Solutions to Exercises[1]

1. (i) See Appendix 1.A.

(ii) It is a dominant strategy for each bidder to bid up to her value. Hence, conditional on winning, bidder i expects to pay the expected highest of the other $n-1$ bidders' values, conditional on her own value, v_i, being highest, that is, $[(n-1)/n]v_i$.[2]

(iii) Bidder i wins with probability $(F(v_i))^{n-1} = (v_i/\bar{v})^{n-1}$. So in an ascending auction, her unconditional expected payment is $[(n-1)/n](v_i^n/\bar{v}^{n-1})$. By revenue equivalence, this is her (unconditional) expected payment in the "all-pay" auction, and, hence, her bid.

(iv) Assuming the other $n-1$ bidders bid according to the equilibrium bidding strategy $b(v)$, i's surplus from bidding as type $\tilde{v}$ is $S_i = v_i(F(\tilde{v}))^{n-1} - b(\tilde{v})$. Bidder i's optimal choice of $\tilde{v}$ satisfies

$$\frac{\partial S_i}{\partial \tilde{v}} = 0 \quad \Rightarrow \quad v_i(n-1)(F(\tilde{v}))^{n-2}f(\tilde{v}) - b'(\tilde{v}) = 0.$$

In equilibrium, each bidder behaves as her own type, that is, $\tilde{v} = v_i$, so $b'(v_i) = v_i(n-1)(F(v_i))^{n-2}f(v_i)$. For $F(v)$ uniform on $[0, \bar{v}]$, we have

$$b'(v) = (n-1)\frac{v^{n-1}}{\bar{v}^{n-1}},$$

which, since $b(0) = 0$, has solution

$$b(v) = \frac{(n-1)}{n}\frac{v^n}{\bar{v}^{n-1}}.$$

(See chapter 1, note 121 for a more detailed explanation of applying a similar procedure to a first-price auction.)

2. (i) Type Ls always reject, because their value is less than a. For Hs, given that an H opponent accepts

$$S(H \mid \text{reject}) = \tfrac{1}{2}(0) + \tfrac{1}{2}\tfrac{1}{2}(1) = \tfrac{1}{4}$$

[1] I am very grateful for the assistance of Eric Budish and Marco Pagnozzi in preparing these solutions.

[2] Recall from Appendix 1.D that the expected value of the kth highest among n values independently drawn from a uniform distribution on $[0, \bar{v}]$ is $[(n+1-k)/(n+1)]\bar{v}$.

$$S(H \mid \text{accept}) = \tfrac{1}{2}(1-a) + \tfrac{1}{2}\tfrac{1}{2}(1-a) = \tfrac{3}{4}(1-a) > \tfrac{1}{4}.$$

So Hs always accept, and

$$S(H) = \tfrac{3}{4}(1-a), \quad S(L) = 0, \quad \pi(\text{seller}) = \tfrac{3}{4}a$$

[In fact, if $a \in [\frac{1}{2}, \frac{2}{3})$ it is also an equilibrium for Hs always to reject, and there is also a mixed strategy equilibrium in which Hs accept a fraction $q = (4 - \frac{2}{a})$ of the time.]

(ii) The distribution of types is not strictly increasing and atomless. So types' relative surpluses are not pinned down as they are in Appendix 1.A, figure 1.1—in terms of that analysis, equation (4) is not well defined (see part (iii) of this question). [The relative surpluses would be pinned down, and so revenue equivalence would apply, if the distribution of types were *either* strictly increasing *or* atomless; in our context a more restricted set of auction forms satisfies revenue equivalence; see, e.g., Maskin and Riley (1985).]

(iii) Type Hs win with probability $p_H = \frac{3}{4}$, and type Ls win with probability $p_L = \frac{1}{4}$. The incentive compatibility constraints are:

$$S(H) \geq S(L) + p_L(v_H - v_L) \quad \Rightarrow \quad S(H) \geq S(L) + \tfrac{1}{4},$$

$$S(L) \geq S(H) + p_H(v_L - v_H) \quad \Rightarrow \quad S(H) \leq S(L) + \tfrac{3}{4}.$$

Since L's individual rationality constraint requires $S(L) \geq 0$, an optimal mechanism has $S(L) = 0$, $S(H) = \frac{1}{4}$. The mechanism in (i), with $a = \frac{2}{3}$, is one such mechanism.

(iv) A take-it-or-leave-it offer of 1 to both buyers (with a lottery if both accept, and a commitment not to sell if neither does) is accepted by Hs and captures the entire surplus.

3. (i) See Appendix 1.A. (The fact that a bidder's probability of winning an object, $P_i(v)$, is different in a multi-unit context does not affect the standard argument.)

(ii) If after the first sealed-bid auction the second unit were unexpectedly sold by ascending auction (with the five remaining bidders), the expected payment, conditional on winning, of a bidder with value v would be $E({v_2}^5 \mid {v_1}^5 = v)$. Since the other four bidders' values, conditional on being below v, are uniformly distributed below v, this expectation equals $\frac{4}{5}v$ (see note 2). By revenue equivalence this must be the player's bid in the second sealed-bid auction.

(iii) If both objects are sold simultaneously in an ascending auction, a bidder with value v wins if she has one of the two highest values, and pays the thirdhighest actual value when she wins. Her expected payment is then

$$P(v \text{ is highest})\, E({v_3}^6 \mid v \text{ is highest})$$

$$+P(v \text{ is second highest})\, E({v_3}^6 \mid v \text{ is second highest}).$$

If the objects are sold in sequential sealed-bid auctions, the bidder still wins an object if her value is in the top two, and her expected payment is

$$P(v \text{ is highest})(v\text{'s bid in 1st auction})$$

$$+P(v \text{ is second highest})(v\text{'s bid in 2nd auction}).$$

By revenue equivalence, the expected payments from these two different two-unit auctions are equal. Furthermore, from part (ii) her bid in the second sealed-bid auction is $\frac{4}{5}v$, which equals $E({v_3}^6 \mid {v_2}^6 = v)$. Therefore, v's bid in the first sealed-bid auction is $E({v_3}^6 \mid {v_1}^6 = v)$ which equals $\frac{4}{6}v$.

(iv) Extending the argument, when $(m + 4)$ bidders remain for m objects, v's bid is $[4/(m + 4)]v$. Since the winner of the rth auction has the actual value v_r^{n+4}, and $m = n + 1 - r$, the actual winning bid in the rth auction is $[4/(m + 4)]v_r^{n+4} = [4/(n + 5 - r)]v_r^{n+4}$. But $E(v_r^{n+4}) = [(n + 5 - r)/(n + 5)]\bar{v}$, so the ex-ante expected price in the rth auction equals $[4/(n + 5)]\bar{v}$, which is constant. [It is not hard to check the price is a martingale.]

(v) See section 1.10.3.

4. (i) In an ascending auction, bidder i's unconditional expected payment is

$$\Pr(v_i > v_j)E[v_j \mid v_i > v_j] = F(v_i)\int_{\underline{v}}^{v_i} x\frac{f(x)}{F(v_i)}dx = \int_{\underline{v}}^{v_i} xf(x)dx$$

(or, integrating by parts, $v_iF(v_i) - \int_{\underline{v}}^{v_i} F(x)dx$).

By revenue equivalence, i's unconditional expected payment is the same in an "all-pay" auction (i.e., under current US rules).

We assumed that $F(\cdot)$ is strictly increasing and atomless, that bidders are risk-neutral, and that the "all-pay" equilibrium is symmetric and increasing with type $\underline{v}$ bidding zero.

(ii) Revenue equivalence implies that the expected legal expenses of a bidder with value v are the same under both rules, assuming Quayle's rules also result in a symmetric and increasing equilibrium with type $\underline{v}$ bidding zero (this is justified in part iv).

(iii) Now the expected payments of the lowest type are positive, since that type will lose and pay some of the opponent's cost. So every type's expected legal expenses are higher (by the expectation of the lowest type's payment, since the incentive compatibility condition is unchanged for every type; see Appendix 1.A) (assuming a symmetric, increasing, equilibrium, etc.; see part v).

(iv) Under Quayle's rules, spending $l(v_i)$ yields total expected payment

$$E(i\text{'s payment}) = \underbrace{l(v_i)}_{\text{Direct Cost}} + \underbrace{(1 - F(v_i))l(v_i)}_{\text{Pr(Lose)} * \text{Transfer}} \underbrace{- F(v_i)\int_{\underline{v}}^{v_i} l(x)\frac{f(x)}{F(v_i)}dx}_{\text{Pr(Win)} * E(\text{Transfer}|\text{Win})}.$$

Using parts (i) and (ii) we also have $E(i\text{'s payment}) = \int_{\underline{v}}^{v_i} xf(x)dx$. Equating these two expressions yields a differential equation which is satisfied by the (increasing) function

$$l(v_i) = \frac{{v_i}^2}{3}\frac{3 - v_i}{(2 - v_i)^2}$$

if $F(v_i) = v_i$ (noting that $\int_0^{v_i} l(x)dx = [{v_i}^3/6(2 - v_i)]$).

(v) A party's deservingness may influence the judgment, as may other asymmetries, risk aversion, etc.

Furthermore, *which* lawsuits are brought is a crucial issue not captured in the model. In particular, under the European systems low types will prefer not to participate (see part iii). This suggests the number of trials will be lower. (But note that if in our model low types are permitted to withdraw, there can be no increasing equilibrium, so we no longer have revenue equivalence with an ascending auction.)

However, the model does capture the intuition that the larger stakes in the European and the Quayle systems increase the incentives to spend. In the European system this means higher expenses. In Quayle's it counteracts the fact that paying your lawyer another \$1 may cost you \$2. [See section 2.2.1 and Appendix 2.A, and Baye, Kovenock, and de Vries (1997) for further discussion.]

5. (i) Bidder i's expected utility from behaving as type $\tilde{v}$, given her opponents behave according to the (to be determined) equilibrium bidding function $b(v)$ is

$$EU_i = \tilde{v}^{n-1}\left(v_i - b(\tilde{v}) - k\left[b(\tilde{v}) - E\left[\max_{j\neq i} b(v_j) \mid v_j < \tilde{v},\ \forall j \neq i\right]\right]\right).$$

Bidder i's optimal bidding choice of $\tilde{v}$ satisfies $\partial EU_i/\partial\tilde{v} = 0$, so:

$$\begin{aligned}&(n-1)\tilde{v}^{n-2}\left(v_i - b(\tilde{v}) - k\left[b(\tilde{v}) - E\left[\max_{j\neq i} b(v_j) \mid v_j < \tilde{v},\ \forall j \neq i\right]\right]\right)\\ &\quad + \tilde{v}^{n-1}\left(-b'(\tilde{v}) - k\left[b'(\tilde{v}) - \frac{\partial}{\partial\tilde{v}}E\left[\max_{j\neq i} b(v_j) \mid v_j < \tilde{v},\ \forall j \neq i\right]\right]\right)\\ &\quad = 0\end{aligned}$$

and in equilibrium $\tilde{v} = v_i$. Assuming there is a linear equilibrium $b(v) = \beta v$, and noting $E[\max_{j\neq i} b(v_j) \mid v_j < v_i, \forall j \neq i] = \beta[(n-1)/n]v_i$ (see note 2), we have

$$(n-1)v_i^{n-2}\left(v_i - \beta v_i - k\left[\beta v_i - \frac{n-1}{n}\beta v_i\right]\right)$$

$$+v_i^{n-1}\left(-\beta - k\left[\beta - \frac{n-1}{n}\beta\right]\right) = 0$$

$$\Rightarrow \quad \beta = \left(\frac{n-1}{n+k}\right).$$

(ii) Bidder i's unconditional expected utility is

$$EU_i = v_i^{n-1}\left(v_i - \beta v_i - k\left(\beta v_i - \frac{n-1}{n}\beta v_i\right)\right) = \frac{v_i^n}{n}.$$

The seller's expected revenue is

$$E\left[\max_i \ (\beta v_i)\right] = \frac{n-1}{n+k}\frac{n}{n+1}.$$

(iii) EU_i is independent of k: since the highest type wins, the lowest type makes zero surplus, and the other conditions for revenue equivalence are satisfied for all k, the bidders are equally well off for all k. $E[\max_i \ (\beta v_i)]$ decreases in k: since there is social waste (embarrassment), and the allocation and bidders' expected utilities are unchanged, the seller is worse off than if $k = 0$.

[The revenue equivalence theorem might more helpfully have been named the bidders'-surpluses equivalence theorem, since in a case like this (or in many wars of attrition, lobbying games, etc.) bidders' utilities are equivalent, but revenue is only equivalent if we think of the social waste as a part of revenue. That is, we have revenue equivalence in our problem if we think of revenue as $b_i + k(b_i - \max_{j\neq i} b_j)$.]

The intuition is that the risk of winning by a large amount depresses bids so that the seller is worse off. But for the buyers, the positive effect of lower bids and the negative cost of embarrassment exactly cancel.

(iv) If losers suffer embarrassment also, then the lowest type makes strictly negative surplus. This raises the issue of whether the lowest possible type would actually bid at all. If not, then under reasonable assumptions the seller is worse off.

If we assume that all bidders participate, then by the usual revenue equivalence argument (the incentive compatibility condition is unchanged for every type) every type's expected utility is reduced by the same amount. Bids are increased by losers' embarrassment costs (the lowest type bids a strictly positive amount, and so pushes up others' bids), so the seller might be better off than with no embarrassment. [For example, if a loser's (dis)utility is $u_i = -l(\max_{j\neq i} b_j - b_i)$ and $n = 2$, then equilibrium bidding strategies are $b(v_i) = [(v_i + l)/(k + l + 2)]$, so if $2l > k$ the seller is better off than with no embarrassment, while if $0 < 2l < k$ both buyers and seller are worse off.]

(v) In an ascending auction the winner pays just a little more than the runner-up bid so suffers no embarrassment. But part (iii) shows the view expressed in the first sentence of the question is wrong in the model of part (i). The view is correct in the model of part (iv) (and, I conjecture, also for some bidders in asymmetric versions of model i). However, more important reasons for bidders to lobby against sealed-bid auctions are discussed in parts C and D of this volume.

6. (i) Conditional on a bidder with value x winning, her opponent's value, v_{opp}, is uniform on $[0, x]$, so bidder x expects to pay $\frac{1}{2}x$.

(ii) Conditional on winning, a bidder pays v_{opp} if $v_{\text{opp}} < p(x)$, and b otherwise. Thus, her conditional expected payment is

$$\Pr\Big(v_{\text{opp}} < p(x) \mid v_{\text{opp}} < x\Big)E\Big(v_{\text{opp}} \mid v_{\text{opp}} < p(x)\Big) + \Pr\Big(v_{\text{opp}} > p(x) \mid v_{\text{opp}} < x\Big)b$$

$$= \frac{p(x)}{x}\,\tfrac{1}{2}p(x) + \frac{x - p(x)}{x}b = \frac{1}{x}\Big(\tfrac{1}{2}(p(x))^2 + b\big(x - p(x)\big)\Big).$$

(iii) By revenue equivalence between (i) and (ii) (note $p'(x) > 0$ ensures that the highest value bidder wins),

$$\tfrac{1}{2}x = \frac{1}{x}\Big(\tfrac{1}{2}(p(x))^2 + b\big(x - p(x)\big)\Big) \quad\Rightarrow\quad \big(p(x) - x\big)\big(p(x) + x - 2b\big) = 0.$$

The relevant root is $p(x) = 2b - x$.

[Note that $b \geq 0.5$ ensures $p(x) > 0$. If $b < 0.5$ a set of types would bid b immediately, resulting in possible inefficiency, and meaning that revenue equivalence with the ascending auction no longer holds.]

(iv) (a) Conditional on type x winning the auction, if the runner-up's value is below $p(x)$ both the buy-price and the ascending auction yield the same actual revenue, but if the runner-up's value exceeds $p(x)$ the buy-price auction yields b while the ascending auction revenue is variable with the same average (since conditional on type x winning expected revenue is $\frac{1}{2}x$ in both auctions). Therefore, conditional on type x winning the auction, the distribution of revenues from the ascending auction can be derived by a sequence of mean-preserving spreads from the distribution of revenues from the buy-price auction. So any risk-averse seller will have a higher expected utility from the buy-price than from the ascending auction, for each x, by Rothschild and Stiglitz's (1970) standard result. Since this expected utility ranking of the auctions holds for all x, it also holds for the ex ante expected utilities, that is, when the expectation is taken before knowing x.

(b) The argument parallels that for (a): conditional on *any* given x winning, the buy price auction has the same expected revenue, $\frac{1}{2}x$, as the first-price auction, but is riskier (the first-price auction always yields exactly $\frac{1}{2}x$), so a risk-averse seller prefers the (unconditional) distribution of revenues from the first-price auction to that from the buy-price auction.

(v) Bidders' risk-aversion does not affect bids in the ascending auction, but will result in more aggressive bidding and higher profits in buy-price and first-price auctions (see question 11 and section 1.5). A natural conjecture is that the first-price auction will be even more profitable than the buy-price auction (since the bidding of low-value bidders is unaffected in the latter auction; also with sufficient risk-aversion a set of types might immediately offer the buy-price, resulting in inefficiency). [Budish and Takeyama (2001) provide the first analysis of "buy-prices" of which I am aware, and compare the revenues from the buy-price and the first-price auctions when the type space is discrete.]

7. (i) Marginal revenues in markets 1 and 2 are $MR_1 = 1 - 2q_1, MR_2 = 2 - 2q_2$ (this can be written $MR_1 = 2p_1 - 1$, $MR_2 = 2p_2 - 2$, see (ii)). To maximize total revenue, set $MR_1 = MR_2$, with $q_1 + q_2 = 1$. So $q_1 = \frac{1}{4}, q_2 = \frac{3}{4}$ and $p_1 = \frac{3}{4}$, $p_2 = \frac{5}{4}$ (since this yields $MR_i > 0$, the monopolist wants to sell the whole unit).

(ii) In an auction context "price" $p = v$ and $MR_i = v_i - [(1 - F(v_i))/f(v_i)]$ (see Appendix 1.B). Since $F(v_1) = v_1$, $F(v_2) = v_2 - 1$, we again have $MR_1 = 2v_1 - 1$, $MR_2 = 2v_2 - 2$. The optimal auction allocates the unit to the bidder who has the highest marginal revenue (bidder 1 if $2v_1 - 1 \geq 2v_2 - 2$, that is $v_1 \geq v_2 - \frac{1}{2}$, and bidder 2 otherwise), at a price equal to the lowest valuation he could have had and still won (since $MR_2 \geq 0$ the auctioneer always wants to sell the unit). (With this pricing rule, bidders will report their *MR*s truthfully; see Bulow and Roberts, 1989.)

(iii) A monopolist sells to its highest marginal revenue customers, and an optimal auctioneer sells to the highest marginal revenue bidder, in both cases at the highest prices that satisfy incentive compatibility. The difference is that in (i) expected sales to market 1 are $\frac{1}{4}$, and total revenue is $\frac{9}{8}$; in (ii) expected sales to buyer 1 are $\frac{1}{8}$ (bidder 2 always wins if $v_2 > \frac{3}{2}$ or $v_1 < \frac{1}{2}$, and wins in half the cases where both $v_2 < \frac{3}{2}$ and $v_1 > \frac{1}{2}$), and total revenue is $\frac{25}{24}$. Price discrimination is like an auction with a flexible capacity constraint; an auctioneer who was permitted to give out units to both bidders or to neither bidder, such that the same number of units was given out on average, would face the same problem (and so earn the same revenue) as the price-discriminating monopolist (our auctioneer would give units to both bidders if $v_1 \geq \frac{3}{4}$ and $v_2 \geq \frac{5}{4}$, to neither bidder if $v_1 < \frac{3}{4}$ and $v_2 < \frac{5}{4}$, and sell one unit otherwise).

8. (i) See Appendix 1.B. The result extends to other auction forms whenever the revenue equivalence theorem applies.

(ii) The optimal auction allocates the item to the bidder who reports the highest marginal revenue, $v_i - [(1 - F(v_i))/f(v_i)]$, at the lowest valuation she could have had and still won (since $MR_A \geq 0$ the auctioneer always wants to sell the unit). (With this pricing rule, bidders will report their *MR*s truthfully; see Bulow and Roberts, 1989.) The bidders' marginal revenues are $MR_A = 10$, $MR_B = 2v_B - 30$, and $MR_C = 2v_C - 50$. So if $v_B < 20$ and $v_C < 30$, *A* gets the item at price $p = 10$. If $v_B > 20$ and $v_C < 30$, *B* wins at $p = 20$. If $v_B < 20$ and $v_C > 30$, *C* wins at $p = 30$. Finally, if $v_B > 20$ and $v_C > 30$, then if $v_B - 20 > v_C - 30$, *B* wins at $p = v_C - 10$, otherwise C wins at $p = v_B + 10$.

(iii) This is the "Maximum Game" of Bulow and Klemperer (2002). Bidder *i*'s marginal revenue is

$$MR_i = v - \frac{1 - F_i(t)}{f_i(t)} \frac{\partial v}{\partial t_i}.$$

Writing $t_{\max}$ for the highest signal,

$$MR_i = \begin{cases} t_i - (1 - t_i) & \text{if } i \text{ has the highest signal} \\ t_{\max} & \text{if } i \text{ has one of the two lower signals (since then } \dfrac{\partial v}{\partial t_i} = 0). \end{cases}$$

So, since $E(t_{\max}) = \frac{3}{4}$ (see note 2), the expected marginal revenue of the winning bidder equals $\frac{1}{2}$ conditional on the highest-signal bidder winning, and equals $\frac{3}{4}$ conditional on the highest-signal bidder not winning. By part (i), therefore:

(a) The bidder with the highest signal wins: $E[\pi(\text{seller})] = \frac{1}{2}$.

(b) The bidder with the highest signal wins $\frac{2}{3}$ of times while a bidder with a lower signal wins $\frac{1}{3}$ of times (when the highest is excluded):

$$E[\pi(\text{seller})] = \tfrac{2}{3}\tfrac{1}{2} + \tfrac{1}{3}\tfrac{3}{4} = \tfrac{7}{12}.$$

(c) The bidder with the highest signal wins $\frac{1}{3}$ of times while a bidder with a lower signal wins $\frac{2}{3}$ of times:

$$E[\pi(\text{seller})] = \tfrac{1}{3}\tfrac{1}{2} + \tfrac{2}{3}\tfrac{3}{4} = \tfrac{2}{3}.$$

(d) The price is *decreasing* in the number of bidders because higher-signal bidders have lower marginal revenues (and higher information rents). Reducing the number of bidders increases the probability of selling to a lower-signal bidder. (See also chapter 1, note 63.)

[A direct, but more cumbersome, approach to part (iii) is: (a) the symmetric equilibrium bidding strategy is $b(t) = t$ (no bidder will drop out before her signal and, since the bidder with the highest signal wins, a bidder who stays in past her signal, and wins, will have negative profits), so $E\pi = E$(second-highest signal) $= \frac{1}{2}$. (b) If bidders i and j participate, then bidder i with signal t_i bids up to $E(v \mid t_i = t_j)$ (see Appendix 1.D). This equals $t_i t_i + (1 - t_i)\frac{1}{2}(1 + t_i) = \frac{1}{2}(1 + t_i^2)$ (because, conditional on $t_i = t_j$, with probability $1 - t_i$ the excluded signal is the highest with expected value $\frac{1}{2}(1 + t_i)$). Since the density of the lower of two uniform signals is $2(1 - t)$, the expected price at which the lower signal of the two participants quits is $E\pi = \int_0^1 2(1 - t)\frac{1}{2}(1 + t^2)dt = \frac{7}{12}$. (c) The lowest type estimates the value as the expectation of the highest of the other two signals, which is $\frac{2}{3}$. So, this is the take-it-or-leave-it price.]

9. (i) Bidder i remains in the bidding until the price where, if $z_i = z_j$, she is exactly indifferent between winning and quitting, that is, she bids up to $3z_i + z_i = 4z_i$. (See Appendix 1.D, for discussion.)

(ii) (a) Conditional on i winning, j's signal is uniformly distributed on $[0, z_i]$. Thus, i's conditional expected payment in the ascending auction is $4z_i/2$. By revenue equivalence, the expected payment conditional on winning is the same in the Dutch auction, hence i bids $b(z_i) = 2z_i$.

(b) If type z_i bids $b(\tilde{z})$, she wins with probability $\tilde{z}$, and her prize is worth $3z_i + (\tilde{z}/2)$ on average (since she beats types below $\tilde{z}$), so her surplus is $S = \tilde{z}(3z_i + \frac{1}{2}\tilde{z} - b(\tilde{z}))$. Her optimal choice of $\tilde{z}$ satisfies

$$\frac{\partial S}{\partial \tilde{z}} = 0 \quad \Rightarrow \quad \left(3z_i + \tfrac{1}{2}\tilde{z} - b(\tilde{z})\right) + \left(\tfrac{1}{2} - b'(\tilde{z})\right)\tilde{z} = 0.$$

In equilibrium, $\tilde{z} = z_i$, so $b'(\tilde{z}_i) = 4 - (b(z_i)/z_i)$ and $b(0) = 0$ (in symmetric equilibrium type $z_i = 0$ wins only when $z_j = 0$, so type 0 cannot bid more). So, $b(z_i) = 2z_i$.

(iii) (a) Buyers' strategies in ascending auctions are not affected by their risk preferences. However, in a Dutch auction, risk-neutral bidders bid $b(z_i) = 2z_i \leq 3z_i + z_j$, so are guaranteed profits, conditional on winning. So risk-averse bidders bid a little higher, trading lower payoffs for a higher probability of winning. [For example, if bidders' utilities are $\sqrt{v_i - b(v_i)}$, then in a Dutch auction they bid up to $b(z_i) = (\frac{3}{2} + \sqrt{\frac{11}{12}})z_i$.] So the seller prefers a Dutch auction.

(b) Conditional on type z_i winning, in the Dutch auction *actual* revenue is $2z_i$, whereas in the ascending auction *expected* revenue is also $2z_i$ but *actual* revenue is variable (since it depends on the loser's signal), so the distribution of ascending auction revenues is a mean-preserving spread of the Dutch revenues. Therefore any risk-averse seller will have a higher expected utility from the Dutch auction than from the ascending auction, for each z_i, by Rothschild and Stiglitz's (1970) standard result. Since this expected utility ranking of the auctions holds conditional on *any* z_i, it also holds for the ex ante expected utilities, that is, when the expectation is taken with respect to z_i. So a risk-averse seller prefers the Dutch auction.

10. (i) See Appendix 1.A.

(ii) This is a version of the "Wallet Game" analyzed in Klemperer (1998) and Bulow and Klemperer (2002). Write $t_{(j)}$ for the jth highest actual signal. In a symmetric equilibrium, type t expects to win if her signal is not lowest, so remains in the bidding until the price at which, if she is tied with another bidder for lowest signal, she is exactly indifferent between winning and losing. (See Appendix 1.D for a similar argument.) Since $E\left[t_{(1)} \mid t_{(2)} = t_{(3)} = t\right] = \frac{1}{2}(1 + t)$, this price is $b(t) = \frac{1}{3}(t + t + \frac{1}{2}(1 + t)) = \frac{1}{6} + \frac{5}{6}t$.

(iii) Type t's expected surplus is

$$S(t) = \Pr(t \text{ is highest}) \cdot E\left[v - b(t_{(3)}) \mid t_{(1)} = t\right]$$
$$+ \Pr(t \text{ is second highest}) \cdot E\left[v - b(t_{(3)}) \mid t_{(2)} = t\right].$$

Type t is highest with probability t^2, and in this case $E[t_{(2)}] = \frac{2}{3}t$, $E[t_{(3)}] = \frac{1}{3}t$ (see note 2), and hence $E[v] = \frac{2}{3}t$ and $E[b(t_{(3)})] = (\frac{1}{6} + \frac{5}{6}(\frac{1}{3}t))$. Type t is second-highest with probability $2t(1 - t)$, and in this case $E[t_{(1)}] = \frac{1}{2}(1 + t)$, $E[t_{(3)}] = \frac{1}{2}t$, and hence $E[v] = \frac{1}{6}(4t + 1)$ and $E[b(t_{(3)})] = (\frac{1}{6} + \frac{5}{6}(\frac{1}{2}t))$. Therefore

$$S(t) = t^2\left(\frac{2}{3}t - \left(\frac{1}{6} + \frac{5}{6}\left(\frac{1}{3}t\right)\right)\right) + 2t(1 - t)\left(\frac{1}{6}(4t - 1) - \left(\frac{1}{6} + \frac{5}{6}\left(\frac{1}{2}t\right)\right)\right)$$
$$= \frac{1}{9}t^2(3 - t).$$

(iv) Similarly, the surplus of type t in a first-price auction is

$$t^2\left(\frac{2}{3}t - b(t)\right) + 2t(1 - t)\left(\frac{1}{6}(4t + 1) - b(t)\right)$$

(in which $b(t)$ is now the first-price bid).

By revenue equivalence, and part (iii), this equals $\frac{1}{9}t^2(3 - t)$, so

$$b(t) = \frac{-5t^2 + 6t + 3}{9(2 - t)}.$$

11. (i) (a) It is a dominant strategy for each bidder to remain in the bidding until the price reaches her value. So the bidder with the highest value wins and pays the second highest value, and expected revenue is $E(\text{2nd highest } v_i) = \frac{1}{3}$ (see note 2).

(b) Assuming the other bidder bids according to the (to be determined) equilibrium bidding strategy $b(v)$, i's surplus from bidding as type $\tilde{v}$ is $S = (v_i - b(\tilde{v}))\tilde{v}$ (since i then wins with probability $\tilde{v}$). Bidder i's optimal choice of $\tilde{v}$ satisfies $\partial S/\partial\tilde{v} = 0 \Rightarrow (v_i - b(\tilde{v})) + \tilde{v}(-b'(\tilde{v})) = 0$. In equilibrium, $\tilde{v} = v_i$, so $b'(v_i) = [(v_i - b(v_i))/v_i]$ which, since $b(0) = 0$, has solution $b(v_i) = \frac{1}{2}v_i$. So expected revenue is $\frac{1}{2}E(\text{highest } v_i) = \frac{1}{3}$.

(c) Expected revenue is the same across auction types, illustrating the revenue equivalence theorem.

(ii) (a) Bidding up to v_i remains the dominant strategy, so expected revenue is unaffected by the risk-aversion.

(b) Following the same method as in (i)(b):

$$S = \left(\sqrt{v_i - b(\tilde{v})}\right)\tilde{v}.$$

$$\frac{\partial S}{\partial \tilde{v}} = 0 \quad \Rightarrow \quad \sqrt{v_i - b(\tilde{v})} + \frac{\tilde{v}\left(-b'(v_i)\right)}{2\sqrt{v_i - b(\tilde{v})}} = 0.$$

For $\tilde{v} = v_i$, $b'(v_i) = 2(v_i - b(v_i))/v_i$ which, using the condition $b(0) = 0$, has solution $b(v_i) = \frac{2}{3}v_i$. So expected revenue is $\frac{2}{3}E(\text{highest } v_i) = \frac{4}{9}$.

(c) Risk-averse players bid uniformly higher in first-price auctions than their risk-neutral counterparts. Taking one's opponent's bid as fixed, a higher bid is less risky—it gives a higher probability of a lower prize—so is preferred by a risk-averse bidder. Since (i) satisfied revenue equivalence, first-price auctions are more profitable than ascending ones in (ii).

12–20. Outlines for some of the essays can be found at www.paulklemperer.org.