A1. (i) See section 1.6, section 4.2.1 page 128, and Appendix 1.C.

(ii) [Appendix 1.D pp. 55-57 contains a common-values example closely related to the private-values example described here.]

(a) Bidders bid up to their value, so the price is the expected second highest actual value, i.e. $\theta + \left(\frac{n-1}{n+1}\right)$ (see Appendix 1.E). Profits are $\left(\frac{n^2-n}{n^2+n}\right)$.

(b) Each bidder bids $v_i - x$ for some x (since the diffuse prior means she has no knowledge of how high or low her value is relative to others).

A bidder who deviates to bid $v_i - x + \epsilon$ would earn $x - \epsilon$ if she won, and wins with probability $(t_i + \epsilon)^{n-1}$ and, so she would expect to earn

$\pi(\epsilon) = \int_{t=0}^{1}(x-\epsilon)(t+\epsilon)^{n-1}dt = (x-\epsilon)\left[\frac{(t+\epsilon)^n}{n}\right]_{t=0}^{1} = \frac{(x-\epsilon)}{n}\left[(1+\epsilon)^n - \epsilon^n\right] \implies$
$\frac{\partial \pi(\epsilon)}{\partial \epsilon} = -\frac{[(1+\epsilon)^n - \epsilon^n]}{n} + \left(\frac{x-\epsilon}{n}\right)\left[n(1+\epsilon)^{n-1} - n\epsilon^{n-1}\right]$.

In equilibrium $\frac{\partial \pi(\epsilon)}{\partial \epsilon} = 0$ at $\epsilon = 0$, i.e. $x = \frac{1}{n}$.

The highest bidder (with expected value $\theta + \frac{n}{n+1}$) wins so profits are $\frac{n}{n+1} - \frac{1}{n} = \left(\frac{n^2-n-1}{n^2+n}\right)$, confirming part (i)'s result that sealed-bid profits are below ascending profits when bidders' private signals (here v_i) are affiliated.

(iii)(a) Ascending auction behaviour is unaffected.

(b) If θ is common knowledge, bidders' private information are now t_i, and independent, so revenue equivalence with (i) applies and profit is again $\left(\frac{n^2-n}{n^2+n}\right)$.

The difference between (ii)b and (iii)b illustrates the linkage principle (see section 1.6).

(iv)(a) An equilibrium is for ascending auction behaviour to be unaffected. But it is not clear why bidders who know they do not have the highest value will bother to bid; if they don't, then profit will be much lower.

(b) The equilibrium is for the highest-value bidder to bid (and win) at (just above) the second-highest value. There is the same difficulty that low-value bidders may not bother to play.

(v) If the t_i are also affiliated we have

π(iva) $=$ π(ivb) (assuming the equilibria described above),

π(iiia) $>$ π(iiib) (affiliation),

π(iiib) $>$ π(iib) (linkage), and

π(iva) $=$ π(iiia)$=$ π(iia) (trivially).

Summarising: π(iia) $=$ π(iiia) $=$ π(iva) $>$ π(ivb) $>$ π(iiib) $>$ π(iib).

A2. (i) See Appendix 1.A.

The key is equation (1): $S_i(v) \geqslant S_i(\widetilde{v}) + (v - \widetilde{v})P_i(\widetilde{v})$. This assumes private values. [It's not hard to do the common values case.]

(ii) [The Oxford examination candidates were not expected to write all of what follows!]

(a) (1) fails because a type v who deviates to behave as if he had type $\widetilde{v}$ does *not* earn the right hand side of (1) – because type v expects to face competitors with values that are conditional on her having type v, while type $\widetilde{v}$ expects to face competitors with values that are conditional on her having type $\widetilde{v}$. Appendix 1.C describes informally how to proceed, though candidates were not expected to go into this. A candidate might note that revenue equivalence still holds between the first-price sealed-bid and Dutch auctions, and also between the second-price sealed-bid and ascending auctions if either values are private or there are just two bidders.

(b) (1) fails because it is generally no longer true that

$$S_i(v) = vP_i(v) - i\text{'s expected payment.}$$

[One rather special case is that $S_i(v) = vP_i(v) - EU(v\text{'s payment})$

i.e., the bidder is risk-averse with respect to money, but the prize is not (equivalent to) money. In this case (1) holds as usual, hence $S(\cdot)$ is pinned down by $P(\cdot)$ and $S(\underline{v})$ as usual, so $EU(\text{type } v\text{'s payment})$ is equivalent across auctions under the usual revenue equivalence conditions. (And since risk-averse buyers are equally well off, a risk-neutral seller prefers the auction that stabilises their payments in order to give them a given expected utility most cheaply. So all-pay auctions raise more money than first-price auctions, which themselves raise more money than second-price auctions.)]

Again, of course, the first-price sealed-bid and Dutch are revenue equivalent, as are the second-price and ascending.

(c) This depends on the model of collusion – how do bidders share information and collude? But if, for example, all the bidders agree to (jointly) win the object at a price of zero, and then share the object among themselves à la Cramton, Gibbons and Klemperer (1987) then (1) holds and revenue equivalence follows as usual — it's just that each bidder's surplus is higher by the same amount $S(\underline{v}) > 0$.

A special case of CGK is McAfee and McMillan's (1992) suggestion that the n colluders allocate the object among themselves using a first-price "knockout" auction, the winner of which pays $1/n^{\text{th}}$ of his bid to all n colluders (including himself). This is incentive compatible (since a loser's payoff does not depend on his bid, each colluder makes his usual bid) and budget balanced ex post. Or the colluders can allocate the object among themselves in dominant strategies by running a second-price auction, the winner of which pays a risk-neutral ring-center who previously paid all the colluders $1/n^{\text{th}}$ of the expected second-highest of their n values (again a loser's payoff does not depend on his bid, so each colluder makes his usual bid), but this is only budget balanced in expectation – Graham and Marshall (1987). (The GM result extends to cases when not all bidders collude if the main auction is a second-price auction; if the knockout winner wins the main auction, he pays the ring-center any excess of the knockout price over the main auction price, and the ring-center previously pays all the colluders $1/n^{\text{th}}$ of the expected value of this.)

(Note, of course, that a colluder may want to cheat at the main auction. However, collusion is easier to sustain in second-price and ascending auctions than in first-price auctions, because in the former the designated winner can bid infinitely high and other colluders have no incentive to cheat and try to win the auction (Robinson, 1985). Also, (tacit) collusion is easier in ascending (multi-unit or repeated) auctions because bidders can use bids to signal (Chapters 3, 4, of my book; Klemperer, 2000; Brusco and Lopomo, 2002; Cramton and Schwartz, 2000). With non-independent values, an ascending auction allows colluding bidders to induce non-colluding opponents to bid less aggressively, by having some colluders drop out at a low price, thus signalling a low valuation (Pagnozzi, 2003). See also Section 1.9.)

(d) If each bidder is restricted to a single unit only, then equations (1) – (5), and hence revenue equivalence, hold as usual. The only change is that the $P_i(v)$ function is different to reflect the higher probability of winning that more units provides.

If a bidder's value is linear in the number of units won, we can just write $P_i(v) =$ (expected number of units i wins) and (1) – (5), and revenue equivalence, again hold as before.

With multiple identical indivisible units, and if bidders' values are not necessarily linear in the number won, a generalisation of the usual argument applies: let $\underline{v} \equiv (v_1, ..., v_N)$ represent the type whose *marginal* value of winning a j^{th} unit is v_j, and $\underline{P}(\underline{v}) \equiv (P_1(\underline{v}), ..., P_N(\underline{v}))$ be the vector of type $\underline{v}$'s equilibrium probabilities of winning at least j units (i.e., his value of winning exactly j units is $\sum_{k=1}^{j} v_k$, and his probability of doing so is $P_j(\underline{v}) - P_{j+1}(\underline{v})$). Then equation (1) becomes

$$S(\underline{v}) \geqslant S(\underline{\widetilde{v}}) + (\underline{v} - \underline{\widetilde{v}}) \cdot \underline{P}(\underline{\widetilde{v}}) \qquad (1')$$

(in which $(\underline{v} - \underline{\widetilde{v}}) \cdot \underline{P}(\underline{\widetilde{v}})$ is the dot product of $(\underline{v} - \underline{\widetilde{v}})$ and $P(\underline{\widetilde{v}})$) and the standard kind of argument now applies.[1]

Even more generally, if the units are not identical, let $\underline{v} \equiv (v_1, ..., v_L)$ be the type whose valuation of the j^{th} possible *allocation* of the units is v_j, and we again get equation (1′). (So with N distinct indivisible units and I bidders, $L = N^{I+1}$ if the auctioneer can retain units; a bidder's valuations may be different for allocations which differ only in the assignments to other bidders, so externalities between bidders can also be taken into account in this formulation.) (See Engelbrecht-Wiggans, 1988 and Krishna and Perry, 2000.)

So revenue equivalence applies fairly generally in the multi-unit context, in the sense that any two auctions that allocate the objects in the same way (i.e., have the same $\underline{P}(\underline{v})$ function) and give the same surplus to a particular type, are revenue-equivalent under the usual kinds of conditions.

However, it is important to note that standard auction forms will *not* in general result in the same allocation, $\underline{P}(\underline{v})$. In particular, while most standard auctions achieve the efficient allocation and are therefore revenue equivalent in the symmetric single-unit case, this is not true in the multi-object case because

[1]Substituting $\underline{v}$ by $t\underline{y}$ and $\tilde{\underline{v}}$ by $(t+dt)\underline{y}$ into (1′), and then making the converse substitution, and taking $dt \to 0$, and integrating up in the usual way, yields

$$S(t\underline{y}) = S(\underline{0}) + \int_{\tau=0}^{t} \underline{y} \cdot \underline{P}(\tau \underline{y}) d\tau \qquad (5')$$

So the surplus function is pinned down by the $\underline{P}(\cdot)$ function, and we have revenue equivalence as usual.

For a simple example, consider 2 identical items for sale when each bidder has a two-dimensional type (v, k), values a single object at v, and values a second object at kv ($0 \leqslant k \leqslant 1$). Then

$$S(v,k) \geqslant S(\tilde{v},\tilde{k}) + (v-\tilde{v})P_1(\tilde{v},\tilde{k}) + (kv - \tilde{k}\tilde{v})P_2(\tilde{v},\tilde{k}) \qquad (1')$$

If we fix $k = \tilde{k}$ and write $\tilde{v} = v + dv$ then we can obtain an expression for $dv \to 0$:

$$\frac{dS(v,k)}{dv} = P_1(v,k) + kP_2(v,k).$$

Similarly, we can fix $v = \tilde{v}$ and write $\tilde{k} = k + dk$ to obtain: $\frac{dS(v,k)}{dk} = vP_2(v,k)$.

These two equations are sufficient to pin down the surplus function, and hence ensure revenue equivalence across auctions that induce the same $P_1(\cdot,\cdot)$ and $P_2(\cdot,\cdot)$, for example, efficient auctions.

To illustrate, if $n = 2$, k is constant across bidders, v is distributed uniformly on [0,1], and the auction is efficient, the probability that bidder v wins at least 1 object is $\min(v/k, 1)$, and the probability that he wins both objects is kv. This gives $\frac{dS(v,k)}{dv} = \min(v/k, 1) + k^2 v$. For example, for $k = 0$, $\frac{dS}{dv} = 1$, because each bidder is guaranteed to win exactly one object; for $k = 1$, $\frac{dS}{dv} = 2v$, because $P_1 = P_2 = F(v) = v$.

the demand-reduction and other problems both affect them and affect them differently (see Section 1.10 and the Afterword to Chapter 1).

(e) Equation (1) holds for types that exist with positive probability, but (4) (i.e., $\frac{dS(v)}{dv}$) may not be defined everywhere. However, the argument will go through, and so revenue equivalence will apply, unless the distribution is neither strictly increasing nor atomless, and fails only if there is an atom at the edge of a "gap":

When the distribution of bidders' feasible valuations has a gap from x^- to x^+ (i.e., the distribution is not strictly increasing in this range), the equation after (3) becomes: $P(x^+) \geqslant \frac{S(x^+)-S(x^-)}{x^+-x^-} \geqslant P(x^-)$

$$\Longrightarrow S(x^+) \in [S(x^-) + (x^+ - x^-)P(x^-), S(x^-) + (x^+ - x^-)P(x^+)]$$

When there is a gap but not an atom, $P(x^+) = P(x^-)$, so $S(x^+)$ is determined by $S(x^-)$ (and everywhere else it is determined by (4): $\frac{dS(v)}{dv} = P(v)$), so revenue equivalence holds.

Similarly, the surplus function is pinned down when there is an atom but not an adjacent gap. Suppose there is an atom at x. Then there will be a discontinuity in the $P(\cdot)$ function at x: bidders with values slightly higher than x win against the positive measure set of bidders with value x, whereas bidders with values slightly lower than x do not. When there is not also an adjacent gap, all this means is that there is a discontinuity in $\frac{dS(v)}{dv}$ and hence a kink in the $S(\cdot)$ function at x, so the entire surplus function is pinned down, and revenue equivalence again holds.

When there is an atom at the edge of a gap, however, it is impossible to pin down the surplus function: $P(x^-) < P(x^+)$, so $S(x^+)$ is not determined by

$S(x^{-})$, and revenue equivalence fails. See (the solution to) Exercise 2 of *Auctions: Theory and Practice*, which we discuss below.[2]

Nevertheless the standard auction forms generally remain revenue equivalent even when there are atoms at the edges of gaps (and hence also mixed strategy equilibria in, for example, first-price auctions).

For example, consider the case of N symmetric bidders each of which has a private value independently drawn from a finite set of types, $v_1 < .... < v_H$ with probabilities $p_1, ..., p_H$ respectively. Let $P(v_k) = \left(\sum_{j=1}^{k-1} p_j\right)^{N-1}$ (so $P(v_k)$ is the probability that a bidder with value v_k has a strictly higher value than all his $N-1$ competitors). In a first-price auction, type v_1 bids v_1, while other types randomise. By standard arguments there can be no atoms in the randomisation, nor can there be gaps (since the lowest type above a gap would lower his bid), nor can the ranges of different types overlap (since an overlap would imply that a higher and a lower type of the same bidder are both happy choosing these different strategies that give the higher type a lower probability of winning[3]). So type v_k is indifferent about mimicking type v_{k-1} by bidding at the top of type v_{k-1}'s range (which is the bottom of type v_k's range). But in an ascending auction, also, type v_k is indifferent about (almost) mimicking type v_{k-1} by dropping out at (just above) v_{k-1}. So in both these auctions

$$S(v_k) = S(v_{k-1}) + (v_k - v_{k-1})P(v_k)$$
$$\Longrightarrow S(v_k) = S(v_1) + \textstyle\sum_{j=1}^{k}(v_j - v_{j-1})P(v_j) \qquad (5')$$

That is, bidder surplus is pinned down and equal across the standard auctions in the "discrete" (finite number of types) case, just as in the "continuous" case (in which the

[2]A simple illustration is a single bidder whose value is L or H, to whom the seller sells with probabilities $P(L) = 0$ and $P(H) = 1$, respectively, by offering price $r \in (L, H]$. Then $S(L) = 0$, and $S(H) \in [0, H - L]$ and the left and right ends of the interval are achieved by $r = H$ and $r = L$ respectively.

[3]To see this is a contradiction, see the equation below (3) (which does not require small dv) in Appendix 1A.

distribution from which bidders' types are drawn is strictly increasing).

The point is that these auctions have the property that the highest type always wins so (5′) approximates (5) in a continuous approximation to the discrete case. Although we can find examples of auctions in the discrete case that are efficient but not revenue equivalent to the standard auctions (and may raise more money than the standard auctions), these examples don't yield efficiency in a continuous approximation to the discrete model that "fills in the gaps".

To illustrate all these issues, consider Exercise 2's analysis of efficient mechanisms which always assign a unit to just one of two bidders who each have value v_H with probability p_H and value v_L with probability p_L. Efficiency implies types v_H and v_L win with probabilities $P(v_H) = \left(\frac{p_H}{2} + p_L\right)$ and $P(v_L) = \frac{P_L}{2}$ respectively, and (1) $\Longrightarrow$

$$S(v_H) \geqslant S(v_L) + (v_H - v_L)P(v_L) \qquad (1^{*})$$

$$\text{and } S(v_H) \leqslant S(v_L) + (v_H - v_L)P(v_H) \qquad (1^{**})$$

giving a range of solutions from the seller-optimum characterised by (1*) holding with equality, to the buyer-optimum characterised by (1**) holding with equality. (See solution to Exercise 2.)

Now consider a continuous approximation of this discrete case in which most types are very close to v_H or v_L with a tiny density everywhere else. Revenue equivalence holds, of course, for this distribution for any given allocation. Furthermore, for the types v_H^- (the lowest of the types who are very close to v_H) and v_L^+ (the highest of the types who are very close to v_L), for example

$$S(v_H^-) \geqslant S(v_L^+) + P(v_L^+)(v_H^- - v_L^+) \qquad (1^{\sim})$$

$$\text{and } S(v_H^-) \leqslant S(v_L^+) + P(v_H^-)(v_H^- - v_L^+) \qquad (1^{\sim\sim})$$

Of course, $S(v_H^-) \approx S(v_H)$ and $S(v_L^+) \approx S(v_L)$ (using equation (1), since $v_H - v_H^- \approx 0$ and $v_L^+ - v_L \approx 0$). But pinning down the allocation in the discrete case (all v_H's beat all v_L's) does *not* pin down a unique allocation in the continuous analog: an auction can treat types such as v_H^- and v_L^+ in several different ways while still corresponding to the (symmetric) efficient allocation of the discrete case.

To find the seller-optimum in the continuous case, note that the marginal revenue is initially just below v_H, then turns very negative at v_H^- when the density suddenly becomes very thin, then becomes barely positive again at v_L^+. So to maximise revenue, all types with a value of v_H^- or below are pooled and given an equal chance of winning.[4] (See Bulow and Roberts (1989); as always, we cannot give higher-value types lower chances of winning than lower-value types.) So $P(v_H^-) = P(v_L^+) = P(v_L)$. (Of course, $P(v_H)$ and $P(v_L)$ are unchanged from the discrete case.) So $(1^{\sim})$ and $(1^{\sim\sim})$ both hold with equality, and therefore $S(v_H) = S(v_L) + (v_H - v_L)P(v_L)$, just as in the seller-optimum of the discrete case.

The "seller-pessimum" of the continuous model, by contrast, minimises revenue by pooling all the negative marginal revenues in with the high marginal revenue bidders at the top of the distribution, i.e., pools everyone at v_L^+ or above. (Again, we cannot give higher-value types lower chances of winning than lower-value types.) So now $P(v_L^+) = P(v_H^-) = P(v_H)$ and $(1^{\sim})$ and $(1^{\sim\sim})$ again hold with equality, but in this case $S(v_H) = S(v_L) + (v_H - v_L)P(v_H)$ as in the buyer optimum of the discrete model.

Finally consider the continuous model for any standard auction (e.g., ascending or first-price sealed-bid) in which the highest-value bidder wins. In this case

[4]In the optimal auction the price starts at the pooling price of v_L, and then jumps to $\approx \frac{v_H+v_L}{2}$. (This makes v_H equally well off as if he joins the pool by bidding v_L and then wins with probability 1/2 when the other bidder bids v_L. He (almost) always makes no money if the other bidder is not a v_L.) When two bidders jump, the price rises continuously to (usually) about v_H.

This optimal auction can alternatively be implemented by first offering a price, p, that makes v_H just indifferent between pooling at v_L and accepting (and possibly being rationed at) p, i.e., $\left(\frac{p_H}{2} + p_L\right)(v_H - p) = \frac{p_L}{2}(v_H - v_L)$; and then offering price v_L if no-one accepts p. This is the form in which we described the mechanism in the solution to Exercise 2. Strictly, this alternative implementation is slightly suboptimal for an atomless distributions of types. But it is arbitrarily close to optimal for distributions of types that arbitrarily closely approximate the "discrete" two-type distribution. And it *is* optimal if (as suffices for our purposes) we restrict attention to strictly increasing distributions of types that put atoms at v_H and v_L and a small density everywhere else.

$P(v_H^-) \approx P(v_L^+) = p_L > P(v_L)$ (recall $P(v_L) = \frac{p_L}{2}$) since the intermediate types beat all v_L's whereas a v_L beats another v_L only half the time. So (1˜) and (1˜˜) both hold with (approximate) equality and $S(v_H) = S(v_L) + (v_H - v_L)p_L$. Again this corresponds exactly to a standard auction such as a first-price or ascending auction in the discrete model in which, as noted above: (a) a type v_H is just indifferent about (almost) mimicking a v_L, and (b) a type v_H who (almost) mimicks a v_L beats all v_L's and no v_H's so wins with probability p_L, so again $S(v_H) = S(v_L) + (v_H - v_L)p_L$.

The way I think of all this, then, is that revenue equivalence holds in the discrete case over mechanisms that would induce the same allocation across all types including those types that do not actually exist (but would exist if we filled the gaps with a continuous approximation): if a type v_H^- were to exist in our discrete example he would pool with the L's in the seller-optimum; if a type v_L^+ existed he would pool with the H's in the seller-pessimum; but either of these types would beat the L's and lose to the H's in a "standard" auction.

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A3. [A closely related question is Problem 9. See also Problem 1 (and Part 3 of my first-year course notes).]

(i) By symmetry i will win if $z_i \geqslant z_j$.

So i will quit at $p = z_i + z_i = 2z_i$.

[i is just indifferent about quitting or not at this point. It is easy to see that if i were to win later she would lose money (assuming j is following her equilibrium strategy) and quitting earlier would mean quitting when i's value exceeds the asking price (again assuming j is following her equilibrium strategy). See Appendix 1.D (or section 3.2 of my first-year course notes).]

(ii) If i has type z, then conditional on i winning, j is uniformly distributed as $[0, z]$. So on average j will quit at $2\left(\frac{z}{2}\right)$ conditional on i winning.

i.e. i's expected payments are z conditional on winning.

By the Revenue Equivalence Theorem, the expected payment of type z is the same in the English and Dutch auctions, so it is the same conditional on winning (since type z has the same probability of winning either auction).

So type z bids z in the Dutch auction.

(iii) Type z's expected payments are z^2 unconditional on winning in the English (and Dutch) auction.

By the Revenue Equivalence Theorem, the expected payment of type z is the same in the English (and Dutch) and All-Pay auctions, so type z bids z^2 in the All-Pay auction.

(iv) We look for a symmetric equilibrium in which type z bids $b(z)$.

If z bids $b(\widetilde{z})$, she beats types $\leqslant \widetilde{z}$, i.e. with average signal $\frac{\widetilde{z}}{2}$. So conditional on winning type z's value is $z + \frac{\widetilde{z}}{2}$, and she wins with probability $\widetilde{z}$, so type z's expected surplus from bidding $b(\widetilde{z})$ is $S = (z + \frac{\widetilde{z}}{2})\widetilde{z} - b(\widetilde{z})$.

$\frac{\partial S}{\partial \widetilde{z}} = z + \frac{\widetilde{z}}{2} + \frac{\widetilde{z}}{2} - b'(\widetilde{z})$

$= 0$ at $\widetilde{z} = z$ in equilibrium

$\implies b'(z) = 2z$ with boundary condition $b(0) = 0$ (since $z = 0$ never wins in equilibrium). The solution is $b(z) = z^2$ which agrees with part (iii).

(v) See Klemperer (1998) (or section 3.5 of my first-year course notes): bidder 2 quits at z_2 in the ascending auction (if she bothers to enter at all); the Dutch auction solution is almost unaffected from part (ii).

For revenue equivalence between two auctions to hold, we need that the probability that any given type of bidder will win is the same in the two auctions. See Appendix 1.A. (It would be sufficient for the purposes here that the object always went to the bidder with the highest signal.) But here type z of bidder 1 wins an ascending auction with probability 1, but wins a Dutch auction with probability only (slightly above) z, see section 1.7.2 of the book or Klemperer (EER 1998) (or section 3.5 of my first-year course notes).

[Oxford, 1st Year Micro 2004]

A4. (i)(a) Initially each bidder stays in until it would be just indifferent if it suddenly found itself a winner, i.e., if two others quit simultaneously. So at the point at which a bidder with value t_i quits the price must be

$$p = t_i + [2t_i + E(t \mid t \geq t_i)] \tag{1}$$

(i.e., if two others were to quit, their signals would be assumed to be t_i and the remaining signal would be assumed to be somewhere above t_i). So after the first quit at price p, the signal of the quitter is inferred to be the t_i that satisfies (1). Call this signal $t_{(4)}$. Remaining bidders with signals t_i then quit at prices such that they would be just indifferent about finding themselves winners, i.e., at prices

$$p = t_i + \left[t_{(4)} + t_i + E(t \mid t \geq t_i)\right] \tag{2}$$

(i)(b) If a bidder with value t_i were suddenly to find himself a winner at the point at which he was himself about to quit he would infer the quitter's signal would also be t_i, and the two bidders excluded from his auction had signals $E(t)$. i.e. he quits at price

$$p = t_i + [t_i + 2E(t)] \tag{3}$$

(i)(c) In the single-auction case, (a), expected revenue $= 2E\{t_{(3)} + \left[t_{(4)} + t_{(3)} + E(t \mid t \geq t_{(3)})\right]\}$ using (2). For the uniform distribution, $Et_{(i)} = \frac{5-i}{5}$, so expected revenue $= 2(\frac{2}{5} + \frac{1}{5} + \frac{2}{5} + \frac{3\frac{1}{2}}{5}) = 3\frac{2}{5}$. And in the two-auction case, (b), expected revenue $= 2E\{t_{(2 \text{ of } 2)} + t_{(2 \text{ of } 2)} + 2E(t)\}$ in which $t_{(2 \text{ of } 2)}$ is the second-highest of two signals, using (3), so for the uniform distribution expected revenue $= 2(\frac{1}{3} + \left[\frac{1}{3} + 2(\frac{1}{2})\right]) = 3\frac{1}{3}$.

(ii) (a) In the single-auction case the expected revenue is $E(MR_{(1)} + MR_{(2)})$. In the two auction case the two high-signal bidders are in the same auction one-third of the time (in which case the winner are the highest and third-highest signal bidder), so the expected revenue is $E(MR_{(1)} + \frac{2}{3}MR_{(2)} + \frac{1}{3}MR_{(3)})$. So revenue is higher in the single-auction case if $E(MR_{(2)}) > E(MR_{(3)})$.

(b) (I) For $F(t) = t, MR(t) = v + t - 1$, so $MR_{(2)} > MR_{(3)}$ always, so the single-auction case is more profitable.

(II) For $F(t) = 1 - t^{-2}$, $MR(t) = v - \frac{t}{2}$, so $MR_{(2)} < MR_{(3)}$ always, so the two-auction sales mechanism is more profitable.[5]

[5]Note that MR's are the easier way of computing expected revenues. For the uniform case and a single-auction,

$$\begin{aligned} \text{expected revenue} &= E(MR_{(1)} + MR_{(2)}) \\ &= E([v + t_{(1)} - 1] + [v + t_{(2)} - 1]) \end{aligned}$$

(III) Since the problem is pure common values, any allocation is equally efficient, so bidders' preferences are precisely the opposite of the seller's.

(IV) In the private-value case, $v_i = t_i$, marginal revenue $MR_i = t_i - \frac{1-F(t_i)}{f(t_i)}$ is "downward sloping" for both $F(t) = t$ $(MR(t) = 2t - 1)$ and $F(t) = 1 - t^{-2}$ $(MR(t) = \frac{t}{2})$, so the seller prefers the single auction for either distribution.

(c) A good exam answer would just be that non-decreasing MR is much more likely with common than with private values, so it's much more likely that separate auctions are more profitable with common than with private values.

A good answer might also point out why: with pure common values (our problem) non-decreasing MR just means $(-\frac{1-F(t_i)}{f(t_i)})$ is non-*in*creasing in t_i which is obviously much easier than having $(t_i - \frac{1-F(t_i)}{f(t_i)})$ non-*in*creasing. We could note that non-decreasing MR is even easier in common-value contexts if values are non-additive (e.g., we could discuss the "maximum game" see Bulow and Klemperer "Prices and the Winner's Curse", *Rand Journal*, 2002). With private values, by contrast, we know $t_{(2)} = E(MR(t) \mid t \geq t_{(2)})$ so $E(t_{(2)}) = E(MR_{(1)})$ and, of course, $t_{(2)} > MR_{(2)}$, so $E(MR_{(1)}) > E(MR_{(2)})$ which is at least suggestive of the difficulty of having sufficiently non-decreasing MR for split auctions to be more profitable (but see below).

I did not expect any examinee to write any of what follows:

One might even wonder whether separate auctions could ever be more profitable in the private case since always $E(MR_{(1)}) > E(MR_{(2)})$. However, it is *not* always true that $E(MR_{(2)}) > E(MR_{(3)})$, because $t_{(3)} = E(MR(t) \mid t \geq t_{(3)})$, so $E(t_{(3)}) = \frac{1}{2}E(MR_{(1)} + MR_{(2)})$ (since $t_{(3)}$ is the price, i.e. average revenue, obtained from selling to everyone above $t_{(3)}$ who on average are $t_{(1)}$ and $t_{(2)}$), so $E(MR_{(2)}) = E(2t_{(3)} - t_{(2)}) \neq E(t_{(3)})$.

A simple example which give separate auctions being more profitable with private values is a two-point support: probability 10% of α; probability 90% of $\beta < \alpha$, so Prob $(t_{(4)} = \alpha) = 0.0001$, Prob $(t_{(3)} = \alpha) = 0.0037$, Prob $(t_{(2)} = \alpha) = 0.0523$, etc.[6] For this private-values example, it's actually easier to note directly that expected

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$$= \quad ([2 + \frac{4}{5} - 1] + [2 + \frac{2}{5} - 1]) = 3\frac{2}{5}$$

while with two auctions, expected revenue is reduced by $\frac{1}{3}E(MR_{(2)} - MR_{(3)})$, i.e., by $\frac{1}{15}$ - we obtain the result without needing to compute exactly where bidders quit the auction.

For $F(t) = 1 - t^{-2}$, it is equally straightforward to compute profits using marginal revenues but considerably more tiresome (for me at least) to compute prices directly as in part (i).

[6]Another example is inelastic constant-elasticity demand. If $F(t) = 1 - t^{\eta}$ then $MR(t) = t(1 + \frac{1}{\eta})$ which is negative and therefore increasing everywhere (as t falls) if $\eta = -\frac{1}{2}$, say. This is rather confusing, especially since I've just pointed out that $E(MR_{(1)}) > E(MR_{(2)})$. The problem is that marginal revenue is really infinite at $t = 1$ in this case, reflecting the infinite expected profits that can be made by selling with arbitrarily low probability. This can all be made clear and sensible (i.e.

revenue from a single auction is $E(2t_{(3)})$ while that from two separate auctions is $E(\frac{1}{3}(t_{(2)}+t_{(4)}+\frac{2}{3}(t_{(3)}+t_{(4)}))$ so the separate auctions win iff $E(t_{(2)}-t_{(3)}) > 3E(t_{(3)}-t_{(4)})$. (Of course it's easy to check that the MR formula yields this too.[7])

Returning to private values *vs* common values: since $E(t \mid t \geq t_{(3)}) = E\left(\frac{t_{(1)}+t_{(2)}}{2}\right)$ while $E(t_{(2 \text{ of } 2)}) = E(\frac{1}{6}(t_{(2)}+t_{(4)})+\frac{1}{3}(t_{(3)}+t_{(4)}))$ we find (from part (i)) that two separate auctions are more profitable in our common value case iff $2E(t_{(2)}-t_{(3)}) > 3E(t_{(3)}-t_{(4)})$ which is clearly easier to satisfy than the private-value condition.

It's also easy to look at bidders' preferences in the private-value case. Bidder surplus is welfare less revenue, i.e., $v - MR$ of the winning bidders (in expectation). So the bidders prefer the single auction iff

$$E(t_{(2)} - MR_{(2)}) > E(t_{(3)} - MR_{(3)})$$

or $E\left(\frac{1-F(t_{(2)})}{f(t_{(2)})}\right) > E\left(\frac{1-F(t_{(3)})}{f(t_{(3)})}\right)$. Note that this is *precisely* the condition for the seller to prefer the two auctions in our additive common-value model. With linear demand this is false (because MR is twice as steep as demand everywhere) but with CE -2 demand ($F(t) = 1 - t^{-2}$) it is true (because MR is half as steep as demand everywhere). With CE -2 demand, bidders with private values dislike the split auction because the efficiency losses outweigh the small reduction in payments to the seller.

Those who are truly fascinated by this problem may wish to look at Ellison, Fudenberg and Möbius' paper "Competing Auctions", *Journal of the European Economic Association* (2004) though this paper is somewhat differently focused. (Möbius is a former Oxford Economics M.Phil student who is now (2006) a Harvard Professor).

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expected profits always finite) by considering a close approximation that hits the price axis at a finite price.

[7] $E(t_{(2)}) = E(MR_{(1)}); E(t_{(3)}) = E\left(\frac{MR_{(1)}+MR_{(2)}}{2}\right)$;
$E(t_{(4)}) = E\left(\frac{MR_{(1)}+MR_{(2)}+MR_{(3)}}{3}\right)$.

A5. SKETCH ANSWER

The bare bones of an answer are discussed in the book (*Auctions: Theory and Practice*, Klemperer (2004)) see e.g., pages 33 (including note 91) and 63-4 (including note 7). Other useful references are Rothkopf et al (JPE 1990), Milgrom's 2005 *Clarendon Lectures* (given in Oxford this academic year), and Ausubel and Milgrom "The Lovely but Lonely Vickrey Auction" (Chapter 2 of *Combinatorial Auctions*, Cramton, Shoham and Steinberg,eds.(2006)).

The efficiency properties in the private-values context are standard.

Problems that apply even in the single-unit context include
- others can see the winner's value (so the winner may be vulnerable to ex-post exploitation (Rothkopf, JPE, 1990), or auctioneer exploitation (the auctioneer may have a confederate place a bid just less good than the winner's to leave the winner (almost) zero surplus), or even absent foul play the result may be a PR disaster (as in the celebrated New Zealand auction, pg 110 of the book)
- the efficiency of the auction can facilitate entry deterrence by strong players and be disastrous for revenue (see the general discussion of entry in 1st vs 2nd price auctions in section 4.3.1 of the book and elsewhere)
- disadvantages of 2nd price auctions vs 1st price auctions more generally can be discussed (revenue properties with risk-aversion, asymmetries, budget constraints, etc; see book)
- efficiency properties are limited outside the private-value context.

In the multi-unit context there are more problems including:
- the auction is "non-transparent", that is, payments are hard to understand for non-experts, as is the mechanism itself for many people
- different bidders pay different amounts for identical objects and often those who value an object more pay less
- in multi-unit contexts the mechanism "favours larger bidders over smaller bidders" (e.g. relative to a standard ascending auction)
- when there are complementarities, there are curious and difficult-to-guard-against collusive possibilities (even for bidders who would all lose absent collusion)
- collusion may be facilitated even absent complementarities (though ascending auctions are worse for collusion -see the book)
- demerging, or setting up new bidders, can help a bidder
- revenues can be very low, when goods are complements (and, relatedly, adding bidders can sometimes *reduce* revenues) as discussed in, for example, Milgrom's Clarendon lectures.

A6. [I have ignored subscripts.]

(i) Total bidder surplus is the expected high value of 2/3, since expected transfers have to sum to zero. So surplus per bidder is on average 1/3 versus 1/6 in a "regular" auction.

(ii) The revenue equivalence theorem applies exactly, and for the same reasons, as "normal". So

$$S(v) = S(0) + \int_0^v P(x)dx$$

Clearly, $P(v) = v$. Also, given the symmetry of the problem, $S(0) = 1/6$ because average surplus is 1/6 greater than in the "regular" auction (1/3 versus 1/6) and we know from the revenue equivalence theorem that the difference is a constant type-by-type since $dS/dv = P(v)$ (the same amount) in both games, for all types. So

$$S(v) = S(0) + \frac{v^2}{2} = \frac{1}{6} + \frac{v^2}{2}.$$

(If necessary, candidates could work out that $S(0) = \frac{1}{6}$, by using the fact that $E(S(v)) = \frac{1}{3}$.)

(iii) In the sealed-bid auction the bidder will pay his bid when he is highest and receive the competitor's bid when that bid is higher. For a bidder with value v we have

$$1/6 + v^2/2 = S(v) = v^2 - vb(v) + \int_v^1 b(x)dx$$

since the right-hand side computes expected surplus as value times probability of buying, less expected payments when a buyer, plus expected receipts when a seller. We can now see directly which linear function solves this equation, or we can differentiate both sides to obtain

$$v = 2v - b(v) - v\frac{\partial b}{\partial v} - b(v)$$

which yields $\frac{\partial b}{\partial v} = 1 - 2\frac{b(v)}{v}$. Since the question makes clear that we will find $\frac{\partial b}{\partial v}$ is a constant it is easy to see that a solution is $b(v) = \frac{v}{3}$. (It is also clear directly from the formula for $S(v)$ that $b(1) = 1/3$, since a bidder with value 1 must be a winner.)

(iv) In the ascending auction the analogous surplus equation is:

$$1/6 + v^2/2 = S(v) = v^2 - \int_0^v b(x)dx + (1-v)b(v)$$

which produces (through differentiation of both sides)

$$v = 2v - 2b(v) + (1-v)\frac{\partial b}{\partial v}$$

so $\frac{\partial b}{\partial v} = \frac{2b(v)-v}{1-v}$. The formula immediately yields $b(0) = 1/6$, and the solution is therefore $b(v) = \frac{1}{6} + \frac{v}{3}$.

Bidders go beyond their value, and take the risk of buying at a price that exceeds their value, in the hope of selling out at a higher value (as in Bulow, Huang and Klemperer (JPE 1999) though in a private value context).

(v) Type v of a partner with share α earns $1/6+v^2/2-\alpha v$, so it is simple to compute that the worst off type of a partner with share α has value α and earns $1/6-\alpha^2/2$ (or $1/24$ for $\alpha = 1/2$ — up to now we are assuming $\alpha = 1/2$, but we will want general α in the next part.)

(vi) Nothing is different in the analysis of general shares. If we are to run an efficient mechanism, so $P(v) = v$, incentive compatibility requires us to maintain $S(v) = S(0)+\frac{v^2}{2}$. All we can do is to alter $S(0)$ by having the smaller partner pay the larger partner for participating in the mechanism.

Given shares α and $(1-\alpha)$, the worst-off types of the two partners earn surpluses of $1/6-\alpha^2/2$ and $1/6-(1-\alpha)^2/2$ respectively. Therefore, we can have the smaller partner pay $\left|\frac{1}{4}-\frac{\alpha}{2}\right|$ (i.e., half the difference between these two surpluses) to the larger partner, so that both worst-off types make (the same) non-negative surplus if $(1/6-\alpha^2/2) + (1/6-(1-\alpha)^2/2) \geq 0$ — i.e., $\alpha \in [1/2-\sqrt{3}/6,\ 1/2+\sqrt{3}/6]$. For α outside this interval, efficient individually-rational dissolution is impossible if the partners know their own values before agreeing to a mechanism. (This confirms Myerson and Satterthwaite's (JET 1983) result on the impossibility of efficient bilateral trading when one player owns 100% of the good to be traded.)

See Cramton, Gibbons, and Klemperer (Econometrica 1987) for the general case; note also that the problem is identical to one in which the two bidders collude in buying from a seller, by bidding against each other in a pre-auction knock-out for the right to buy the asset from the seller for zero.

A7. [Sketch Answer]

(i) [Paul Klemperer was quoted in the Economist 9/9/06, p77.] He was probably suggesting that auctioning larger blocks of spectrum might have worked better in this particular case.

(ii) concern about "exposure" among some bidders. That is, bidders who had a strong preference for complete coverage of a region were reluctant to bid on smaller blocks for fear that they would be "stranded" winning some but not all parts of the region. The result could be that bidders who either wanted only small blocks, or were willing to risk winning only some parts of the region could have picked up those areas cheaply. The fear of stranding would be reinforced if resale was not permitted, or was hard for some other reason. Larger bidding increments are perhaps slightly more likely to create this problem.

(iii) a Vickrey auction would sell private-value objects efficiently at disparate prices assuming a fixed number of non-colluding bidders, private values, and no budget constraints.

A sequential auction (by English/Japanese, Dutch, first-price sealed bid, second-price sealed bid, etc) or a discriminatory auction of identical objects is, in theory, efficient assuming symmetric bidders.

Perfect price discrimination is efficient, and imperfect price discrimination may be more efficient than a fixed price.

(iv) if there are no complementarities at all between the smaller areas, then the objects should be sold independently, but if the smaller areas are perfect complements (individually useless to an owner without the whole region) then they should be sold together. More generally, selling a small number of large licences makes life very difficult for bidders who only want smaller areas, especially if resale is hard. So it depends upon whether it is more efficient to encourage bidders who are interested in winning smaller areas (these bidders may be new entrants) or bidders interested in winning larger regions.

(v) in a package auction, a bidder can specify a bid for a group of properties and then wins all of these if his bid beats the sum of the best bids for the individual properties. This still makes life very difficult for smaller bidders who have difficulty coordinating their bids to beat the package bidder, and also face an "free-rider" problem (each small bidder wants the other small bidders to bid higher prices that will contribute to defeating the package bidder). There are other problems: a bidder who wants to bid for a package as well as for individual units is bidding against himself; if bids must be left on the table in subsequent rounds then bidders may be reluctant to make offers, but if bids need not be left on the table coordination is particularly hard, etc.

A8. [Sketch of a possible answer.]

1. The candidate might begin by showing that under the appropriate conditions, all auctions make the same expected profits

(see sec 2 of my first-year class notes on the course website)

2. But the profits are only the same in expectation,

and under the assumptions that:

- The number of bidders is fixed
- They play noncooperatively
- There is a single indivisible unit (or if there are multiple identical objects, bidders want a single indivisible unit each)
- Bidders' values are independent
- They are risk neutral
- There are no budget constraints
- There are no externalities between bidders
- Bidders are drawn from symmetric distributions, and the auction is a standard one, so that the highest actual type is the winner (or some alternative assumptions so that the the lowest type's payoff is constant (e.g., zero), and a given type of a given bidder wins any auction with equal probability).

Note that setting a reserve price affects types' probabilities of winning – so setting a reserve price is an important part of auction design, even if all the other assumptions above hold.

(Not all these issues were discussed in the first-year lectures, but many were)

3. The candidate might then discuss how auctions' expected profits differ if these assumptions are relaxed. (With the exception of independence, probably externalities, and in some cases symmetry, relaxing the assumptions above generally favours first price over second price mechanisms.)

(only some of this was discussed in the first-year lectures)

4. The auctioneer's objective may not be (only) revenue. (In particular, efficiency considerations tend to favour second price mechanisms.)

A9. (i) See Appendix 1.C of *Auctions: Theory and Practice*. See also Section 1.6.

(ii) References, and the bare bones of an answer are in Section 1.6.

(iii) See Section 1.6, Appendix 1.C and also, e.g., Section 4.2. See also question 17 of the book and its sketch solution.

A10. (i) A. We run a "second marginal-revenue auction". The woman's demand curve is $80-40q$, so her marginal revenue is $80-80q = 80 - 80((80-v)/40) = 2v - 80$, while the man's demand is $40 - 40q$, so his $MR = 2v - 40$. So we ask the bidders to state their values, and give the object to the bidder with the higher marginal revenue, provided that marginal revenue exceeds the seller's valuation, at a price equal to the lowest value at which that bidder could have won given the other bidder's report (and given the seller's cost).

B. Since the woman's marginal revenue beats the man if and only if her value exceeds his by at least 20, we could give the man a discount of 20 conditional on him being the winner, and simply run a standard ascending auction with a reserve price of 45.

C. An optimal auction maximises expected marginal revenue, just as a price discriminating monopolist maximises the sum of marginal revenues. Because the man is drawn from a distribution that is lower by a constant, he has a higher marginal revenue at any given value.

(ii) A. This is the problem described in Bulow and Roberts (1989), pages 1078-81, but with 20 deducted from all the valuations. The demand curve for $q < 20$ is $80 - 200q$, so $MR = 80 - 400q = 2v - 80$ (as before); demand for $q > 20$ is $50 - 50q$, so $MR = 50 - 100q = 2v - 50$. Because marginal revenue is not downward sloping, selling to the highest value bidder does not sell to the highest marginal-revenue bidder, so the seller would do better to "iron" the marginal revenue curve and can do this best by treating all bidders with values between 35 and 50 equally, as if they all had marginal revenues of 20 (since the triangular areas cut off by the $MR = 20$ line are then equal in a diagram like Bulow and Roberts' figure 3). The reserve price is 30, since a bidder with value 30 has $MR = 10$.

The higher-value bidder is sold to at the price of the lower-value bidder except that the randomisation (if it is necessary) is at a price of 35, and that if the winner's value exceeds 50 and the runner-up's value is between 35 and 50, the winner pays 42.5 (so that if the winner had a value of 50, as was just necessary to guarantee victory, it would have been indifferent between its outright victory and participating in the lottery).

[The question did not ask this, but this outcome could be achieved by running a simple ascending auction, but skipping all the prices between 35 and 42.5. In fact, no-one will then quit between 35 and 50; any bidder with a value between 35 and 50 will quit at 35–if its competitor quits simultaneously, it prefers a fair lottery at 35 to winning for sure at 42.5, and if its competitor doesn't quit simultaneously, it is doomed anyway. (With n bidders still in the auction at a price of 35, the auctioneer would need to jump the price to $50 - (15/n)$.)]

B. It doesn't matter whether or not we treat the man and woman fairly, provided that for each individual we make it equally likely that all values between 35 and 50 are equally likely to win, so that the expected MR of the winner is as in the case of "fair" allocation (and we must choose prices so that this outcome is achieved given the incentive compatibility constraints).

If we always favour the man in the "ironing" range, then the higher-value bidder is sold to at the price of the lower-value bidder (as before), except that: if both values are between 35 and 50 the man wins at a price of 35; if the man's value exceeds 50 and the woman's value is between 35 and 50, the man pays 35 (so if the man had a value of 50, he would have been indifferent between stating that value and stating a somewhat lower value); and if the woman's value exceeds 50 and the man's value is between 35 and 50, the woman pays 50 (so if the woman had a value of 50, she would have been indifferent between stating that value and stating a somewhat lower value).

A11. (i) see part 2 of the course notes on the web (or Appendix 1A of my book).

1.(iia) $V/2$ (= expected second highest signal).

profit = $V/4$ (= expected difference between highest and second highest signal).

[Recall that the expected value of the k^{th} highest of n independent draws from a uniform distribution on $[X, Y]$ equals $X + [(n+1-k)/(n+1)](Y - X)$ from the course notes on the web, or Appendix 1A of my book, or this is not hard to compute directly, for $n = 3$ firms.]

(iib) $V/2$, and profit $= V/4$ by revenue equivalence theorem with ascending auction.

[Recall revenue equivalence applies to expected bidders' surpluses as well as to expected total payments.]

(iiia) the contest for the uncommitted consumers is a sealed bid auction which yields $(\theta - \varphi)C/4$, by revenue equivalence with ascending auction. All three firms also make $\varphi(1 - c_i)$ automatically on their locked in customers, and average costs are $C/2$, so expected industry profits are $3\varphi(1 - C/2) + (\theta - \varphi)C/4$.

(iiib) prize is $(\theta - \varphi)(1 - c_i)$, independent – of course – of the bids. Payment is $\theta(1 - Q_i)$ for winner, $\varphi(1 - Q_i)$for losers. (Any firm with the highest possible cost would choose $Q_i = 1$, hence making expected surplus of zero as required for the revenue equivalence theorem to hold.) So this auction yields $(\theta - \varphi)C/4$ by direct analogy with (iib). In addition, all three firms also make $\varphi(1 - c_i)$ automatically. So expected industry profit $= 3\varphi(1 - C/2) + (\theta - \varphi)C/4$.

(iv) losing the ability to price discriminate means firms no longer rip-off their locked in consumers so much - but it also dulls competition for the uncommitted consumers. In this example the effects cancel. More generally, either could dominate.

[See Klemperer (*Review of Economic Studies*, 1995, pp. 515-539).]

The model of the question is from appendix B of Bulow and Klemperer (*Brookings Papers*: *Microeconomics,* 1998, pp. 323-394) which applies this model to the cigarette market.

A 12.

(*Note that those in charge of the exam slightly altered the wording of the question.*)

i. a. A is indifferent about winning at price p if $\alpha z_A + (p/k_B)(1-\alpha) = p$, or $p = k_B \alpha z_A/(k_B + \alpha - 1)$, so A stays in the bidding until the price reaches $p = k_A z_A$, in which $k_A = k_B \alpha/(k_B + \alpha - 1)$, and then quits.

b. When $\alpha = 1$, we have pure private values so A bids up to its private value. When $\alpha < 1$, we have common-value elements to valuations, so A faces a winner's curse and therefore bids less, the more aggressively that B bids (i.e., the larger is k_B). [One way of thinking about this equilibrium is that it illustrates the "strategic substitutes" and "strategic complements" analysis developed in Bulow, Geanakoplos and Klemperer (*JPE*, 1985). With pure private values, your bidding is unaffected by how aggressive your opponent is. But with a common values element, the more aggressively your opponent bids, the worse is your winner's curse, so the less aggressively you will want to bid. That is, we have strategic substitutes. And this effect is greater, the greater is the common value element, that is the smaller is α.]

c. Clearly, $k_B = k_A \alpha/(k_A + \alpha - 1)$. It is easy to see that $k_A = k_B = 1$ is a Nash equilibrium.

[there are imperfect equilibria in general (one bids to ∞, the other bids 0); assuming the z's have positive density everywhere, there are other perfect Bayesian equilibria only in the pure common values case, $\alpha = 1/2$.]

ii. a. A is indifferent about winning at price p if $\phi\{\alpha z_A + (p/k_B)(1-\alpha)\} = p$, or $p = \phi k_B \alpha z_A/(k_B + \phi\alpha - \phi)$, so A stays in the bidding until the price reaches $p = k_A z_A$, in which $k_A = \phi k_B \alpha/(k_B + \phi\alpha - \phi)$, and then quits. [If $\phi > \alpha/(1-\alpha)$, A bids ∞]

b. As before, $k_B = k_A \alpha/(k_A + \alpha - 1)$. Substituting the claimed equilibrium values of k_A and k_B into this expression for k_B, and into the expression for k_A that was derived in the previous part, confirms the claimed equilibrium.

c. $k_A = \infty$; $k_B = \alpha$; i.e., A stays in forever; B bids up only to the minimum possible value it can have conditional on its own signal. The reason is that it is common knowledge that A's value exceeds B's, so it can never be rational for B to buy if A is unwilling to.

iii a. As $\phi \to \alpha/(1-\alpha)$, B quits at αz_B so profit $= \alpha E(z_B)$. At $\phi = 1$, profit $= E\min(z_A,\ z_B)$, which is greater if z_i and z_j are independent draws from the same uniform distribution, and $\alpha < 2/3$.

b. Run a first price auction; use a reserve price; give a toehold or options or other subsidy to the weaker player (e.g., counting B's bid as ϕ times what it actually is–equivalently giving B a discount equal to a fraction $(\phi - 1)/\phi$ of its bid–yields symmetric behaviour, so profit $= ((\phi + 1)/2)\ E\min(z_A,\ z_B)$, since the auctioneer has to pay the subsidy half the time.)