, d) \neq s$.
   - *Interpretation:* The solution should distribute all of the potential benefits obtainable from the cooperation.

NBS: Symmetry and Pareto Efficiency



NBS: Independence of Irrelevant Alternatives



Nash Bargaining Solution

Theorem

There is a unique bargaining solution $f^N : \mathcal{B} \to \mathbb{R}^2$ satisfying Nash's 4 axioms, and f^N satisfies

$$f^N(S, d) = \arg \max_{d \leq s \in S} (s_1 - d_1)(s_2 - d_2).$$

$f^N(S, d)$ is called the Nash Bargaining Solution (NBS).

Strategy of proof:

- Show that $f^N(S, d)$ as defined above exists and is unique.
- Show that $f^N(S, d)$ satisfies Nash's 4 axioms.
- Show that any bargaining solution $f(S, d)$ satisfying the 4 axioms coincides with $f^N(S, d)$.

Illustration of Nash Bargaining Soln: dividing a pie of size 1

- Set of feasible physical agreements:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 + x_2 \leq 1, \, x_i \geq 0 \text{ for } i = 1, 2\}.$$

- $u_i(x_i) = x_i$ for $i = 1, 2$: players' utilities are linear in share
- Hence $S = \{(u_1, u_2) \in \mathbb{R}^2 \ : \ u_1 + u_2 \leq 1, \, u_i \geq 0 \text{ for } i = 1, 2\}$
- disagreement utility pair $= (d_1, d_2)$

- Then $f^N(S, d) = \arg\max_{d \leq s \in S}(s_1 - d_1)(s_2 - d_2)$ yields

$$f_i^N(S, d) = d_i + \frac{1}{2}(1 - d_1 - d_2), \qquad i = 1, 2$$

Mapping B-S bargaining into division of a pie of size 1

- Buyer (B) and seller (S) bargain over (q, p), where q is quantity produced and traded and p is payment from B to S.
- Given an agreement on (q, p),
  - S's payoff is $p - C(q)$, where $C'(q) > 0$, $C''(q) > 0$
  - B's payoff is $R(q) - p$, where $R'(q) > 0$, $R''(q) < 0$
- A pie of the maximum size would be generated by S producing q^*, where q^* solves $R'(q) = C'(q)$.
- An arbitrary agreement (q, p) maps into

$$x_1 = \frac{R(q) - p}{R(q^*) - C(q^*)} \quad \text{and} \quad x_2 = \frac{p - C(q)}{R(q^*) - C(q^*)},$$

  and we can take $s_i = x_i$, since players' utilities are linear in x_i.
- The set of (q, p) pairs representing agreements on the Pareto frontier of set S is $\{(q, p) | \, q = q^*, C(q^*) \leq p \leq R(q^*)\}$.

Rubinstein's (1982) model of alternating-offer bargaining

- 2 players, $i = 1, 2$, bargain over "pie" of size 1.
- An agreement: $x \equiv (x_1, x_2)$, where x_i is player i's share.
- Set of possible agreements
  $X = \{x \in \mathbb{R}^2 \ : \ x_1 + x_2 = 1, \ x_i \geq 0 \text{ for } i = 1, 2\}$.
- Game (bargaining procedure):
  - players can take actions only at times $0, 1, 2, 3, \ldots$
  - players alternate in making offers, with 1 making the 1st offer
  - player who receives offer may accept or reject
  - acceptance ends bargaining
  - rejection means game continues till next round, with rejector making the next offer
  - and so on, without any limit of number of rounds

Rubinstein's model (cont.)

- Outcome: either agreement (x, t) or perpetual disagreement D
  - NB: $(x, 0)$ denotes agreement on split (x_1, x_2) at $t = 0$.
- Assumptions about commitment embedded in rules:
  - players committed to any agreement once an offer is accepted
  - players committed to rules determining who offers when (each player can commit not to listen to a counteroffer until 1 period has elapsed following rejection of his offer)
  - players are not constrained by past offers
- Preferences over outcomes (x, t) : $U_i(x_i, t) = \delta_i^t x_i$, $0 < \delta_i < 1$, $U_i(D) = 0$
  - easily generalized to $U_i(x_i, t) = \delta_i^t u_i(x_i)$ with $u_i(\cdot)$ nonlinear
- Information:
  - players have complete information about each other's preferences
  - players are certain about set of feasible agreements
  - players know, at each stage, all previous moves
- Strategies: specify action at every node in game tree where it is player's turn to move

Nash equilibria in the bargaining game

Definition: A pair of strategies, σ for player 1 and τ for player 2, is a *Nash equilibrium* (NE) if, given that 2 plays τ, no strategy for 1 results in an outcome that 1 prefers to the outcome generated by (σ, τ) and, given σ, no strategy of 2 results in an outcome that 2 prefers to the outcome generated by (σ, τ).

- Nash eqm. concept evaluates optimality of a strategy *only* along the putative eqm. path; it does not test optimality of actions at nodes of the game tree that would not be reached under the putative eqm. strategies.

Claim: For every agreement $x \in X$, outcome $(x, 0)$ (split x at time 0) is generated by a NE of bargaining game.

Claim: There are NE outcomes which involve delayed agreements.

Implication of the claims: NE concept puts hardly any restrictions on outcomes in a bargaining game with alternating offers.

Nash equilibria in the bargaining game (cont.)

- **Claim:** For every agreement $x \in X$, outcome $(x, 0)$ is generated by a NE of bargaining game.
- **Proof:** Consider an arbitrary $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ and strategies:

$$\begin{aligned} \bar{\sigma}: &\quad 1 \text{ always proposes } \bar{x} \text{ and accepts offer iff } x_1 \geq \bar{x}_1 \\ \bar{\tau}: &\quad 2 \text{ always proposes } \bar{x} \text{ and accepts offer iff } x_2 \geq \bar{x}_2 \end{aligned}$$

  - If 1 uses $\bar{\sigma}$ and 2 uses $\bar{\tau}$, outcome is: 1 proposes $\bar{x}$ at $t = 0$; 2 accepts.
  - Given 2's strategy, 1 can not obtain a larger share than $\bar{x}_1 (= 1 - \bar{x}_2) \geq 0$, and, because $\delta_1 < 1$, prefers to receive a given share earlier rather than later. Since $\bar{x}_1 \geq 0$, the equilibrium outcome is no worse than perpetual disagreement, which results in payoff 0.
  - By a similar argument, given 1's strategy, 2 can do no better than by accepting 1's initial offer.

- **Claim:** There are NE outcomes which involve delayed agreements.
  - *Example*: After period 0, players use $(\bar{\sigma}, \bar{\tau})$ given above. In period 0, 1 offers $(x_1, x_2) = (1, 0)$, and 2 rejects every offer.

Subgame perfect equilibrium in the bargaining game

Players' rules for accepting/rejecting offers can be seen as *threats* to reject specific types of offers. Hence it is natural to strengthen the equilibrium concept to subgame perfect equilibrium (SPE).

- In a SPE, only *credible* threats (i.e. only threats that it would be in the threatener's interest to carry out) can influence the other player's behavior.

Definition: A strategy pair is a *subgame perfect equilibrium* (SPE) if the strategy pair it induces in every subgame is a NE of that subgame.

Claim: The NE strategies $(\bar{\sigma}, \bar{\tau})$ defined above do not constitute a SPE.

Proof: Suppose $\bar{x}_1 < 1$. Consider how 2 could respond if 1 made a larger demand at $t = 0$, i.e. if 1 offered $(\bar{x}_1 + \varepsilon, \bar{x}_2 - \varepsilon)$:

- 2 gets payoff
  - $\bar{x}_2 - \varepsilon$ if accepts at $t = 0$; or
  - $\delta_2 \bar{x}_2$ if rejects and continues to use $\bar{\tau}$, with 1 using $\bar{\sigma}$ (outcome is $(\bar{x}, 1)$)
- given $\delta_2 < 1$, $\exists\, \varepsilon$ sufficiently small that 2 would be better off accepting 1's offer of $(\bar{x}_1 + \varepsilon, \bar{x}_2 - \varepsilon)$
- hence $\bar{\tau}$ is not a best response in this subgame, so $(\bar{\sigma}, \bar{\tau})$ is not a SPE.

The threat implicit in 2's strategy $\bar{\tau}$ is not credible.

Existence and uniqueness of SPE

Strengthening the eqm. concept from NE to SPE has a dramatic effect: from essentially no prediction to a unique prediction.

Proposition

The strategies σ^* and τ^* constitute the unique SPE of the alternating-offer bargaining game:

$$\begin{aligned} \sigma^*: &\quad \textit{1 always proposes } x^* \textit{ and accepts } y \textit{ iff } y_1 \geq y_1^* \\ \tau^*: &\quad \textit{2 always proposes } y^* \textit{ and accepts } x \textit{ iff } x_2 \geq x_2^* \end{aligned}$$

where x^* and y^* are the unique solutions to the "indifference conditions"

$$y_1^* = \delta_1 x_1^* \quad \textit{and} \quad x_2^* = \delta_2 y_2^*. \tag{*}$$

The SPE outcome is that 1 proposes x^* at $t = 0$, and 2 accepts.

- Because players are impatient, proposer has some short-term monopoly power: there is a "first-mover advantage"
  - $x_1* > y_1^*$ since $\delta_1 < 1$
  - $y_2* > x_2^*$ since $\delta_2 < 1$

Sketch of proof of SPE proposition

1. Confirm that (σ^*, τ^*) yields outcome $(x^*, 0)$: straightforward
2. Confirm that (σ^*, τ^*) constitutes a SPE, i.e. that (σ^*, τ^*) induces a NE in every subgame:
   - In subgames following a rejection, player making offer would do strictly worse
     - if asked for less, since would be accepted
     - if asked for more, since would be rejected, and a less attractive agreement would be reached at a later date
   - In subgames following an offer, the acceptance/rejection rules in (σ^*, τ^*) are optimal, because given (*), each responder's rejection threshold is the offer that makes him indifferent between accepting and rejecting, given anticipated continuation behavior.
3. Confirm that (σ^*, τ^*) are the unique SPE strategies

Sketch of proof of SPE proposition (cont.)

3. Confirm that (σ^*, τ^*) are the unique SPE strategies:

- First consider a modified version of the game: Suppose players know that, if no agreement has been reached after $t = 1$, bargaining will stop and outcome will be $x = (z, 1 - z)$, for some *exogenously* given $z \in (0, 1)$.
- Since modified game has a *finite* horizon, SPE can be found by backward induction and is unique:
  - at $t = 1$, player 2 makes offer leaving 1 indifferent between accepting and rejecting: $y_1 = \delta_1 z$
  - at $t = 0$, player 1 makes offer leaving 2 indifferent between accepting and rejecting: $x_2 = \delta_2 y_2$
- So 1's SPE payoff would be $1 - \delta_2 y_2 = 1 - \delta_2[1 - \delta_1 z] \equiv x_1^*(z)$. Note that $x_1^*(z) \uparrow$ in z.
- Now recognize that z is actually endogenous....

Sketch of proof of SPE proposition (cont.)

3. Confirm that (σ^*, τ^*) are the unique SPE strategies (cont.):

- Now recognize that z is actually endogenous:
  - Let z^H be 1's highest continuation payoff in any SPE starting from $t = 2$. Then $x_1^*(z^H)$ is 1's highest payoff in any SPE from $t = 0$.
  - But the game beginning at $t = 2$ is identical to the game as a whole. Therefore $x_1^*(z^H) = z^H$, and the solution to this equation is 1's highest SPE payoff.
  - Now let z^L be 1's lowest continuation payoff in any SPE from $t = 2$. Analogous reasoning yields $x_1^*(z^L) = z^L$.
  - Hence $z^H = z^L$, so 1's SPE payoff is unique. Check that solution to $x_1^*(z) = z$ matches solution to $(*)$.
- Show that (σ^*, τ^*) are the only strategies forming a SPE and yielding the unique SPE payoffs.

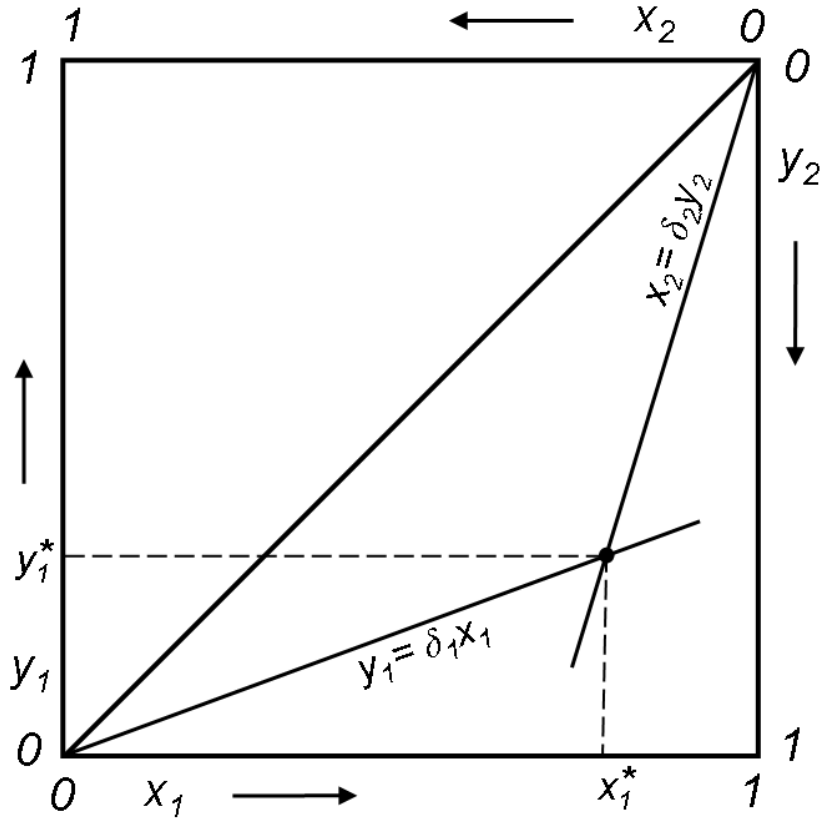
Q.E.D.

Discussion of the unique SPE

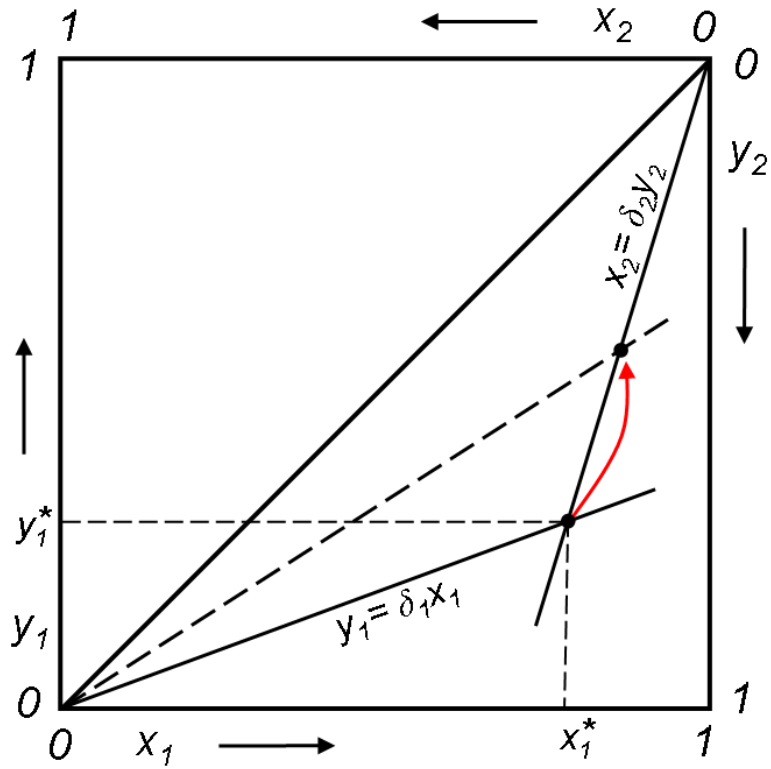
Bargaining ends immediately, so the outcome is Pareto efficient

- No costs of delay are incurred in equilibrium, but potential costs of delay (i.e. sizes of δ_1 and δ_2) do affect equilibrium payoffs
- How do SPE payoffs depend on δ_1 and δ_2?
  - $\delta_1 \uparrow \Longrightarrow x_1^* \uparrow$: player 1 benefits when becomes more patient, because 2 must offer more to make 1 indifferent between accepting and rejecting
  - $\delta_2 \downarrow \Longrightarrow x_1^* \uparrow$: player 1 benefits when 2 becomes less patient, because 1 needs to offer 2 less to make 2 indifferent between accepting and rejecting

Effect of players' patience on SPE strategies



Effect of increasing δ_1 on SPE strategies



$y_1^* = \delta_1 x_1^*$ and $x_2^* = \delta_2 y_2^*$, along with $x_1^* + x_2^* = 1$ and $y_1^* + y_2^* = 1$, yield

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2} \quad \text{and} \quad y_2^* = \frac{1-\delta_1}{1-\delta_1\delta_2}$$

In special case where $\delta_1 = \delta_2 \equiv \delta < 1$, $x_1^* = y_2^* = \frac{1}{1+\delta} > \frac{1}{2}$, and as $\delta \to 1$, $x_1^* = y_2^* \to \frac{1}{2}$ (first-mover advantage vanishes).

Significance of complete information

In contrast to this complete-information model, bargaining models with **incomplete information** do predict delay in reaching agreement.

- A player who is uncertain about opponent's preferences might find it optimal to make an offer that is accepted by some types of opponent and rejected by other types.
- So clearly incomplete information can generate a positive probability of delay. But more subtle to determine how much delay will arise.

Alternative motivation to agree: risk of breakdown

Players may be motivated to reach agreement by exogenous risk that opportunity will disappear ("breakdown"):

- Suppose that, after each rejection, "breakdown" B occurs with prob. $q \in (0,1)$; with prob. $1-q$, bargaining continues.
- Strategies that lead to agreement x at time t if no breakdown occurs generate a lottery $\langle\langle x, t\rangle\rangle$:
  - outcome x with probability $(1-q)^t$
  - outcome B with probability $1-(1-q)^t$
- Suppose players have utility functions $u_i(x_i)$, with $u_i(0)=0$ and $u_i(B)=0$, and preferences satisfy expected utility:

$$EU\langle\langle x, t\rangle\rangle = (1-q)^t u_i(x_i)$$

- Compare with $\delta_i^t u_i(x_i)$: $(1-q)$ plays same role as δ_i for each player

Relationship between SPE and NBS

From now on, assume $\delta_1 = \delta_2 \equiv \delta$. We want to study SPE outcome as time between offers gets small (formally, as $\delta \to 1$).

Let $U_i(x_i, t) = \delta^t u_i(x_i)$, with $u_i(0) = 0$, $U_i(D) = 0$, and u_1, u_2 strictly increasing and weakly concave. In the unique SPE, $x^*(\delta)$ and $y^*(\delta)$ yield utility pairs on the Pareto frontier and satisfy the "indifference conditions"

$$u_1(y_1^*(\delta)) = \delta u_1(x_1^*(\delta)) \quad \text{and} \quad u_2(x_2^*(\delta)) = \delta u_2(y_2^*(\delta)) \qquad (**)$$

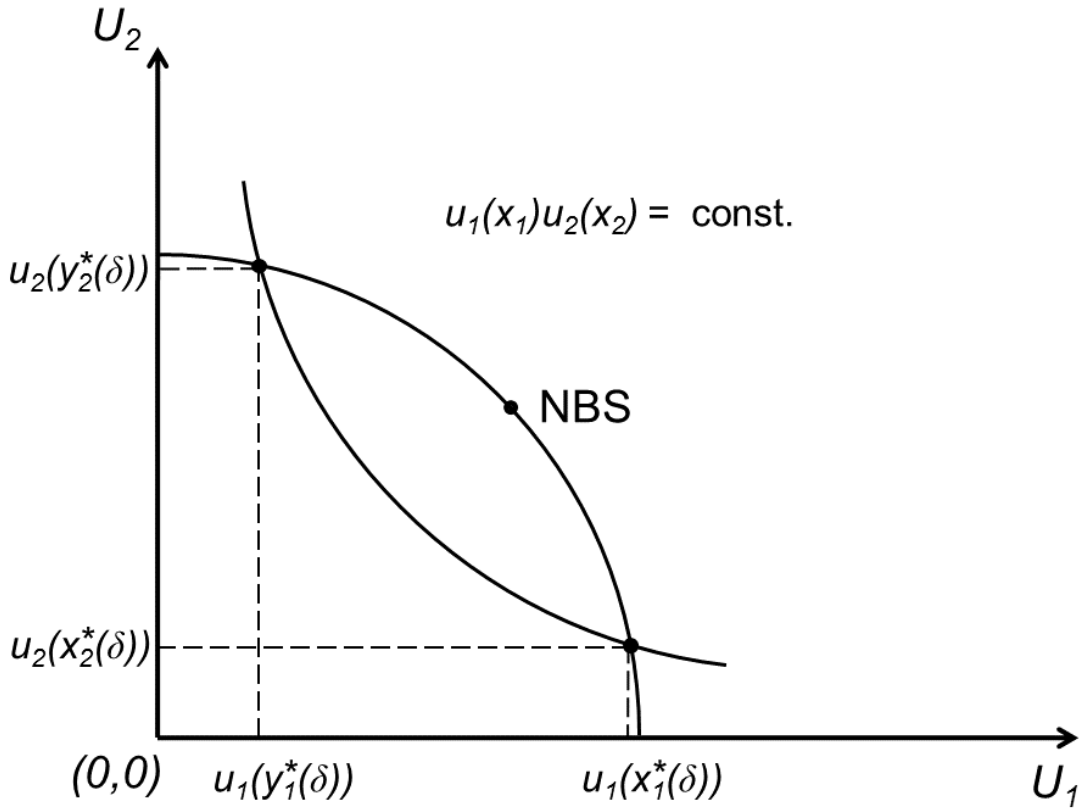
Proposition

In the limit as $\delta \to 1$, both $x^*(\delta)$ and $y^*(\delta)$ approach the Nash Bargaining Solution corresponding to $d = (0, 0)$ and $S = \{(s_1, s_2) \in \mathbb{R}^2 : (s_1, s_2) = (u_1(x_1), u_2(x_2))$ for (x_1, x_2) such that $x_1 + x_2 \leq 1$ and $x_1, x_2 \geq 0\}$.

Discussion:

- The NBS treats players symmetrically, but the alternating-offer game yields a first-mover advantage, hence an asymmetry between players.
  - As $\delta \to 1$, the first-mover advantage disappears.
- The proposition holds for any weakly concave $u_1(\cdot)$, $u_2(\cdot)$.

SPE approaches NBS as $\delta \to 1$

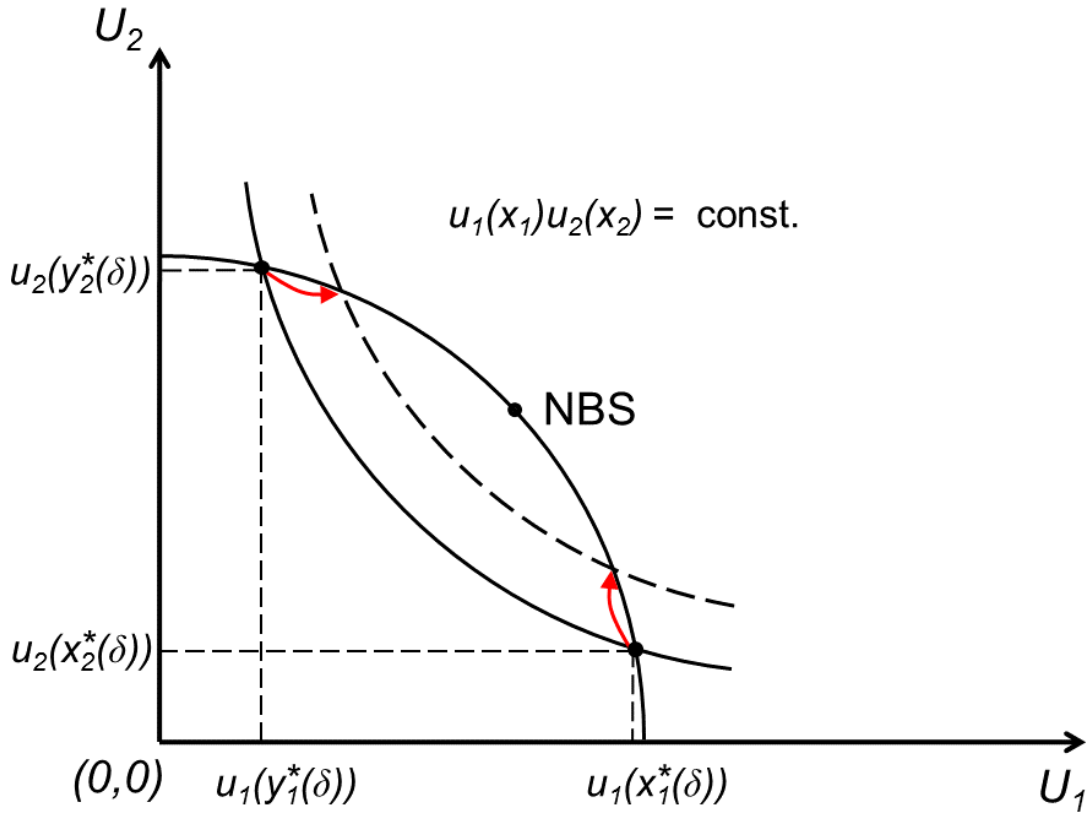


Proof: The NBS for $\langle S, d \rangle$ above is

$$\arg \max_{d \leq s \in S} (s_1 - d_1)(s_2 - d_2) = \arg \max_{x \in X} u_1(x_1) \cdot u_2(x_2)$$

From $(**)$, $u_1(x_1^*(\delta)) \cdot u_2(x_2^*(\delta)) = u_1(y_1^*(\delta)) \cdot u_2(y_2^*(\delta))$ and $\lim_{\delta \to 1}[u_i(x_i^*(\delta)) - u_i(y_i^*(\delta))] = 0$, i=1,2. Therefore, as $\delta \to 1$, $x^*(\delta)$ approaches the maximizer of $u_1(x_1)u_2(x_2)$ over S.

SPE approaches NBS as $\delta \to 1$



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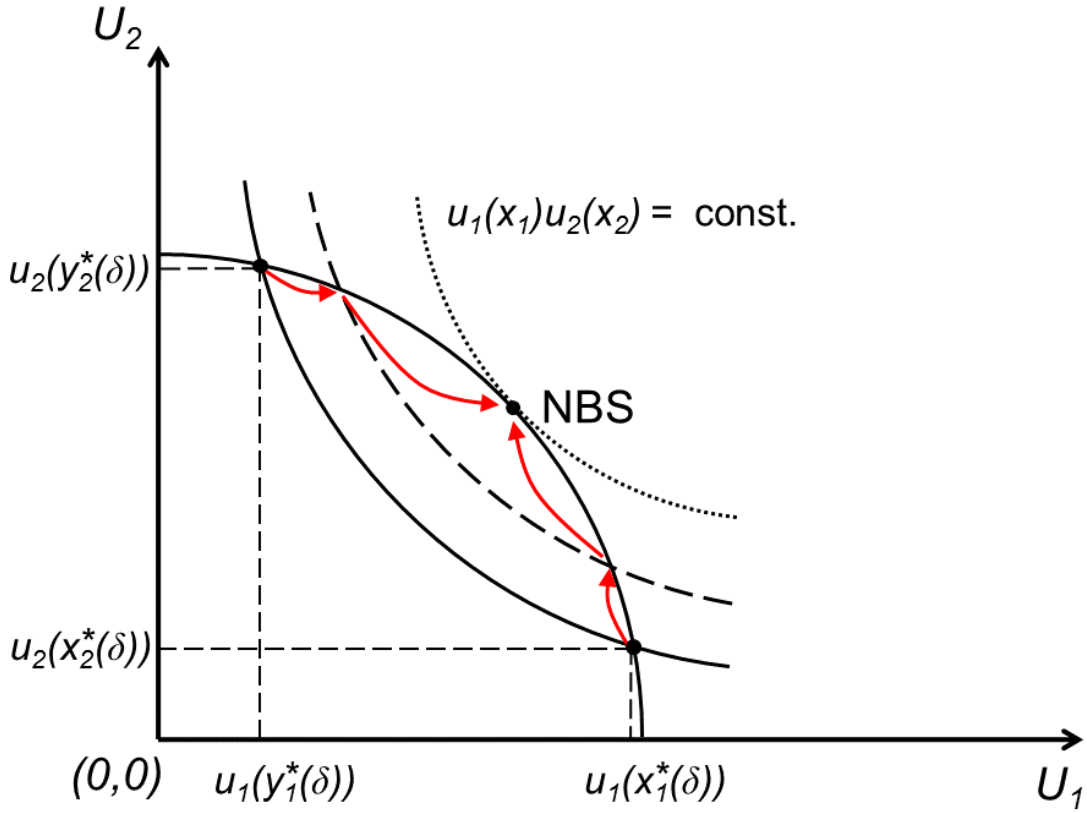

Proof: The NBS for $\langle S, d \rangle$ above is

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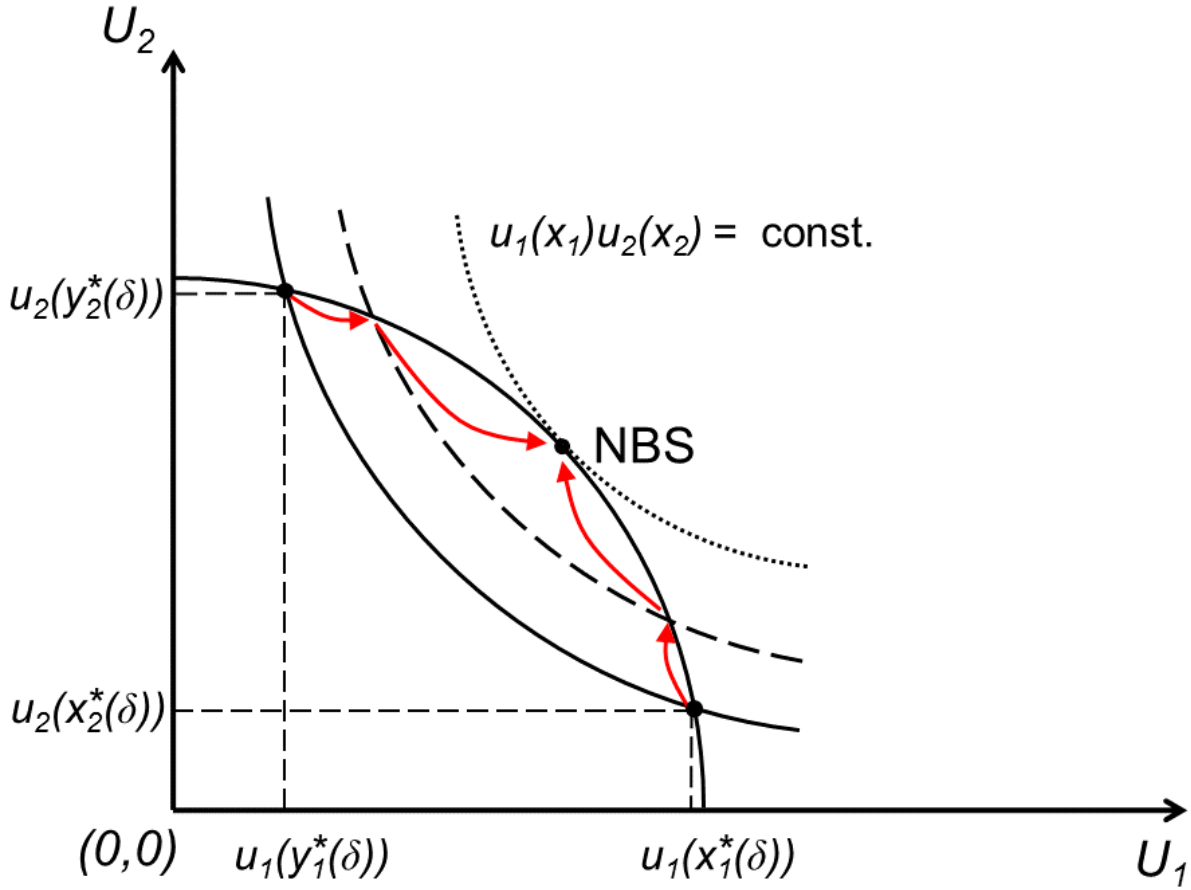
SPE approaches NBS as $\delta \to 1$



For any δ, the points $(u_1(x_1^*(\delta)), u_2(x_2^*(\delta)))$ and $(u_1(y_1^*(\delta)), u_2(y_2^*(\delta)))$ lie on a common iso-value curve of $u_1 \cdot u_2$. As $\delta \uparrow$, these two points approach each other and, in the limit as $\delta \to 1$, they coincide. Where they coincide must be a point of tangency between an isovalue curve of $u_1 \cdot u_2$ and the Pareto frontier: this is precisely the NBS.

SPE approaches NBS as risk of breakdown $\to 0$

- Return to the version of the model where, after each rejection, "breakdown" B occurs with probability q.
- Let $u_i(x_i) = x_i$, and suppose that $u_i(B) = d_i$: If breakdown occurs, player i gets utility d_i.
  - Assume breakdown is inefficient: $d_1 + d_2 < 1$.
- The unique SPE still satisfies the "indifference conditions":

$$x_2^* = (1-q)y_2^* + qd_2 \quad \text{and} \quad y_1^* = (1-q)x_1^* + qd_1$$

- Denote, and solve for, the solutions $x_1^*(q)$ and $y_1^*(q)$.
- *In the limit as the risk of breakdown disappears ($q \to 0$), $x_1^*(q)$ and $y_1^*(q)$ both approach $d_1 + \frac{1}{2}(1 - d_1 - d_2)$, which is player 1's payoff in the Nash Bargaining Solution.*
  - So the disagreement utilities in the Nash bargaining problem correspond, in the strategic setting, to the utilities from **involuntary** bargaining breakdown.

Extensions

- Outside options (Binmore et al (1986); Sutton (1986)):
  - If one player has a **voluntary** "outside option", how is SPE behavior affected? How should Nash Bargaining Solution be modified?
- Incomplete information (Kennan and Wilson (1993)):
  - When does bargaining under incomplete information result in delay in reaching agreements? How much delay? Does delay persist as time between offers becomes small?
- Strategic uncertainty (Friedenberg (2019)):
  - Even with complete information, players may be uncertain about how others will play the game after unexpected moves. When can such strategic uncertainty itself generate delays in reaching agreement?
- More general time preferences (Schweighofer-Kodritsch (2018)):
  - If bargainers do not use exponential discounting but rather display some "present-bias", do predictions of the alternating-offer model change?