# Two examples of structural modelling. Notes for "Structural modelling".

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Revised, February 3 2012

## 1 Introduction.

Structural models are models for empirical work that have 'deep' parameters that are invariant to changes in the economic environment. Structural modelling always involves theoretical modelling; my own feeling is that it is impossible to do good applied microeconometrics without also doing good theory. The polar extreme to structural modelling is quasi-experimental (or natural experiment) modelling which seeks to model *atheoretically* reactions to exogenous changes. The contrast between structural modelling and quais-experimental analysis is covered briefly in the second part of this course.

The elements of a (micro) structural model include:

1. well defined objectives for agents (utility maximisation, profit maximisation, cost minimisation, election to office, loss aversion etc.);
2. well defined constraints (budget constraints, imperfect capital markets, production possibilities; constraints on information processing etc.);
3. an explicit statement of uncertainty and what the agents know and the beliefs of the agents about uncertain outcomes;
4. how constraints, preferences and beliefs vary across agents (heterogeneity);
5. a model of interactions between agents;
6. some idea of when the above elements will be invariant to changes in the economic environment.

Examples include: demand analysis; auctions; labour supply and taxes; schooling choices (stop now or continue with a DPhil?); marriage and fertility decisions; a small number of firms competing in the same market; bargaining between husband and wife; choosing which crops to plant; etc. .

To take a particular example, consider two firms producing the same good for the same market (duopoly). We might model assuming: (1) profit maximisation;

(2) production functions and the demand curve they face; (3) what each firm knows or believes about the other firm; (4) different production functions; (5) static or dynamic interactions with noncooperative behaviour or colluding and acting as a monopolist, choosing quantities (Cournot) or prices (Bertrand) or some other interaction; (6) the economic environment here is the demand curve, changing this would not change (1), (3), (4), or (5).

Some of the uses of structural models are:

- understanding the motivation for behaviour. This includes 'testing the theory'. For the duopoly example, we might want to test for profit maximisation.
- prediction in environments never seen before. For the duopoly example: what would happen to prices and quantities if the demand curve changed - perhaps because of changes in taxes.
- policy analysis. For the duopoly example: how would firms react if the government changed competition policy?
- welfare analysis. For the duopoly example: how do profits and consumer surplus depend on the way firms interact?

These notes provide a couple of examples of structural modelling. An extended simple example on time use and household production is given in the second set of notes.

## 2 A demand model.

When considering the demand for individual goods, we are often interested in the income and price elasticities for the goods. For example, policy makers would like to know which goods have a low own price elasticity since increasing taxes on these goods increases tax revenue. As another example, goods that have a low income elasticity ('necessities') should be less taxed since they are bought disproportionately by the poor. In a more sophisticated vein, optimal tax formulæ require information on price and income responses.

We have $n$ goods with prices $(p_1, p_2, ...p_n)$ and a single agent who spends $x$ in total on the goods.[1] The demand system for quantities, $q_i$, is given by:

$$q_i = f^i\left(p_1, p_2, ...p_n, x\right) \text{ for } i = 1, 2...n \tag{1}$$

[1] In demand analysis we take total expenditure as given, even though it is a choice for the agent. (This is the subject of the next section). Sometimes researchers refer to $x$ as 'income'. This is very misleading since income and total expenditure in any month can be very different.

The objects of interest are the elasticities:

$$\frac{\partial q_i}{\partial x}\frac{x}{q_i} = \frac{\partial \ln q_i}{\partial \ln x}, \text{ the income elasticity for good } i \tag{2}$$

$$\frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = \frac{\partial \ln q_i}{\partial \ln p_i}, \text{ the own price elasticity for good } i \tag{3}$$

$$\frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i} = \frac{\partial \ln q_i}{\partial \ln p_j}, \text{ the cross price elasticity of good } i \text{ for price } j \tag{4}$$

The question is, how to choose functional forms for $f^i(.)$ so that we can estimate these elasticities?

A (parametric) structural approach starts from a utility function $u(q_1, q_2, ...q_n; \boldsymbol{\theta})$ where the form of $u(.)$ is known but the parameter vector $\boldsymbol{\theta}$ is unknown. If we maximise the utility function subject to the budget constraint:

$$p_1 q_1 + p_2 q_2 + ...p_n q_n = x \tag{5}$$

we will end up with a set of demand functions as in (1). The important thing about these functions is that they will satisfy the four Slutsky conditions: adding-up, zero homogeneity and symmetry and negative semidefiniteness of the Slutsky matrix. Moreover, these four conditions ensure that we can find a unique preference ordering that generates the observed demands.

If we consider nonparametric identification, then we usually use revealed preference conditions. These do not require the specification of a parametric utility function. However, finite data and revealed preference usually only set identifies preferences.

The first of the Slutsky conditions, *adding-up*, requires:

$$p_1 f^1(\mathbf{p}, x) + p_2 f^2(\mathbf{p}, x) + ...p_n f^n(\mathbf{p}, x) \equiv x \tag{6}$$

This says that if we add up individual expenditures we will have total expenditures.[2] This is such a basic requirement that we would always want to impose it on our demand functions. This is the 'theory' that we shall be concerned with in this illustration.

A more down to earth empirical researcher might announce that they are not terribly concerned with theory but require only a 'sensible' specification. For example a non-structural approach might start with the observation that since we are interested in elasticities, it would be a good idea to have demand functions that have elasticities as parameters. Thus we could take the so-called double-log form:

$$\ln q_i = \alpha_i + \beta_i \ln x + \sum\nolimits_{j=1}^{n} \gamma_{ij} \ln p_j + \nu_i \text{ for } i = 1, 2...n \tag{7}$$

[2] The four Slutsky conditions are sometimes known as 'rationality' conditions. Adding-up is more than this - it is a sanity condition. If you reject it then you (implicitly) believe that $1 + 1 = 3$.

This advantage of this specification is that the unknown parameters $(\alpha_i, \beta_i, \gamma_{i1}, ...\gamma_{in})$ are also the objects of interest, the elasticities. For example:

$$\frac{\partial q_i}{\partial x}\frac{x}{q_i} = \beta_i \tag{8}$$

However, this specification does have one drawback: it can only satisfy adding-up under very restrictive circumstances. To see this, take the exponential of both sides, multiply by prices and sum over $i$ to give total expenditure:

$$x = \sum\nolimits_{i=1}^{n} p_i q_i = \sum\nolimits_{i=1}^{n} p_i \exp\left(\alpha_i + \beta_i \ln x + \sum\nolimits_{j=1}^{n} \gamma_{ij} \ln p_j + \nu_i\right) \tag{9}$$

Since this has to hold for all parameters and values of prices and total expenditure, we can take partial derivatives with respect to $x$ and $p_k$ respectively:

$$\begin{aligned} 1 &= \sum\nolimits_{i=1}^{n} p_i \exp(.) \frac{\beta_i}{x} = \frac{1}{x}\sum\nolimits_{i=1}^{n} p_i q_i \beta_i \Rightarrow x = \sum\nolimits_{i=1}^{n} p_i q_i \beta_i \\ 0 &= q_k + \sum\nolimits_{i=1}^{n} p_i q_i \frac{\gamma_{ik}}{p_k} \Rightarrow 0 = p_k q_k + \sum\nolimits_{i=1}^{n} \gamma_{ik} p_i q_i \end{aligned} \tag{10}$$

Sufficient conditions for these equations to hold are:

$$\begin{aligned} \beta_i &= 1 \text{ for } i = 1, 2...n \\ \gamma_{kk} &= -1 \text{ for } k = 1, 2...n \\ \gamma_{ik} &= 0 \text{ for } k \neq i \text{ for } i = 1, 2...n \end{aligned} \tag{11}$$

With no restrictions on the values of prices and total expenditure these conditions are also necessary for (10). Thus taking a simple specification and imposing the most basic theory restriction (adding-up) leads us to conclude that the expenditure elasticities are all unity (no good is a necessity or a luxury) and that all goods have unit own price elasticity and zero cross price elasticity. Since these elasticities are the objects of interest, it seems that we do not need data![3] This is, of course, a nonsense. The correct conclusion from this analysis is that the double-log form is worthless for theory based empirical analysis. We have to find some other demand system which does allow us to impose the four Slutsky conditions and that fits the data well. Necessarily any other functional form than the double-log will have that elasticities that vary with prices and total expenditure (and other household characteristics such as sex and age). There is a very large literature on generating demand systems that have these properties.

You might ask, what happens if we go ahead and fit the double-log form to our data anyway? Suppose we have a sample of households and their annual expenditures on a group of goods such as 'food at home', 'food outside the home', 'household services', 'transport' etc.. Suppose we also have the price indices for these broad categories; that is the price (index) of food at home etc.. From this we can construct quantities (as expenditures divided by price) and

[3] Moreover, these coefficient values will give a demand system that satisfies the other three Slutsky conditions. The demands are generated by a Cobb–Douglas utility function.

total expenditure. If we have enough variation in prices and total expenditure, we can estimate the coefficients of the double-log demand system using OLS. These estimates will *not* be equal to the values given in (11). Rather we will see what look like sensible demands for each good, taken one at a time. For example, we would certainly have that $\beta_i < 1$ for food at home; that is, food at home is a necessity. Moreover, if we formally test for whether the coefficients satisfy the conditions in (11) we will almost certainly reject. For example, we will find that the food total expenditure elasticity is 'significantly' different from unity (that is, it is many standard errors below unity). If you formally test (11) you will certainly reject. *This is not a rejection of adding-up.* It is telling you that the original functional form is wholly unsuited for its purpose. Always remember that most 'tests of the theory' are also tests for ancillary assumptions, including the functional form assumed.

The moral of this story is that *economic theory is not an optional extra for empirical work*.

# 3 Allocation over time: consumption.

Demand analysis is about how agents allocate a given total expenditure in a given period between individual goods. Consumption analysis is about how agents decide on the total expenditures (now called 'consumption') for each period. Thus demand is within period (or, intratemporal) analysis and consumption is intertemporal analysis. Consumption analysis is one of the most important areas of policy analysis and for understanding growth and cyclical fluctuations. Among the questions of interest, we have: how does consumption react to transitory changes in income (how effective is Keynesian policy?) or changes in interest rates? To focus the discussion we shall consider a very specific policy proposal.

Suppose the government proposed the following (radical) policy. On their eighteenth birthday, every citizen receives a loan of £30,000. Agents are free to reject this loan or to take only a part of it. This is financed by the government borrowing on world markets at the world interest rate. No interest is paid on this loan by individuals until age 30, but the principal grows to reflect the interest the government pays. From age 30 to age 60 the government taxes back the principal and interest from individuals so that the debt is cleared at age 60. The question of interest is: what would this do to the consumption of young people (aged $18 - 30$)?

In the simplest life-cycle model with no liquidity constraints this policy leaves lifetime wealth at age 18 (assets plus the discounted flow of future income, net of taxes) unchanged, so that there would be no change in consumption behaviour.[4] In such a model, there would also be no effect on choices such as schooling, career choice, marrying and having children, buying a home, emigrating at age 29 and retirement. My own guess is such a policy would have major impact on

[4] This assumes that anyone can borrow on world markets at the same rate of interest as the government.

all of these. But guesses are not very useful here, we need empirically based analyses. The important thing about analysis of this policy is that it is unlike any ever seen and we cannot do much with extrapolating from the reaction to past policies. This requires structural modelling.

One strand of the consumption literature dominated analysis up until about 1978. This is the Keynesian model which posited that changes in consumption are due to recent changes in income.[5] The best empirical specification in this strand is due to Davidson, Hendry, Srba and Yeo (DHSY) and was published in 1978. The original specification was developed for the analysis of aggregate time series, but we can also think of applying it to the consumption paths of individuals. DHSY recommend the following functional form for consumption changes

$$\Delta \ln C_{t+1} = \beta_1 \Delta \ln Y_{t+1} + \beta_2 \Delta\Delta \ln Y_{t+1} + \beta_3 \ln \left( \frac{C_t}{Y_t} \right) + u_{t+1} \tag{12}$$

where $C_t$ is consumption in period $t$ and $Y_t$ is net of tax real income. This is explicitly an atheoretical approach. The only hint of theory is the 'error-correction' term $\beta_3$ which captures that consumption cannot stray too far from income indefinitely (consumption and income are cointegrated). Thus, if consumption was high relative to income last period then consumption this period will be lower ($\beta_3$ is negative). Although minor improvements have been made to the DHSY formulation it has shown remarkable robustness in modelling aggregate time series data. Despite this, it is difficult to see how we would use these estimates in *ex ante* analysis of the loan policy of the previous paragraph. No one, presumably, believes that agents would treat the loan as current income. It seems we need a framework that explicitly allows us to think about people making lifetime plans (or, at least, engaging in forward looking behaviour).

In the same year as DHSY, Hall published a seminal paper that put life-cycle theory at front and centre in the analysis of consumption changes. The central feature of any forward looking rational model is that agents will seek to keep the marginal utility of expenditure constant from period to period. After all, if you expect money to the 'worth' more next period relative to today, then you should decrease consumption today and save more for tomorrow. You will do this until the marginal utilities of money in the two periods are equal.

Although Hall considered a quadratic utility function, most subsequent researchers have used an iso-elastic within period utility function:

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma} \tag{13}$$

(with a log form if $\gamma = 1$).[6] This is then embedded in an intertemporally

[5] Empirical analyses based on household level data also emphasise changes in personal circumstances, such as losing a job or having a child

[6] This iso-elastic form is not arbitrary. It is the consequence of two assumptions. The first is that preferences over lifetime consumption are homothetic - that is, if lifetime wealth is increased by 10% then agents will increase consumption in each period by 10%. The second assumption is that preferences are additive over time. The iso-elastic form is the only form that allows for these assumptions simultaneously.

additive framework by taking lifetime expected utility from period $t$ to be:

$$U_t = E_t \left( \sum_{\tau=t}^{T} \beta^{\tau-t} u(C_\tau) \right) \tag{14}$$

where $E_t(.)$ is the expectations operator conditional on information at time $t$, $\beta$ is a discount factor (usually taken to be less than but close to unity) and $T$ is the end of life.[7] Denote assets at the beginning of period $\tau$ by $A_\tau$ and income in period $\tau$ by $Y_\tau$. The lifetime budget constraint is given by:

$$\begin{aligned} A_{\tau+1} &= (1 + r_{\tau+1})(A_\tau + Y_\tau - C_\tau) \text{ for } \tau + t, ...T \\ A_{T+1} &= 0 \end{aligned} \tag{15}$$

with $A_t$ given. The first element of the budget constraint states that the agent can lend or borrow between periods $\tau$ and $\tau + 1$ at an interest rate of $r_{\tau+1}$. The second constraint states that the agent cannot die leaving debts (so that $C_T = A_T + Y_T$). We assume sequential planning in which the agent makes a decision on consumption in the current period conditional on all of the information available.

The parameters of interest are the discount factor $\beta$ and the coefficient of relative risk aversion $\gamma$. Given these we can predict how the agent will react to, for example, changes in income paths or future interest rates. There are two broad approaches to estimating these two parameters. One is to set up a dynamic program for the problem and then to simulate this using models for uncertain income streams and interest rate processes. This requires auxiliary models for these two processes. Hall proposed an alternative, based on the first order condition for optimal intertemporal allocation, which only requires us to observe realised consumption and interest rates. The Hall approach leads to a simple structural model.

The first order condition in period $t$ for maximising $U_t$ subject to the constraints is given by:

$$\left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} (1 + r_t) \beta = \varepsilon_t \tag{16}$$

where $\varepsilon_t$ is a 'surprise' term which has the property that it is expected to be unity before the event: $E_{t-1}(\varepsilon_t) = 1$. This is usually known as the consumption *Euler equation*.[8] In this simple model the only surprises are for the income stream or for interest rates. For example, an *unanticipated* increase in income would lead to a higher $\varepsilon_t$ and hence to higher current consumption, relative to consumption last period.[9] A change in the interest rate has a direct effect and an indirect (wealth) effect. The direct effect (which can, in turn, be decomposed into an income effect and a substitution effect) comes directly through the presence of

[7] We are allowing that future income and interest rates are uncertain but the date of death is known.

[8] This is because Euler was the first to systematically develop the calculus of variations. Euler himself never did any economics (more's the pity).

[9] It is critical to note here that changes in income that were anticipated based on lagged information have no effect.

$r_t$ in (16). The indirect effect arises since an unanticipated rise in the real rate makes debtors worse off and creditors better off; this is reflected in the surprise term $\varepsilon_t$. If we take logs of (16), we have:

$$\begin{aligned} \Delta \ln C_t &= \left(\frac{\ln \beta}{\gamma}\right) + \frac{1}{\gamma} r_t - \frac{\ln \varepsilon_t}{\gamma} \\ &= \pi_0 + \pi_1 r_t + u_t \end{aligned} \tag{17}$$

Here the $\beta$ and $\gamma$ are 'structural' parameters whereas the $\pi_0$ and $\pi_1$ are the parameters that will be estimated (*not* 'reduced form' parameters). The parameters $\pi_0$ and $\pi_1$ can be estimated by linear GMM using the orthogonality condition that $u_t$ is uncorrelated with any variables dated $t-1$ or earlier.[10]

This is all rather magical. It seems that by taking a simple theory model we can generate a linear equation for changes in log consumption that identifies all of the relevant parameters without any need to observe incomes, assets or beliefs. The contrast between the right hand sides of (17) and (12) is striking. The latter has current and lagged income and consumption on the rhs whereas the former only has real interest rates. The simplicity of the Hall formulation and its tight link to theory lead to an almost universal overnight change in modelling and a huge Euler equation empirical literature that allowed for, for example, liquidity constraints, habits and aggregate shocks. This is not the place to discuss it, but the results from this literature have been disappointing. Nowadays, the trend is to modelling structural models using the dynamic program approach. Versions of Keynesian consumption functions along the DHSY lines have largely disappeared, except as stable characterisations of aggregate consumption. Presumably because they have only very weak theoretical backing and the interpretation of the parameters $(\beta_1, \beta_2, \beta_3)$ in (12) is obscure.

# 4 Structural models.

These two examples are designed to illustrate what we mean by a structural model. One feature of a structural model is that it states explicitly what should be in the estimated equation and what should not. For example, changes in income are central to Keynesian consumption functions but they do not appear in the (structural) Euler equation. All that remains are unanticipated changes that work through the general surprise term. Structural models that specify parametrically utility and production functions also give functional forms for estimating equations. In these equations *every* parameter has a theory based interpretation. A second feature of structural models is that they impose a discipline on explanations. It is easy enough to think up models and assumptions that rationalise any particular empirical regularity. But these assumptions have implications for other observables and these also have to be rationalised with the same parameters. For example, an agent who is found to be risk averse

[10] Note however, that $E_{t-1}(\ln \varepsilon_t)$ is not equal to zero even if $E_{t-1}(\varepsilon_t) = 1$. This makes identifying $\beta$ from estimates of $\pi_0$ and $\pi_1$ problematic.

in their consumption behaviour should also display risk aversion concerning schooling choices, occupational choice and choosing a mate. It is this 'integrity' of personality across different contexts that provides the real challenge in using structural life-cycle models in empirical work.

# 5 Exercises.

1. Think of a particular context for applied microeconometric modelling (not demand or consumption behaviour) and discuss what data you might use; how you would formulate an *ad hoc* model and how you would start to formulate a structural model. Think of a policy question and address how you would use the two models to undertake an *ex ante* analysis of the policy. It is best if you can think of a context that interests you. If you are out of ideas, either have a look at the examples given in the course outline or think about any part of microeconomic theory that particularly interests you.

2. Take the consumption example and identify the different elements of a structural model discussed in the introduction above.

3. What are the restrictions on the parameters $(\pi_0, \pi_1)$ in equation (17) that arise from theory conditions on the structural parameters $(\beta, \gamma)$? Using 16, what restrictions do we have on $\beta$ and $\gamma$ if an *anticipated* increase in the real interest rate is to lead to greater saving between periods $t-1$ and $t$. Comment on the implication of the latter for a Central Bank that is trying to stimulate current consumption. How would you incorporate an uncertain lifetime into equation (14)?

Answer to question 3. The restrictions on the structural parameters are that the discount factor is between zero and unity: $0 < \beta < 1$ and the coefficient of relative risk aversion is positive: $\gamma > 0$. From equation (17), this implies that $\pi_0 < 0$ and $\pi_1 > 0$. The former states that if the interest rate is zero then the agent will, on average, have a declining consumption path $(C_t < C_{t-1})$. The second restriction implies that a higher interest rate will lead to higher consumption growth; that is, $(C_t > C_{t-1})$.