Toeholds and Takeovers: General Characterization, Existence, and Uniqueness of Equilibrium

Jeremy Bulow, Ming Huang, and Paul Klemperer

February 1996

Preliminary Version–Please let us know of any errors.

Abstract: This paper generalizes and extends Bulow-Huang-Klemperer (1995) (BHK), which analyses the effect of toeholds on competition among bidders in a takeover battle. We consider a generalization in which each bidder's valuation of the target company has both private and common-value components. We obtain necessary and sufficient conditions for equilibrium. We prove the existence of a unique equilibrium under mild technical conditions. Specifically, the equilibrium strategies of two bidders with toeholds are shown to be pure strategies. The bidding functions of the two bidders are continuous, strictly increasing, and satisfy a pair of coupled differential equations. These differential equations are shown to admit a unique solution under mild technical conditions.

This is a technical supplementary paper to "Toeholds and Takeovers," J. Bulow, M. Huang, and P. Klemperer, Journal of Political Economy, June 1999.

1. Introduction

Takeover contests sometimes occur among bidders with prior stakes, often referred to as "toeholds," in the target company. Bradley, Desai, and Kim (1988), for example, report that more than half of acquiring firms have toeholds prior to takeover bidding. Few studies, however, analyze the effect of toeholds on the competition between bidders.[1]

In Bulow, Huang, and Klemperer (1995) (BHK), we analyze the effect of toeholds on competition among bidders and on the expected revenue in a takeover battle. We also study the policy implication of such effects. In order to focus on the strategic effects of toeholds on bidders' competition, we model a takeover contest as an ascending-bid auction between two bidders with private information and common valuation. Our results show that, in a pure common-values setting, even small toeholds can have large effects on bidders' equilibrium strategies and on shareholders' expected revenue. This paper extends this study to a more general setting in which bidders' valuations of the target company have both private- and common-value components.

BHK shows that toeholds have two conflicting effects on bidders' equilibrium bidding strategies. The first effect is that toeholds give bidders an incentive to bid more aggressively. Given the opponent's strategy, a bidder with a toehold bids more aggressively than he would without any toehold because he takes into account the fact that a slightly higher final price increases his profit in the event that he loses and sells his stake. Each bidder thus chooses to "overbid" in order to maximize his ex-ante expected profit, even though he might make negative profit ex-post.[2] In fact, in equilibrium, any bidder who narrowly wins the auction loses money. A bidder may also bid more aggressively because of the increased aggressiveness of his opponent since he feels safer in trying to "bid up the price under" his opponent. This first effect dominates when the two bidders have sufficiently symmetric toeholds, and results in higher expected revenue for other shareholders.

The second effect, as shown by BHK, is due to the common-value component of bidders' valuations of the target company. Since a bidder's ex post valuation of the target company also depends on his opponent's private signal, beating a more aggressive opponent at a given price implies a higher winner's curse. A bidder facing a more aggressive opponent may therefore be more cautious. This effect often dominates when the two bidders have

[1] Shleifer and Vishny (1986) show how initial toeholds, as a source of profit, can provide an incentive for a takeover bidder who faces the free-rider problem of Grossman and Hart (1980). Hirshleifer and Titman (1990) and Chowdhry and Jegadeesh (1994) also use toeholds to circumvent the free-rider problem. The former analyzes the tendering decisions of shareholders and the optimal offer of a bidder with private information about the value of the target company. The latter shows how the bidder can use the size of his toehold as a signaling device. All of these papers, however, assume that there is only one bidder and focus on the interaction between the bidder and the shareholders.

[2] Without a toehold, each bidder quits at a price that is equal to his (or her) own expected valuation, conditioning on his opponent quitting at that price.

sufficiently asymmetric toeholds. In a pure common-values setting, a bidder with a large toehold can force his opponent to quit at a relatively low price, leaving non-bidding shareholders with less expected revenue. The management of a target company, when facing a bidder with a large toehold, should then "level the playing field", either by altering the auction format or by helping a second bidder acquire a stake in the company in order to achieve higher expected revenue for non-bidding shareholders.

BHK uses a pure common-values setting so as to focus on the strategic effect of toeholds on bidders' equilibrium strategies. More realistically, we expect that takeover battles have elements of both common- and private-value auctions. Since the bidders would often use the target's assets in similar ways, we expect each bidder to have different information that is useful in assessing the common value of these assets. Some of the gains to a bidder from takeover, however, may be unique to the bidder, and thus independent of other bidders' private signals. A more realistic model would then incorporate a mixed valuation for the bidders: each bidder's valuation of the target company depends on both his own signals and his opponent's private signals, and it depends more on the former than on the latter. In this paper, we extend some of the results of BHK to this more general setting.

Obtaining explicit solutions of the equilibrium of an ascending-bid auction under private information and mixed valuation, however, turns out to be difficult. In this paper, we will instead give a general characterization of the equilibrium and prove the existence of a unique equilibrium under mild technical conditions. We first show that a pair of strategies is an equilibrium if and only if the two bidders' bids, each as a function of his private signal, are single-valued (pure strategy), continuous, strictly increasing, and satisfy a specified pair of coupled differential equations with appropriate boundary conditions. Making use of this characterization, we then go on to show that, under mild technical conditions, there exists a unique equilibrium for an ascending auction between two bidders with toeholds.

Burkart (1995) and Singh (1995) have also analyzed the effects of toeholds on takeover contests.[3] Both papers, however, restrict to the case of independent pure private values and thus ignore the potentially large strategic effects of toeholds. In addition, the equilibria studied in both papers rely on the assumption that one of the bidders does not have a toehold.[4] In this paper, we assume that both bidders have positive toeholds and that each bidder's valuation of the target company has both private- and common-value components.

The rest of the paper proceeds as follows. In Section 2, we introduce the model and specify technical assumptions. Section 3 characterizes an equilibrium by providing necessary and sufficient conditions. Section 4 proves the existence of a unique equilibrium solution. Section 5 concludes.

[3] We became aware of the two papers after developing the results in BHK and in this paper.

[4] Their equilibrium solutions are not robust to giving even an arbitrarily small stake to the bidder with no toehold.

2. The Model

Two risk-neutral bidders, each with exogenously given partial ownership of a corporation in the fractional amounts of θ_1 and θ_2, respectively, compete to take over the corporation. Bidders' prior shares are common knowledge. Each bidder has his (or her) own private information. Prior to the takeover auction, bidder i (with $i = 1, 2$) observes its own type (or signal) T_i, a random variable with outcomes in $[0, 1]$, but has no information on the other's type. The two types, T_1 and T_2, are independently distributed. The distribution function for T_i, for $i = 1, 2$, is $F_i : [0, 1] \to [0, 1]$, with $F_i(0) = 0$ and $F_i(1) = 1$. Bidder i's valuation of the target company, v_i, may depend on the other bidder's type as well as its own. That is, $v_i = V_i(T_1, T_2)$, with $i = 1, 2$.

The target company is sold using a conventional ascending-bid (i.e. English) auction. That is, the price starts at zero and rises continuously until one of the bidders drops out at some price $b \geq 0$. The winner buys the loser's shares as well as all of non-bidders' shares at the drop-out price b.[5] A tie is broken at random.[6] The payoff to the loser i (with toehold θ_i) is $\theta_i b$, while the payoff to the winner j (with toehold θ_j) is $V_j(t_1, t_2) - (1 - \theta_j)b$ if the types T_1 and T_2 are t_1 and t_2, respectively.

2.1. Bidders' Strategies and Equilibrium

A strategy for bidder i specifies his bidding plans for every possible type in $[0, 1]$. For a given type, bidder i may decide to quit at a certain price if the other bidder has not yet done so, or he may randomize among multiple bids. A strategy for bidder i, therefore, gives his bid in the form of a probability measure that depends on his own type. Formally, we define *a strategy* for bidder i as a function from the Cartesian product of the set of possible types $[0, 1]$ with the family $\mathcal{B}([0, \infty))$ of the Borel subsets of the set of admissible bids $[0, \infty)$ to the interval $[0, 1]$. That is, $\mathcal{P}_i : [0, 1] \times \mathcal{B}([0, \infty)) \to [0, 1]$, so that $\mathcal{P}_i(t_i, \cdot)$ is a probability measure over $[0, \infty)$ for all $t_i \in [0, 1]$. A strategy is *pure* if, for any $t_i \in [0, 1]$, the support of probability measure $\mathcal{P}_i(t_i, \cdot)$ is a single point.

Each bidder maximizes his expected payoff by choosing an optimal strategy while taking into account the other bidder's strategy. Define the product measure of $\mathcal{P}_j$ and F_j, for $j = 1, 2$, over the product of the set of possible types $[0, 1]$ with the set of allowable bids

[5] The equilibrium problem is similar if the winner buys only just enough shares so that he has the minimum fraction of the total shares that gives him right of control. We assume that all shareholders, including the two bidders, are willing to sell their shares to the winner of the takeover contest. We thus ignore the free-rider problem of Grossman and Hart (1980) and focus on the effect of toeholds on competition between the bidders. Alternatively, this assumption can be interpreted as an assumption that bidders make offers directly to the management. See also, for example, Giammarino and Heinkel (1986), Fishman (1988), and Hirshleifer and Png (1989).

[6] With a tie, bidder i wins with probability $q_i > 0$, where $q_1 + q_2 = 1$. This assumption serves to complete the rules of the model and does not have any impact on the equilibrium.

$[0, \infty)$ as

$$\mathcal{P}_j {*} F_j\big(\tau \times \mathcal{B}\big) = \int_\tau \mathcal{P}_j(t, B)\, dF_j(t)$$

for any Borel subset τ of $[0,1]$ and any Borel subset B of $[0, \infty)$. Let $\Pi_i(b', t_i; b; \mathcal{P}_j)$, for $(i, j) = (1, 2)$ or $(2, 1)$ and for b' and b in $[0, \infty)$ with $b' \geq b$, denote the conditional expected payoff to bidder i (with type t_i) who decides to quit at price b' if the other bidder has not yet done so, given bidder j's strategy $\mathcal{P}_j$ and the fact that no one has dropped out before bidding level b. Then we have

$$\begin{aligned}
\Pi_1\big(b', t_1; b; \mathcal{P}_2\big) = \frac{1}{\mathcal{P}_2 {*} F_2\Big([0,1] \times [b, \infty)\Big)} & \Big[\theta_1 b' \, \mathcal{P}_2 {*} F_2\Big([0,1] \times [b, b')\Big) \\
& + (1 - q_1)\theta_1 b' \, \mathcal{P}_2 {*} F_2\Big([0,1] \times \{b'\}\Big) \\
& + q_1 \int_{t_2=0}^{1} [V_1(t_1, t_2) - (1 - \theta_1)b'] \, \mathcal{P}_2 {*} F_2\Big(dt_2 \times \{b'\}\Big) \\
& + \int_{t_2=0}^{1} \int_{b=b'+}^{\infty} [V_1(t_1, t_2) - (1 - \theta_1)b] \, \mathcal{P}_2 {*} F_2\Big(dt_2 \times db\Big)\Big],
\end{aligned}$$

where q_i, for $i = 1, 2$, is the probability that bidder i wins in case of a tie at price b'. The conditional expected payoff by bidder 2 is symmetric.[7] The set of prices, $\beta_i(t_i; \mathcal{P}_j)$, at which it is optimal for bidder i with type t_i to quit, is given by

$$\beta_i\big(t_i; \mathcal{P}_j\big) = \Big\{\hat{b} \geq 0 : \Pi_i\big(\hat{b}, t_i; b; \mathcal{P}_j\big) \geq \Pi_i\big(b', t_i; b; \mathcal{P}_j\big), \quad b \in [0, \hat{b}], \ b' \geq b\Big\}, \qquad \text{for} \quad (i,j) = (1,2) \text{ or } (2,1). \tag{2.1}$$

A pair of strategies $(\mathcal{P}_1, \mathcal{P}_2)$ form a Nash equilibrium if, for any $t_i \in [0,1]$ and for $(i, j) = (1, 2)$ and $(2, 1)$, the support of $\mathcal{P}_i\big(t_i, \cdot\big)$ is a subset of $\beta_i\big(t_i; \mathcal{P}_j\big)$. A Nash equilibrium $(\mathcal{P}_1, \mathcal{P}_2)$ is a *pure strategy* Nash equilibrium if $\mathcal{P}_1$ and $\mathcal{P}_2$ are pure.

2.2. Assumptions for General Characterization

We first specify assumptions under which we can give a general characterization of the equilibrium strategies. The first assumption is that both bidders have positive toeholds.[8] That is, $\theta_i > 0$ for each i. The second assumption regards the distributions of the types of the two bidders.[9] The third assumption is on the valuation functions of both bidders.

[7] Note that $\Pi_1\big(b', t_1; b; \mathcal{P}_2\big)$ is (and needs to be) defined only for b satisfying $\mathcal{P}_2{*}F_2\Big([0,1] \times [b, \infty)\Big) > 0$.

[8] The equilibrium in which one bidder does not have any toehold is qualitatively different from the equilibrium in which one bidder has an arbitrarily small toehold.

[9] The assumption that the distributions of T_1 and T_2 are the same can be made without loss of generality.

In order to focus on the impact of asymmetric toeholds, we assume that v_1 and v_2 have distributions with the same support (Assumption 3(ii)), and that the bidder with the higher signal has a (weakly) higher valuation (Assumption 3(iv)).

ASSUMPTION 1. *Toeholds:* $\theta_i \in (0,1)$ *for* $i = 1,2$, *and* $\theta_1 + \theta_2 < 1$.

ASSUMPTION 2. *The types of the two bidders are independent and have the same distribution function* F. *That is,* $F_i = F_j = F$. *The density function* f *of the distribution function* F *exists on* $[0,1]$ *and is continuous and strictly positive (and thus bounded) over the same interval.*

ASSUMPTION 3. *The valuation functions* $V_1 : [0,1] \times [0,1] \to [0,1]$ *and* $V_2 : [0,1] \times [0,1] \to [0,1]$ *have the following properties:*

(i) Continuity: V_1 *and* V_2 *are continuous;*

(ii) Monotonicity: For each i, V_i *is increasing. Moreover,* $V_1(\cdot, t_2)$ *and* $V_2(t_1, \cdot)$ *are strictly increasing, respectively, for each* $t_2 \in [0,1]$ *and* $t_1 \in [0,1]$;

(iii) Bounds: $V_1(0,0) = V_2(0,0) = 0$, *and* $V_1(1,1) = V_2(1,1) = 1$;

(iv) Partial-Private Valuation: whenever $0 \leq t_1 \leq t_2 \leq 1$, $V_1(t_1,t_2) \leq V_2(t_1,t_2)$.

2.3. Some Special Cases

Assumptions 1–3 are mild and allow for general cases of toeholds and valuations. We list here some special cases that are consistent with these assumptions.

(1) Symmetric Toeholds: $\theta_1 = \theta_2$;

(2) Pure Common Valuation: $V_1 = V_2$;

(3) Pure Private Valuation: There exists a continuous and strictly increasing function $v : [0,1] \to [0,1]$ such that $V_i(t_1,t_2) = v(t_i)$ for $i = 1,2$ and for any $(t_1,t_2) \in [0,1] \times [0,1]$.

Burkart (1995) and Singh (1995) studied the special case of pure private valuation under the further assumption that one bidder does not have a toehold. BHK focused on the case of pure common valuation and allow both bidders to have positive toeholds.

2.4. Additional Assumptions for Existence of Unique Solution

In order to ensure the existence of an equilibrium, we add a standard technical assumption on f, V_1, and V_2.

ASSUMPTION 4. *The density function* f *and the valuation functions* V_1 *and* V_2 *are locally Lipschitz.*

For the three special cases listed in Section 2.3, we will show that Assumptions 1–4 are sufficient to ensure the existence and uniqueness of a Nash equilibrium. For general cases satisfying Assumptions 1–3, however, Assumption 4 is not enough to ensure the existence of a unique equilibrium. Fortunately, we only need to make one more mild technical assumption to ensure the existence of an equilibrium. For this, we will strengthen Assumption 3(ii) and assume that each bidder's valuation is strictly monotonic with respect to the other bidder's signal. This is Assumption 5. In order to ensure the uniqueness of a given equilibrium for general cases, we also strengthen Assumption 3(iv) and assume that the partial-private valuation property is strict. In addition, we make an assumption on the partial derivatives of V_1 and V_2 at $(0,0)$. This is Assumption 6.

ASSUMPTION 5. *Strict Monotonicity of Valuation With Respect to Other's Type:*[10] $\frac{\partial V_1}{t_2}(t_1,t_2)$ *and* $\frac{\partial V_2}{t_1}(t_1,t_2)$ *exist and are strictly positive for all* $(t_1,t_2) \in (0,1)\times(0,1)$.

ASSUMPTION 6. *Strict Partial-Private Valuation: whenever* $0 \le t_1 < t_2 \le 1$, $V_1(t_1,t_2) < V_2(t_1,t_2)$, *and, furthermore, the partial derivatives of* V_1 *and* V_2 *exist at* $t_1 = t_2 = 0$ *and* $\dfrac{\partial V_1}{\partial t_1}(0,0) = \dfrac{\partial V_2}{\partial t_2}(0,0) > \dfrac{\partial V_1}{\partial t_2}(0,0) = \dfrac{\partial V_2}{\partial t_1}(0,0)$.

In summary, we will show that Assumptions 1–4 are sufficient for the existence and uniqueness of an equilibrium for the special cases in Section 2.3. For other general cases, we will show that Assumptions 1–5 are sufficient to ensure the existence of an equilibrium, and that Assumptions 1–4 and 6 are sufficient to ensure the uniqueness of any given equilibrium.

3. General Characterization of the Nash Equilibrium

In this section, we characterize the equilibrium by looking for its necessary and sufficient conditions. This extends the general characterization of equilibrium strategies by BHK to cases in which each bidder's valuation of the target company has both private- and common-valuation components.

It is useful to note first that, for our problem, the equilibrium strategies for a two-bidder ascending auction are the same as those of a second-price sealed-bid auction. This follows from the fact that each bidder's optimal strategy, given the other bidder's strategy, is the same for these two kinds of auctions. For a second-price sealed-bid auction, the expected payoff to bidder i (with type t_i) bidding b, given the other bidder's strategy $\mathcal{P}_j$, for $(i,j) = (1,2)$ or $(2,1)$, is given by $\Pi_i(b, t_i; 0; \mathcal{P}_j)$. The set of prices at which it is optimal for bidder i with type t_i to quit is given by

$$\tilde{\beta}_i(t_i;\mathcal{P}_j) = \left\{\hat{b} \ge 0 : \Pi_i(\hat{b}, t_i; 0; \mathcal{P}_j) \ge \Pi_i(b', t_i; 0; \mathcal{P}_j) \text{ for any } b' \in [0,\infty)\right\}. \tag{3.1}$$

[10] This assumption can be relaxed by assuming $\liminf_{\epsilon\to 0} \frac{V_2(t_1+\epsilon,t_2)-V_2(t_1,t_2)}{\epsilon} > 0$ and $\liminf_{\epsilon\to 0} \frac{V_1(t_1,t_2+\epsilon)-V_1(t_1,t_2)}{\epsilon} > 0$ for all $(t_1,t_2) \in (0,1)^2$.

LEMMA 1. *$(\mathcal{P}_1, \mathcal{P}_2)$ is a Nash equilibrium in an ascending-bid auction if and only if it is a Nash equilibrium in a second-price sealed-bid auction.*

PROOF: Take $(i,j) = (1,2)$ or $(2,1)$ and take any $t_i \in [0,1]$. For any strategy $\mathcal{P}_j$, we need to show that $\beta_i(t_i; \mathcal{P}_j) = \tilde{\beta}_i(t_i; \mathcal{P}_j)$. For any $b \in [0, \hat{b}]$ and any $b' \in [b, \infty)$,[11]

$$\Pi_i(\hat{b}, t_i; 0; \mathcal{P}_j) - \Pi_i(b', t_i; 0; \mathcal{P}_j) = \mathcal{P}_j * F_j\Big([0,1] \times [b, \infty)\Big) \Big[\Pi_i(\hat{b}, t_i; b; \mathcal{P}_j) - \Pi_i(b', t_i; b; \mathcal{P}_j)\Big]. \tag{3.2}$$

Comparing this with (2.1) and (3.1), we have $\tilde{\beta}_i(t_i; \mathcal{P}_j) = \beta_i(t_i; \mathcal{P}_j)$. It follows that these two auctions have the same Nash equilibrium strategies. ∎

Lemma 1 shows that the optimal set of prices for bidder i with type t to quit in an ascending-bid auction, for $(i,j) = (1,2)$ or $(2,1)$, is given by

$$\beta_i(t; \mathcal{P}_j) = \arg\max_b \Pi_i(b, t; 0; \mathcal{P}_j).$$

For simplicity, we will, from now on, denote $\beta_i(t; \mathcal{P}_j)$ as $\beta_i(t)$ and $\Pi_i(b, t; 0; \mathcal{P}_j)$ as $\Pi_i(b, t)$. The set of types for which bidder i will find it optimal to drop out at bidding level b is

$$\alpha_i(b) = \{t : b \in \beta_i(t)\}, \qquad b \geq 0.$$

Through a series of lemmas contained in Appendix A, we establish necessary and sufficient conditions for equilibrium strategies. We show that any equilibrium bidding strategies are pure and can thus be represented by a pair of functions, also denoted (β_i, β_j). These functions are continuous and strictly increasing in types and satisfy a pair of coupled differential equations under appropriate boundary conditions. We also show that any two strategies that satisfy these conditions do indeed form an equilibrium.

PROPOSITION 1. *Under Assumptions 1–3, necessary and sufficient conditions for (β_1, β_2) to be Nash equilibrium strategies are:*

(1) β_1 and β_2 are singled-valued, continuous, and strictly increasing;

(2) $\beta_1(0) = \beta_2(0) > V_1(0,0) = V_2(0,0)$;

(3) $\beta_1(1) = \beta_2(1) = V_1(1,1) = V_2(1,1)$;

(4) β_1 and β_2 are differentiable and satisfy the following system of coupled differential equations for $t_1 \in [0,1)$ and $t_2 \in [0,1)$:

$$\begin{cases} \dfrac{d\beta_1}{dt_1} = \dfrac{1}{\theta_2} \dfrac{f(t_1)}{1 - F(t_1)} \left[\beta_1(t_1) - V_2(t_1, \phi_{21}(t_1))\right]; \\ \dfrac{d\beta_2}{dt_2} = \dfrac{1}{\theta_1} \dfrac{f(t_2)}{1 - F(t_2)} \left[\beta_2(t_2) - V_1(\phi_{12}(t_2), t_2)\right], \end{cases} \tag{3.3}$$

[11] Again, we only need to consider b for which $\mathcal{P}_j * F_j\Big([0,1] \times [b, \infty)\Big) > 0$. See footnote 7.

where $\phi_{ij} = \beta_i^{-1} \circ \beta_j$ *for* $(i, j) = (1, 2), (2, 1)$*;*

(5) β_1' *and* β_2' *are continuous and strictly positive on* $[0, 1)$.

PROOF: The proof is based on a series of lemmas found in Appendix A. The necessity of conditions (1)–(3) is proved in Lemmas A1–A7. The necessity of conditions (4) and (5) is proved in Lemma A8. The sufficiency of these conditions for the Nash equilibrium is proved in Lemma A9. ∎

This proposition shows that a bidder with a toehold (who does not have the highest type) optimally "overbids." That is, given the other bidder's strategy, a bidder with a toehold bids strictly higher than what he would bid if he does not have a toehold. Consider the optimal strategy of bidder 1 who faces bidder 2 with strategy β_2. At any price level $\hat{b} \in \big[\beta_1(0), \beta_1(1)\big)$, the lowest type of bidder 2 who still remains in the auction is $\hat{t}_2 = \beta_2^{-1}(\hat{b})$. If bidder 1 does not have a toehold, then his dominant strategy in an ascending auction is to quit at price $\hat{b}$ if his type $\bar{t}_1$ is given by

$$\hat{b} = V_1(\bar{t}_1, \hat{t}_2). \tag{3.4}$$

Bidder 1 chooses to quit at this same price, $\hat{b}$, however, only if his type is $\hat{t}_1 = \phi_{12}(\hat{t}_2)$. According to (3.3), we have

$$\hat{b} = \beta_1(\hat{t}_1) > V_1(\hat{t}_1, \hat{t}_2). \tag{3.5}$$

Comparing (3.4) and (3.5), we have $\hat{t}_1 < \bar{t}_1$. The monotonicity of β_1 then implies that it is optimal for a type-$\bar{t}_1$ bidder 1, who has a toehold, to drop out at a price strictly higher than $\hat{b}$, the price at which he would drop out if he does not have a toehold.

This extends, to our more general setting here, the following result of BHK. In equilibrium, bidders with a toehold overbid in order to maximize their ex ante expected profit. They may, however, lose money ex post if they win the auction by a narrow margin. Burkart (1995) and Singh (1994) also reach the same conclusion under the assumption of independent private valuation and the assumption that one bidder does not have a toehold.

4. Existence and Uniqueness of a Nash Equilibrium

In this section, we prove the existence and uniqueness of an equilibrium for our bidding model. Proposition 1 implies that we only need to show the existence of a unique solution of the differential system (3.3) satisfying boundary conditions (2) and (3) of Proposition 1. The method we use here is similar in spirit to that used by Lebrun (1994) for proving the existence of equilibrium for asymmetric first price auctions under the assumption of independent (pure) private valuation.[12] The two problems, however, are different and require different analysis. In particular, our general assumption of mixed valuation (as opposed to pure private valuation) requires special treatment.

[12] The existence of equilibrium for asymmetric first price auctions under the assumption of independent private valuation can also be shown by topological methods; see Maskin

4.1. Existence of an Equilibrium

Before we give a simple account of the approach that we use for the proof of existence, we prepare for the proof by transforming our differential system (3.3) into a standard form in the following corollary.

COROLLARY 1. *Under Assumptions 1–3, a pair of strategies* (β_1, β_2) *is a Nash equilibrium if and only if the strategies are pure, the bidding functions are strictly increasing, and there exists* $\eta \in (0,1)$ *such that the inverses* $\alpha_1 = \beta_1^{-1}$ *and* $\alpha_2 = \beta_2^{-1}$ *are (i) differentiable over* $[\eta, 1)$*; (ii) solutions over this interval of the differential system (4.1) considered in domain*

$$\mathcal{D} = \left\{(b, \alpha_1, \alpha_2) \in \mathbb{R}^3 : 0 \leq \alpha_1, \alpha_2 < 1;\ \max(V_1(\alpha_1, \alpha_2), V_2(\alpha_1, \alpha_2)) < b < 1\right\}$$

$$\begin{cases} \dfrac{d\alpha_1(b)}{db} = \dfrac{1 - F(\alpha_1(b))}{f(\alpha_1(b))} \dfrac{\theta_2}{b - V_2(\alpha_1(b), \alpha_2(b))}; \\ \dfrac{d\alpha_2(b)}{db} = \dfrac{1 - F(\alpha_2(b))}{f(\alpha_2(b))} \dfrac{\theta_1}{b - V_1(\alpha_1(b), \alpha_2(b))}, \end{cases} \tag{4.1}$$

and (iii) satisfy the boundary conditions:

$$\alpha_1(\eta) = \alpha_2(\eta) = 0; \tag{4.2}$$

$$\alpha_1(1) = \alpha_2(1) = 1. \tag{4.3}$$

PROOF: Suppose that (β_1, β_2) is a Nash equilibrium. Let $\eta = \beta_1(0) = \beta_2(0) \in (0,1)$. Since, from Proposition 1, β_1 and β_2 are continuous, strictly increasing, and differentiable with a positive derivative on $[0,1)$, we can define $\alpha_1 = \beta_1^{-1}$ and $\alpha_2 = \beta_2^{-1}$ on $[\eta, 1)$, and α_1 and α_2 are both differentiable with strictly positive derivatives on $[\eta, 1)$. It is straightforward to show that α_1 and α_2 satisfy (4.1)–(4.3) and that the solution $(b, \alpha_1(b), \alpha_2(b))$, for $b \in [\eta, 1)$, lies inside the domain $\mathcal{D}$.

Next, suppose that α_1 and α_2 on $[\eta, 1)$ for some $\eta \in (0,1)$ are the solution in the domain $\mathcal{D}$ of (4.1)–(4.3). Then it is straightforward to show that $\beta_1 = \alpha_1^{-1}$ and $\beta_2 = \alpha_2^{-1}$ are well defined and satisfy conditions (1)–(5) of Proposition 1 and are thus the equilibrium bidding strategies. ∎

The domain $\mathcal{D}$, illustrated in Figure 1, is designed to ensure that the right-hand sides of the differential system (4.1) are defined, continuous, and locally Lipschitz within its interior. This domain $\mathcal{D}$ is the closed region contained by the following seven 2-dimensional pieces: $AEFD$, $ABCD$ (i.e., I), $DCGF$, ABE, BCG, EBF (i.e., II), and GBF (i.e., II'). The areas represented by II and II' are defined by:

$$II = \{(b, \alpha_1, \alpha_2) \in [0,1]^3 : \alpha_1 \geq \alpha_2,\ b = V_1(\alpha_1, \alpha_2)\};$$

$$II' = \{(b, \alpha_1, \alpha_2) \in [0,1]^3 : \alpha_1 \leq \alpha_2,\ b = V_2(\alpha_1, \alpha_2)\}.$$

and Riley (1994a, 1994b) and Lebrun (1993). One suspects that our problem here might also admit a proof using a topological approach. We have chosen the geometric approach here since it offers more intuition regarding the general properties of the equilibrium, and provides useful guidance for further numerical studies.

The shapes of II and II' depend on the functional forms of V_1 and V_2 and need not be "flat" (although they are flat in Figure 1 for simplicity of illustration).

With this new characterization of the equilibrium strategies, we can restate the existence problem as follows. There exists an equilibrium if and only if there exists a parameter $\eta \in (0, 1)$ for which there exist strictly increasing functions α_1 and α_2 that solve the system (4.1) considered in the domain $\mathcal{D}$ satisfying the boundary conditions (4.2) and (4.3).

Since differential equations are usually solved under one boundary condition only, we need to work with one of the two boundary conditions (4.2) and (4.3), and then check to see if the other one is also satisfied. One potential method of proving the existence of such solutions is by solving the system (4.1) within domain $\mathcal{D}$ under the initial condition (4.3), and then ensuring that the solutions satisfy (4.2) for some η. This approach is, however, infeasible since the system (4.1) has a singularity at $b = 1$. Instead, we consider the system (4.1) with the boundary conditions of (4.2) only. The existence of an equilibrium is then reduced to the existence of a parameter $\eta \in (0, 1)$ for which the solution of (4.1) and (4.2) in domain $\mathcal{D}$ consists of strictly increasing functions α_1 and α_2 defined on $[\eta, 1)$ and such that $\alpha_1(1-) = \alpha_2(1-) = 1$.

For any given $\eta \in (0, 1)$, the trajectory $\{(b, \alpha_1(b), \alpha_2(b)) : b \geq \eta\}$, that represents the solution α_1 and α_2 strictly increases with b and must hit the surface of domain $\mathcal{D}$ somewhere. Since α_1 and α_2 are increasing, it cannot hit surface $AEFD$ or $DCGF$. We also show that it cannot possibly hit surface ABE or BCG (Lemma B1). It can then only hit one of: the interior of I, the interior of $\mathit{II} \cup \mathit{II}'$, and the point B. Our goal is to show that there exists some $\eta \in (0, 1)$ such that the trajectory of the solution of (4.1) corresponding to (4.2) hits B.

In order to prove this, we show that the set of $\eta \in (0, 1)$ corresponding to which the trajectory of the solution of (4.1) and (4.2) hits I is an open set (Lemma B3), and that the set of $\eta \in (0, 1)$ for which the solution trajectory hits $\mathit{II} \cup \mathit{II}'$ is also an open set (Lemma B7). Neither of these two sets is empty (Lemma B8). Furthermore, we show that the solution (α_1, α_2) increases with the starting point η within domain $\mathcal{D}$ (Lemma B5). These results combine to show that there exists $0 < \eta^* \leq \eta^{**} < 1$ such that the trajectories of solutions corresponding to $\eta \in (0, \eta^*)$ hit $\mathit{II} \cup \mathit{II}'$, and that the trajectories of solutions corresponding to $\eta \in (\eta^{**}, 1)$ hit I. We thus conclude that the trajectories of all solutions corresponding to $\eta \in (\eta^*, \eta^{**})$ must hit B and represent the equilibrium solution. These steps then prove the existence of (at least one) equilibrium.

PROPOSITION 2. *Let Assumptions 1–4 be satisfied. Then there exists a Nash equilibrium if any of the following conditions is met:*

(i) Symmetric toehold;

(ii) Pure common valuation;

(iii) *Pure private valuation;*

(iv) *Assumption 5 holds.*

There are two important advantages for the geometric aspect of the proof of existence outlined above. The first is that the existence proof itself provides guidance for future numerical studies of the equilibrium solutions. Numerically, we can use the "shooting method" (see, for example, Marshall, Meurer, Richard, and Stromquist (1994)) by solving (4.1) from the initial condition (4.2), with the equilibrium solution given by a trajectory that hits the point B.[13] The characterization, obtained in the proof, of the dependence of the trajectories on the starting point η can be applied in a numerical study.

The second advantage of such a proof is that we also obtain some general characterizations of the equilibrium strategies. For example, Lemma B4 shows that, in equilibrium, the bidder with a larger toehold always bids more aggressively than the bidder with a smaller toehold, in the sense that, for each $t \in (0, 1)$, we have $\beta_1(t) > \beta_2(t)$ if $\theta_1 > \theta_2$. This result was shown in Bulow-Huang-Klemperer (1995) and is also shown to be valid in our general setting here. This result also shows that we can have an inefficient auction outcome if two bidders have asymmetric toeholds: A bidder with a higher valuation of the target company but lower toehold may lose to an opponent who has a lower valuation but a larger toehold.

4.2. Uniqueness of the Equilibrium Solution

In this section, we establish the uniqueness of any given equilibrium. The main result is stated in Proposition 3. The proof is given in Appendix C.

PROPOSITION 3. *Let Assumptions 1–4 be satisfied. If there exists a Nash equilibrium, then the equilibrium is unique if any of the following conditions is met:*

(i) *Symmetric toehold;*

(ii) *Pure common valuation;*

(iii) *Pure private valuation;*

(iv) *Assumption 6 holds.*

PROOF: See Appendix C. ▮

The basic idea behind the proof of uniqueness is as follows. We first note that, according to the theory of ordinary differential equations (see, for example, Pontriagin (1962)), two solutions of system (4.1) and (4.2) corresponding to the same η must be equivalent. We

[13] Marshall, Meurer, Richard, and Stromquist (1994) used this approach to numerically calculate the the equilibrium solution of asymmetric first price auctions.

therefore only need to rule out, by contradiction, the existence of two solutions, of system (4.1)–(4.3), that correspond to two different η's. Suppose there are two solutions of (4.1)–(4.3) corresponding to two η's. Then we use a monotonicity property of solutions to (4.1)–(4.3) with respect to η to show that they cannot both satisfy boundary condition (4.3).

5. Conclusion

This paper generalizes and extends Bulow-Huang-Klemperer (1995). We study the effect of owning a toehold on the competition between two bidders in a takeover battle by modeling the contest as an ascending-bid auction for two bidders with private information. We extend BHK to a general setting in which bidders' valuations of the target company have both private- and common-value components. We characterize the equilibrium strategies by showing that bidders use pure strategies and that the bidding functions are continuous, strictly increasing, and satisfy a specified pair of differential equations. These differential equations are shown to admit a unique solution under mild technical assumptions.

Appendix A: Lemmas for the General Characterization of Equilibrium

This appendix contains the lemmas that establish necessary and sufficient conditions for equilibrium in Proposition 1.

Define $[\underline{B}_i, \bar{B}_i]$ as the support of bidder i's bidding strategy, that is, the support of measure $\mathcal{P}_i{*}F_i\big([0,1]\times\bullet\big)$ (see Section 2.1 for definition). Then the rationality of the bidders imposes restrictions on the bidders' strategies at price levels below

$$\bar{B} \equiv \min\big(\bar{B}_1, \bar{B}_2\big).$$

In Lemma A1, we show that, for price levels below $\bar{B}$, each bidder's strategy $\beta(t)$ is weakly increasing in t regardless of the other bidder's strategy.

Lemma A1. *Let Assumptions 1 and 3(v) be satisfied. For any given bidder i, let*

$$A_i \equiv \{t_i : \exists b \in \beta_i(t_i) \text{ such that } b < \bar{B}\},$$

and take

$$\hat{t}_i = \begin{cases} 0, & \text{if } A_i \text{ is empty;} \\ \sup A, & \text{otherwise.} \end{cases} \tag{A.1}$$

Then $\beta_i(t_i) < \bar{B}$ for any $t_i < \hat{t}_i$ and $\beta_i(t_i) \geq \bar{B}$ for any $t_i \geq \hat{t}_i$. Furthermore, $\beta_i(\cdot)$, as a set function, is weakly increasing on $[0, \hat{t}_i]$; That is, for any $0 \leq t < t' \leq \hat{t}_i$, with $b \in \beta(t)$ and $b' \in \beta(t')$, we have $b \leq b'$.

Proof: We first claim that, for any $0 \leq t < t' \leq 1$, with $b \in \beta(t)$, $b' \in \beta(t')$, and $\min(b, b') < \bar{B}$, we have $b \leq b'$. We make an observation: for $b > b'$, $\Pi(b,t) - \Pi(b',t)$ is always weakly increasing in t, and is strictly increasing in t if and only if the other bidder drops out with positive probability within bidding range $[b', b]$. We next prove the claim by contradiciton. Suppose $t < t'$, $b \in \beta(t)$, $b' \in \beta(t')$, and $\min(b, b') < \bar{B}$, but $b > b'$. Then $\Pi(b,t') \leq \Pi(b',t')$, and from the observation, $\Pi(b,t) \leq \Pi(b',t)$, and thus $b' \in \beta(t)$. So $\Pi(b',t) = \Pi(b,t)$, implying that the other bidder drops out during $[b', b]$ with positive probability (otherwise, $\Pi(b,t) - \Pi(b',t) > 0$). From the observation again, we have $\Pi(b',t') < \Pi(b,t')$, a contradiction, and thus the claim is proved. It is then easy to show that the above claim implies that, for any $t_i < \hat{t}_i$, $\beta_i(t_i) < \bar{B}$, and for any $t_i \geq \hat{t}_i$, $\beta_i(t_i) \geq \bar{B}$. The proof for the remaining part of the lemma is straightforward. ∎

The following lemma makes use of the intuition that, under appropriate conditions, each bidder strictly prefers overbidding. This result helps with proofs of other lemmas.

Lemma A2. *Suppose there exists $t_2 \in [0,1]$ and $\hat{b}$ such that the probability that bidder 1 bids above $\hat{b}$ is positive and that*

$$\hat{b} \leq V_2\left(\inf\left(\cup_{b \geq \hat{b}} \alpha_1(b)\right), t_2\right). \tag{A.2}$$

Then there exists a small enough positive ϵ such that $\beta_2(t_2) \geq \hat{b} + \epsilon$.

PROOF: We need to show that $\inf \beta_2(t_2) > \hat{b}$. Equation (A.2) implies that bidder 2 does not over-bid at level $\hat{b}$. So we have $\inf \beta_2(t_2) \geq \hat{b}$ and only need to show $\inf \beta_2(t_2) \neq \hat{b}$. Lemma A1 and (A.2) show that bidder 2 with type t_2 bidding $\hat{b}+\epsilon$, with $\epsilon > 0$, over-bids no more than ϵ. Using the fact that P_ϵ, the probability of bidder 1 dropping out between $(\hat{b}, \hat{b}+\epsilon]$ (with $\epsilon > 0$) conditioning on he hasn't dropped out at level $\hat{b}$, goes to zero as $\epsilon \to 0$ (from the Reverse Fatou's Lemma), we have $\Pi(\hat{b} + \epsilon, t_2; \hat{b}) - \Pi(\hat{b}+, t_2; \hat{b}) \geq (1 - P_\epsilon)\theta_2\epsilon - P_\epsilon\epsilon > 0$ for a small enough ϵ. Thus $\inf \beta_2(t_2) \neq \hat{b}$ and the proof is finished. ∎

Lemma A1 and A2 are combined to show that both bidders must have the same positive initial bidding level. It is positive because bidders of the lowest type do not drop out at zero price due to the added aggressiveness from owning a toehold. It is the same level because no bidder is willing to quit at a low price if he knows that the other bidder will not quit until a higher price.

LEMMA A3. *Let Assumptions 1, 2, 3(ii), and 3(iii) be satisfied. Then $\beta_1(0)$ and $\beta_2(0)$ are both single-valued and $\beta_1(0) = \beta_2(0) > V_1(0,0) = V_2(0,0) \equiv 0$.*

PROOF: We first prove $\beta_1(0) = \beta_2(0)$ by contradiction. Suppose $b_1 \in \beta_1(0)$ and $b_2 \in \beta_2(0)$ but $b_1 < b_2$. Then Lemma A1 implies that bidder 2 drops out during $[b_1, b_2]$ only if $t_2 = 0$ and therefore type-0 bidder 1's decision of dropping out at b_1 is not rational. So $\beta_1(0)$ and $\beta_2(0)$ are both single-valued and $\beta_1(0) = \beta_2(0)$. Next, rationality requires $\bar{B} \geq V_2(0,0)$. Moreover, $\bar{B} \neq V_2(0,0)$ since otherwise one bidder, say bidder 1, drops out at $V_2(0,0) = 0$ with probability 1 and bidder 2 with $t_2 = 0$ prefers not to drop out at $V_2(0,0)$, a contradiction. So $\bar{B} > V_2(0,0)$, and applying Lemma A2 to the case of $t_2 = 0$ and $\hat{b} = V_2(0,0)$ shows that $\beta_1(0) > V_1(0,0)$. ∎

Lemma A1 implies that $\beta_i(t-) = \lim_{t' \uparrow t} \beta_i(t')$ is well defined and single-valued for any $t \in [0, \hat{t}_i]$. Similarly defined is $\beta(t+)$. By convention, $0- = 0$ and $1+ = 1$.

With Lemma A1 and A3, we can define the common bidding range of the two parties as $\mathcal{B} = \big[\beta_1(0), \bar{B}\big)$. We can then apply the rationality of both parties to the common bidding range and obtain the following continuity and monotonicity results.

LEMMA A4. *Let Assumptions 1, 2, and 3(v) be satisfied. Then each bidder's bidding function is single-valued and continuous within the common bidding range $\mathcal{B}$.*

PROOF: Suppose β_1 is multi-valued or discontinuous within the common bidding range, then it is multi-valued or discontinuous at some $t_1 < 1$ such that $\beta_1(t_1-) \in \mathcal{B}$ and $\beta_1(t_1+) - \beta_1(t_1-) > 0$. Note that Lemma A1 implies that bidder 1 drops out in the bidding range of $\big[\beta_1(t_1-), \min(\bar{B}, \beta_1(t_1+))\big)$ only if it's of type t_1. Since $t_1 < 1$ and thus the probability of bidder 1 being type t_1, conditioning on its type is no less than t_1, the probability of bidder 1 dropping out within the bidding range of $\big[\beta_1(t_1-), \min(\bar{B}, \beta_1(t_1+))\big)$ is thus zero.

So bidder 2, of any type, would not bid within the range of $\left[\beta_1(t_1-), \min(\bar{B}, \beta_1(t_1+))\right)$. Since $\beta_1(t_1-) \in \mathcal{B}$, there exists a large enough $t_2 < 1$ such that $\beta_2(t_2) > \beta_1(t_1-)$. Therefore bidder 1 with type t_1- bidding $\beta_1(t_1-)$ cannot be rational since he should strictly prefer quitting at $\min(\bar{B}, \beta_1(t_1+))$ given the positive probability of bidder 2 not dropping out until $\min(\bar{B}, \beta_1(t_1+))$. ∎

LEMMA A5. *Let Assumptions 1, 2, 3(i), and 3(ii) be satisfied. Then $\beta(t)$ is strictly increasing within the common bidding range $\mathcal{B}$.*

PROOF: We prove by contradiction. Suppose β_1 is not strictly increasing in the common bidding range. Lemma A1 and A3 imply that β_1 is flat somewhere, say at bidding value $\hat{b}$ from t_{1L} to t_{1H}, with $t_{1L} = \min\{t : \beta_1(t) = \hat{b}\}$, $t_{1H} = \max\{t : \beta_1(t) = \hat{b}\}$, and $t_{1L} < t_{1H} < 1$. Let $\bar{t}_2 = \max\{t : \beta_2(t) = \hat{b}\}$. (Lemma 4 implies that it is well defined.) The rationality of bidder 2 with type $\bar{t}_2+$ bidding $\hat{b}+$ and the continuity of β_i and V_i, for $i = 1, 2$, imply that $\hat{b} \geq V_2(t_{1H}, \bar{t}_2)$. We discuss two cases to show the existence of contradiction. First, if $\hat{b} > V_2(t_{1H}, \bar{t}_2)$, then it is irrational for bidder 2 with type $\bar{t}_2+$ to prefer bidding $\hat{b}+$ to bidding $\hat{b}-$, a contradiction. Next, if $\hat{b} = V_2(t_{1H}, \bar{t}_2)$, then we can apply Lemma A2 to $\hat{b}+$ and $\bar{t}_2+$ to show contradiction. (Strictly speaking, we need to apply the same argument used in Lemma A2 to $\hat{b}+$ and $\bar{t}_2+$.) We have thus finished the proof. ∎

Lemma A6 and A7 establish the boundary conditions of the bidding functions at the highest type.

LEMMA A6. *Let Assumptions 1, 2, and 3 be satisfied. Then $\hat{t}_1 = \hat{t}_2 = 1$ (see (A.1) for definition), and $\bar{B} = \beta_1(1-) = \beta_2(1-) = V_1(1,1) = V_2(1,1)$.*

PROOF: Since $\bar{B} \equiv \min(\bar{B}_1, \bar{B}_2)$ (where $\bar{B}_i$ is the upper bound of the support of bidder i's bidding strategy), we can assume, without loss of generality, that $\bar{B} = \bar{B}_2$. Since the types of bidder 1 who drop out at or above $\bar{B}$ are at least $\hat{t}_1$ (from Lemma A1), the rationality of bidder 2 with type $1-$ dropping out at[14] $\bar{B}$ (or close to $\bar{B}$ in the limit in the case of $\hat{t}_2 = 1$) implies that $\bar{B} \geq V_2(\hat{t}_1, 1)$. The rationality of bidder 1 of type $\hat{t}_1$ (bidding at or above $\bar{B}$) implies that $\bar{B} \leq \mathbf{E}V_1(\hat{t}_1, \tilde{t}_2)$, with the expectation taken over $\tilde{t}_2 \in [\hat{t}_2, 1]$. Combining the above, we have $\mathbf{E}V_1(\hat{t}_1, \tilde{t}_2) \geq \bar{B} \geq V_2(\hat{t}_1, 1)$, which leads to $V_1(\hat{t}_1, 1) \geq \mathbf{E}V_1(\hat{t}_1, \tilde{t}_2) \geq V_2(\hat{t}_1, 1)$. Assumption 3 then implies that

$$\bar{B} = V_1(\hat{t}_1, 1) = \mathbf{E}V_1(\hat{t}_1, \tilde{t}_2) = V_2(\hat{t}_1, 1). \qquad (A.3)$$

We can use (A.3) to prove that $\hat{t}_1 = 1$. Suppose $\hat{t}_1 < 1$. Since $\bar{B} = V_1(\hat{t}_1, 1)$, we have $\beta_1(t_1) > \bar{B}$ for any $t_1 > \hat{t}_1$ and thus bidder 1 remain in the auction above $\bar{B}$ with positive

[14] If $\hat{t}_2 < 1$, bidder 2 with types $\tilde{t}_2 \in [\hat{t}_2, 1]$ drops out at $\bar{B}$ almost surely (under measure $\mathcal{P}_2 * F_2$ (see Section 2.1 for definition)).

probability.[15] We can then apply Lemma A2 to the case of $\hat{b} = \bar{B} = \beta_2(1-)$ and $t_2 = 1-$ to show that[16] $\beta_2(1-) > V_2(\hat{t}_1, 1) + \epsilon$ for some small $\epsilon > 0$, which leads to contradiction. So $\hat{t}_1 = 1$.

Next, suppose $\hat{t}_2 < 1$. Since $\bar{B} = V_2(1,1)$, bidder 2 of type $t_2 \in [\hat{t}_2, 1)$ quitting at $\bar{B}$ would win and overbid almost surely, which leads to contradiction. So we have $\hat{t}_2 = 1$. ∎

LEMMA A7. *Let Assumptions 1, 2, and 3 be satisfied. Then* $\beta_1(1) = \beta_2(1) = V_1(1,1) = V_2(1,1)$.

PROOF: From Lemma A1 and A6, we have $\beta_1(1) \geq V_1(1,1)$ and $\beta_2(1) \geq V_1(1,1)$. So we only need to show that no one with type $t = 1$ bids above $V_1(1,1)$ in equilibrium. Consider the case of $t_1 = t_2 = 1$. Suppose bidder 1 bids above $V_1(1,1)$ with positive probability, then bidder 2 always wants to bid, at least slightly, above $V_1(1,1)$. (The argument is similar to that used for Lemma A3.) Party i's expected wealth in equilibrium should then be strictly larger than $\theta_i V_1(1,1)$ since each at least has the option of dropping out at some level slightly above $V_1(1,1)$. The two parties' total expected wealth should be strictly larger than $(\theta_1 + \theta_2)V_1(1,1)$. On the other hand, the two parties' total wealth in an equilibrium with bidding above $V_1(1,1)$ is equal to $(\theta_1 + \theta_2)V_1(1,1) - (1 - \theta_1 - \theta_2)\big(b - V_1(1,1)\big) \leq (\theta_1 + \theta_2)V_1(1,1)$ (where $b \geq V_1(1,1)$ is the point at which one of the parties finally drops out), which leads to contradiction. It is then impossible that any bidder bids above $V_1(1,1)$. ∎

Lemma A8 shows that equilibrium bidding functions must be differentiable and satisfy a pair of differential equations.

LEMMA A8. *Under Assumptions 1–3, the equilibrium bidding functions β_1 and β_2 are both differentiable on* $[0,1)$ *and satisfy the differential system (1). Furthermore, $\beta_1'(t)$ and $\beta_2'(t)$ are both continuous and strictly positive on* $[0,1)$.

PROOF: Since β_1 and β_2 are continuous and strictly increasing on $[0,1]$ with $\beta_1(0) = \beta_2(0)$ and $\beta_1(1) = \beta_2(1)$, we can define functions $\phi_{ij} : [0,1] \to [0,1]$ by $\phi_{ij} = \beta_i^{-1} \circ \beta_j$ for $(i,j) = (1,2)$ or $(2,1)$ and they are strictly increasing and continuous. Take any $t_1 \in [0,1)$ and any small $\epsilon > 0$. Let $\hat{t}_2 = \phi_{21}(t_1)$, $\Delta b = \beta_1(t_1 + \epsilon) - \beta_1(t_1)$, and let p_ϵ denote the probability that bidder 1's type lies between $(t_1, t_1 + \epsilon)$ conditioning on it being larger than

[15] For the case of $\hat{t}_2 = 1$, this statement relies on the rationality of bidder 1 of type 1 in a zero-probability event of $t_2 = 1$. (Lemma A7 also relies on such a rationality criterion.) If we cannot rely on such a rationality criterion, then Lemma A6 can still be proved under somewhat more restrictive conditions. Specifically, we need to assume one of the following conditions: (i) Pure Common Valuation; (ii) Pure Private Valuation; (iii) Assumption of $V_1(t_1, 1) < V_2(t_1, 1)$ for $t_1 < 1$ (implied by Assumption 6).

[16] Strictly speaking, we can apply the same argument leading to Lemma A2 to show that there exists some small $\epsilon > 0$ and some $\delta > 0$ such that $\beta_2(1 - \delta) > V_2(\hat{t}_1, 1) + \epsilon$. Note that we have assumed away certain "limit" strategies by bidder 1 such as dropping out "immediately after" $\bar{B}$. But such "limit" strategies can be allowed for under some mild (additional) conditions.

t_1. Since bidder 2 with type $\hat{t}_2$ prefers quitting at $\beta_2(\hat{t}_2) = \beta_1(t_1)$ to quitting at $\beta_1(t_1 + \epsilon)$, we have

$$(1 - p_\epsilon)\theta_2 \Delta b \leq p_\epsilon \left[\beta_1(t_1) + \Delta b - V_2(t_1, \hat{t}_2)\right],$$

from which we have

$$\limsup_{\epsilon \downarrow 0} \frac{\beta_1(t_1 + \epsilon) - \beta_1(t_1)}{\epsilon} = \limsup_{\epsilon \downarrow 0} \frac{\Delta b}{\epsilon} \leq \frac{1}{\theta_2} \frac{f(t_1)}{1 - F(t_1)} \left[\beta_1(t_1) - V_2(t_1, \hat{t}_2)\right].$$

Similarly, the fact that bidder 2 with type $\phi_{21}(t_1+\epsilon)$ prefers quitting at $\beta_1(t_1+\epsilon)$ to quitting at $\beta_1(t_1)$ implies that

$$(1 - p_\epsilon)\theta_2 \Delta b \geq p_\epsilon \left[\beta_1(t_1) - V_2(t_1 + \epsilon, \phi_{21}(t_1 + \epsilon))\right],$$

from which we have (we need to use Assumption 3(i) here)

$$\liminf_{\epsilon \downarrow 0} \frac{\beta_1(t_1 + \epsilon) - \beta_1(t_1)}{\epsilon} \geq \frac{1}{\theta_2} \frac{f(t_1)}{1 - F(t_1)} \left[\beta_1(t_1) - V_2(t_1, \hat{t}_2)\right].$$

Combining the above two results, we have

$$\lim_{\epsilon \downarrow 0} \frac{\beta_1(t_1 + \epsilon) - \beta_1(t_1)}{\epsilon} = \frac{1}{\theta_2} \frac{f(t_1)}{1 - F(t_1)} \left[\beta_1(t_1) - V_2(t_1, \hat{t}_2)\right].$$

Next, take any $t_1 \in (0, 1)$ and any small $\epsilon \in (0, t_1)$. Let $\hat{t}_2 = \phi_{21}(t_1)$, $\Delta b = \beta_1(t_1) - \beta_1(t_1 - \epsilon)$, and let q_ϵ denote the probability that bidder 1's type lies between $(t_1 - \epsilon, t_1)$ conditioning on it being larger than $t_1 - \epsilon$. Since bidder 2 with type $\hat{t}_2$ prefers quitting at $\beta_2(\hat{t}_2) = \beta_1(t_1)$ to quitting at $\beta_1(t_1 - \epsilon)$, we have

$$(1 - q_\epsilon)\theta_2 \Delta b \geq q_\epsilon \left[\beta_1(t_1 - \epsilon) - V_2(t_1, \hat{t}_2)\right],$$

from which we have

$$\liminf_{\epsilon \downarrow 0} \frac{\beta_1(t_1) - \beta_1(t_1 - \epsilon)}{\epsilon} \geq \frac{1}{\theta_2} \frac{f(t_1)}{1 - F(t_1)} \left[\beta_1(t_1) - V_2(t_1, \hat{t}_2)\right].$$

Similarly, the fact that bidder 2 with type $\phi_{21}(t_1-\epsilon)$ prefers quitting at $\beta_1(t_1-\epsilon)$ to quitting at $\beta_1(t_1)$ implies that

$$(1 - q_\epsilon)\theta_2 \Delta b \leq q_\epsilon \left[\beta_1(t_1) - V_2(t_1 - \epsilon, \phi_{21}(t_1 - \epsilon))\right],$$

from which we have

$$\limsup_{\epsilon \downarrow 0} \frac{\beta_1(t_1) - \beta_1(t_1 - \epsilon)}{\epsilon} \leq \frac{1}{\theta_2} \frac{f(t_1)}{1 - F(t_1)} \left[\beta_1(t_1) - V_2(t_1, \hat{t}_2)\right].$$

Combining these two results, we have

$$\lim_{\epsilon \uparrow 0} \frac{\beta_1(t_1 + \epsilon) - \beta_1(t_1)}{\epsilon} = \frac{1}{\theta_2} \frac{f(t_1)}{1 - F(t_1)} \left[\beta_1(t_1) - V_2(t_1, \hat{t}_2)\right].$$

Combining this left limit with the previous right limit, we have thus shown that $\beta_1'(t_1)$ exists for $t_1 \in [0,1)$ and

$$\beta_1'(t_1) = \frac{1}{\theta_2}\frac{f(t_1)}{1-F(t_1)}\left[\beta_1(t_1) - V_2(t_1,\hat{t}_2)\right].$$

The existence of β_2' and the second equation of system (1) can be shown similarly.

The continuity of β_1' and β_2' comes from the continuity of f (Assumption 2), V_1, and V_2 (Assumption 3(i)). To show that, in equilibrium, β_1' and β_2' have to be strictly positive on $[0,1)$, we only need to show that the right-hand sides of system (1) are strictly positive for $t \in [0,1)$. Without loss of generality, we need to show that, for any $t_2 \in [0,1)$, $\beta_2(t_2) > V_2(\phi_{12}(t_2), t_2)$. This, however, can be shown by applying Lemma A2 to t_2 and $\hat{b} = V_2(\phi_{12}(t_2), t_2)$. We have thus finished the proof. ∎

Finally, Lemma A9 shows that the above necessary conditions are also sufficient for equilibrium.

LEMMA A9. *Let Assumptions 1–3 be satisfied. Suppose there exist two functions β_1 and β_2 satisfying conditions (1)–(5) in Proposition 1. Then bidding strategies β_1 and β_2 form a Nash equilibrium.*

PROOF: We only need to show that given bidder 2's bidding strategy β_2, bidder 1's optimal strategy is β_1. Let $\alpha_i = \beta_i^{-1}$ for $i = 1,2$. Conditioning on no one dropping out before bidding level $\underline{B} \in [\beta_1(0), 1)$, the expected total wealth of bidder 1 with type $\hat{t}_1$ and a strategy of quitting at $B \in [\underline{B}, 1]$ is given by

$$\begin{aligned}
\Pi_1(B,\hat{t}_1;\underline{B}) &= \theta_1 B\frac{1-F(\alpha_2(B))}{1-F(\alpha_2(\underline{B}))} + \int_{\underline{B}}^{B}\left[V_1(\hat{t}_1,\alpha_2(b)) - (1-\theta_1)b\right]\frac{dF(\alpha_2(b))}{1-F(\alpha_2(\underline{B}))} \\
&= \frac{1}{1-F(\alpha_2(\underline{B}))}\Bigg[\theta_1 B\left[1-F(\alpha_2(B))\right] + \theta_1\int_{\underline{B}}^{B} b\, dF(\alpha_2(b)) \\
&\qquad - \int_{\underline{B}}^{B}\left[b - V_1(\hat{t}_1,\alpha_2(b))\right] dF(\alpha_2(b))\Bigg] \\
&= \frac{1}{1-F(\alpha_2(\underline{B}))}\Bigg[\theta_1 B\left[1-F(\alpha_2(B))\right] + \theta_1 b F(\alpha_2(b))\Big|_{b=\underline{B}}^{b=B} - \theta_1\int_{\underline{B}}^{B} F(\alpha_2(b))\, db\Bigg] \\
&\qquad - \int_{\underline{B}}^{B}\left[b - V_1(\hat{t}_1,\alpha_2(b))\right]\frac{dF(\alpha_2(b))}{1-F(\alpha_2(\underline{B}))} \\
&= \theta_1\underline{B} + \theta_1\int_{\underline{B}}^{B}\frac{1-F(\alpha_2(b))}{1-F(\alpha_2(\underline{B}))}\, db - \int_{\underline{B}}^{B}\left[b - V_1(\hat{t}_1,\alpha_2(b))\right]\frac{dF(\alpha_2(b))}{1-F(\alpha_2(\underline{B}))} \\
&= \theta_1\underline{B} + \theta_1\int_{\underline{B}}^{B}\frac{1-F(\alpha_2(b))}{1-F(\alpha_2(\underline{B}))}\left[1 - \frac{b - V_1(\hat{t}_1,\alpha_2(b))}{b - V_1(\alpha_1(b),\alpha_2(b))}\right] db,
\end{aligned}$$

where we used system (1) to arrive at the last line. Since the integrand of the integral in the last line, which represents the marginal expected profit of staying in the auction at bidding level b, is positive for $b < \beta_1(\hat{t}_1)$ and negative for $b > \beta_1(\hat{t}_1)$, the optimal level for bidder 1 to quit is $\beta_1(\hat{t}_1)$ and this optimal choice does not change as the level of bid $\underline{B}$ increases. We have thus proved the sufficiency of the conditions (1)–(5) for the Nash equilibrium. ∎

Appendix B: Proof of Equilibrium Existence

We first outline the proof for the existence of an equilibrium for the three special cases in Section 2.3 as well as other general cases covered by Assumption 5. All lemmas used in the proof are also contained in this appendix.

Outline of Proof of Proposition 2

PROOF: According to Corollary 1, the existence of an equilibrium can be proved by showing the existence of a parameter $\eta \in (0,1)$ for which there exists 2 strictly increasing functions α_1 and α_2 which form a solution of the system (4.1) considered in domain $\mathcal{D}$ satisfying the boundary conditions (4.2) and (4.3). Since the system (4.1) has a singularity at $b = 1$, we consider the system with boundary conditions (4.2) only. The existence of an equilibrium is then reduced to the existence of a parameter $\eta \in (0,1)$ for which the solution of (4.1) and (4.2) in domain $\mathcal{D}$ consists of strictly increasing functions α_1 and α_2 defined on $[\eta, 1)$ and such that $\alpha_1(1-) = \alpha_2(1-) = 1$.

We first characterize the trajectory within domain $\mathcal{D}$ of the solution of (4.1) for a given $\eta \in (0,1)$. The right-hand sides of (4.1) are strictly positive at $b = \eta \in (0,1)$ with the initial conditions (4.2). The trajectory of the solution then enters into the interior of domain $\mathcal{D}$ as b increases from η. From the theory of ordinary differential equations (see, for example, Pontriagin (1962)), we know that there exists a maximal interval $[\eta, \gamma)$ on which a solution exists in domain $\mathcal{D}$. This solution is called the maximal solution of system (4.1) in domain $\mathcal{D}$ with initial conditions (4.2) and it is unique. Any other solution of (4.1) in domain $\mathcal{D}$ with the same initial conditions can be defined only over an interval that is a subset of $[\eta, \gamma)$ and must coincide with the maximal solution. Since the right-hand sides of (4.1) are strictly positive within $\mathcal{D}$, $\alpha_1(b)$ and $\alpha_2(b)$ of the maximal solution are strictly increasing with respect to b within domain $\mathcal{D}$. Therefore, $\alpha_i(\gamma) = \lim_{b \uparrow \gamma} \alpha_i(b)$, with $i = 1, 2$, is well-defined and the terminal point $\big(\gamma, \alpha_1(\gamma), \alpha_2(\gamma)\big)$ of the trajectory of the maximal solution must be on the boundary of domain $\mathcal{D}$. (Otherwise, it is in the interior of $\mathcal{D}$ and we can extend the solution beyond γ within $\mathcal{D}$, a contradiction.) We call this terminal point the "hitting point" of the solution on the boundary of $\mathcal{D}$. In order to prove existence of a solution for (4.1)—(4.3), we will show that there exists an $\eta \in (0,1)$ such that the maximal solution, corresponding to η, of system (4.1) in domain $\mathcal{D}$ with initial conditions (4.2) hits the boundary of $\mathcal{D}$ at the point $B = (1,1,1)$.

Since α_1 and α_2 are strictly increasing, the hitting point cannot be in $AEFD$ or $DCGF$. Lemma B1 shows that the hitting point cannot be in ABE or BCG except for the point B by showing that there cannot be cases in which only one of $\alpha_1(\gamma)$ and $\alpha_2(\gamma)$ is 1 at the hitting point. So the hitting point of the trajectory can only be either point B with $\gamma = 1$, or in the interior of I with $\gamma = 1$, or in the interior of $II \cup II'$ with $\gamma < 1$. There are therefore three distinct types of solutions. A solution is called, respectively, type-B, type-I, or type-II if the trajectory hits B, the interior of I, or the interior of $II \cup II'$.

We next show that the solution types display certain continuity and monotonicity properties with respect to small changes of the starting point η. Specifically, Lemma B3 shows that Λ_I, the set of $\eta \in (0,1)$ for which the solution is type-I, is an open set. One can also show that Λ_{II}, the set of $\eta \in (0,1)$ for which the solution is type-II, is also an open set (Lemma B7). The monotonicity of solution types with respect to η comes from the following important observation: the solutions (α_1, α_2) of system (4.1) within $\mathcal{D}$ are strictly decreasing in the starting point η and the upper bound of the corresponding maximal interval is non-decreasing in η (Lemma B5). From this, we conclude that, if $\eta \in \Lambda_I$, then $[\eta, 1) \subseteq \Lambda_I$; and if $\eta \in \Lambda_{II}$, then $(0, \eta] \subseteq \Lambda_{II}$.

Before we prove the existence of a type-B solution, we need to show that neither Λ_I nor Λ_{II} is empty. We prove that the solution is type-I for a large enough $\eta \in (0,1)$ and is type-II for a small enough $\eta \in (0,1)$ (Lemma B8). Finally, let $\eta^* = \sup \Lambda_{II}$ and $\eta^{**} = \inf \Lambda_I$, then Lemma B8 and the monotonicity of the solution type with respect to η imply that $0 < \eta^* \leq \eta^{**} < 1$ and that $\Lambda_I = (0, \eta^*)$ and $\Lambda_{II} = (\eta^{**}, 1)$. We therefore have $\Lambda_B = [\eta^*, \eta^{**}]$ and that the solution for any $\eta \in [\eta^*, \eta^{**}]$ is the solution to system (4.1)—(4.3). So there exists an equilibrium. ▮

We next state and prove all the lemmas that were used in the above proof. We first note that it is sometimes useful to make the following change of variables

$$\begin{cases} x_1 = -\ln\left[1 - F(\alpha_1)\right]; \\ x_2 = -\ln\left[1 - F(\alpha_2)\right], \end{cases} \tag{$B.1$}$$

under which equations (4.1) become

$$\begin{cases} \dfrac{dx_1(b)}{db} = \dfrac{\theta_2}{b - \bar{V}_2(x_1(b), x_2(b))}; \\ \dfrac{dx_2(b)}{db} = \dfrac{\theta_1}{b - \bar{V}_1(x_1(b), x_2(b))}, \end{cases} \tag{$B.2$}$$

where $\bar{V}_i : [0, \infty] \times [0, 1] \to [0, 1]$ is such that $\bar{V}_i(x_1, x_2) = V_i(\alpha_1, \alpha_2)$ for $i = 1, 2$. Note that x_i is strictly increasing in α_i, with $x_i = 0$ for $\alpha_i = 0$ and $x_i = \infty$ for $\alpha_i = 1$, and $\bar{V}_i(\infty, \infty) = 1$. For the case of pure private valuation, $\bar{v} : [0, 1] \to [0, 1]$ is defined accordingly.

LEMMA B1. *Let Assumptions 1–3 be satisfied. Let $\eta \in (0, 1)$ be such that (α_1, α_2) is a solution, over interval $[\eta, \gamma)$ (with $\eta < \gamma$), of the problem (4.1) and (4.2) considered in $\mathcal{D}$. Then either $\alpha_1(\gamma) = \alpha_2(\gamma) = 1$ or $0 < \alpha_1(\gamma), \alpha_2(\gamma) < 1$.*

PROOF: We prove by contradiction. First note that strict monotonicity of the solution within $[\eta, \gamma)$ implies that $\alpha_1(\gamma), \alpha_2(\gamma) > 0$. Suppose, without loss of generality, that $\alpha_1(\gamma) = 1$ but $\alpha_2(\gamma) < 1$. That is, $x_1(\gamma) = \infty$ but $x_2(\gamma) < \infty$ under the change of variables (B.1) and (B.2). Continuity of x_1 and x_2 implies that there exists $\hat{b} \in (\eta, \gamma)$ such that $x_2(b) < x_1(b) < \infty$ for all $b \in [\hat{b}, \gamma)$. From (B.2), we have $\frac{dx_2(b)}{db} \geq \frac{\theta_1}{\theta_2} \frac{dx_1(b)}{db}$ for all $b \in [\hat{b}, \gamma)$, and thus $x_2(\gamma) - x_2(\hat{b}) \geq \frac{\theta_1}{\theta_2} \left[x_1(\gamma) - x_1(\hat{b})\right]$. So $x_2(\gamma) = \infty$, a contradiction. We have thus finished the proof. ▮

The following lemma is used in the proof of Lemma B3 and Lemma B7. It is basically an equivalence statement of different ways of looking at the system (4.1) of differential equations by exchanging the independent variable with one of the functions. Here the roles of the independent variable b and function α_2 are exchanged. In the proof of Lemma B3, this transformation makes it possible to apply the continuity property of solutions to initial conditions for the same interval. In the proof of Lemma B7, it helps avoid the singularity of system (4.1) at the hitting point of the solution on the interior of *II*.

LEMMA B2. *Let Assumptions 1–3 be satisfied. Let (α_1, α_2) be a solution over an interval $[\eta, \gamma)$, with $0 < \eta < \gamma \leq 1$, of the differential system (4.1) considered in domain $\mathcal{D}$. Then the functions $\phi_{12} = \alpha_1\alpha_2^{-1}$ and $\beta_2 = \alpha_2^{-1}$ are differentiable and form a solution over the interval $[\alpha_2(\eta), \alpha_2(\gamma))$ of the following system of differential equations considered in the domain*
$\mathcal{D}_2 = \big\{(t_2, \phi_{12}, \beta_2) \in I\!R^3 \,|\, 0 \leq t_2, \phi_{12} < 1;\ \max(V_1(\phi_{12}, t_2), V_2(\phi_{12}, t_2)) < \beta_2 < 1\big\}$,

$$\begin{cases} \dfrac{d\beta_2(t_2)}{dt_2} = \dfrac{1}{\theta_1}\dfrac{f(t_2)}{1-F(t_2)}\left[\beta_2(t_2) - V_1(\phi_{12}(t_2), t_2)\right]; \\ \dfrac{d\phi_{12}(t_2)}{dt_2} = \dfrac{\theta_2}{\theta_1}\dfrac{f(t_2)}{f(\phi_{12}(t_2))}\dfrac{1-F(\phi_{12}(t_2))}{1-F(t_2)}\dfrac{\beta_2(t_2) - V_1(\phi_{12}(t_2), t_2)}{\beta_2(t_2) - V_2(\phi_{12}(t_2), t_2)}. \end{cases} \tag{B.3}$$

Inversely, if (β_2, ϕ_{12}) is a solution over an interval $[v, w)$, with $0 \leq v < w \leq 1$, of the system (B.3) considered in domain $\mathcal{D}_2$, and if $\frac{d\beta_2}{dt_2} > 0$ for all $t_2 \in [v, w)$, then $\alpha_1 = \phi_{12}\beta_2^{-1}$ and $\alpha_2 = \beta_2^{-1}$ are differentiable and form a solution over the interval $[\beta_2(v), \beta_2(w))$ of the system (4.1) considered in domain $\mathcal{D}$.

PROOF: The proof can be done by simple variable transformations. If (α_1, α_2) form a solution over an interval $[\eta, \gamma)$, with $0 < \eta < \gamma \leq 1$, of the differential system (4.1) considered in domain $\mathcal{D}$. Then $\frac{d\alpha_2(b)}{db} > 0$ for all $b \in [\eta, \gamma)$ and functions $\phi_{12} = \alpha_1\alpha_2^{-1}$ and $\beta_2 = \alpha_2^{-1}$ are defined and differentiable, and it is straightforward to verify that their derivatives satisfy (B.3) in domain $\mathcal{D}_2$. Inversely, if (β_2, ϕ_{12}) is a solution over an interval $[v, w)$, with $0 \leq v < w \leq 1$, of the system (B.3) considered in domain $\mathcal{D}_2$, and if $\frac{d\beta_2}{dt_2} > 0$ for all $t_2 \in [v, w)$, then $\alpha_1 = \phi_{12}\beta_2^{-1}$ and $\alpha_2 = \beta_2^{-1}$ are defined and differentiable, and it is straightforward to verify that their derivatives satisfy system (4.1) in domain $\mathcal{D}$ over interval $[\beta_2(v), \beta_2(w))$.
∎

LEMMA B3. *Let Assumptions 1–4 be satisfied. The set Λ_I of all parameters $\eta \in (0,1)$ for which the maximal solution in domain $\mathcal{D}$ is type-I, is an open set.*

PROOF: Let $\eta \in (0,1)$ be such that the solution (α_1, α_2) of system (4.1) in $\mathcal{D}$ with initial conditions $\alpha_1(\eta) = \alpha_2(\eta) = 0$ is type-I (defined over $[\eta, 1)$) with $0 < \alpha_1(1), \alpha_2(1) < 1$. We need to show that there exists $\delta > 0$ such that the solution $(\tilde{\alpha}_1, \tilde{\alpha}_2)$, corresponding to any $\tilde{\eta} \in (\eta-\delta, \eta+\delta)$, of system (4.1) in domain $\mathcal{D}$ is also type-I with $0 < \tilde{\alpha}_1(1), \tilde{\alpha}_2(1) < 1$. From Lemma B2, we know that $\beta_2 = \alpha_2^{-1}$ and $\phi_{12} = \alpha_1\alpha_2^{-1}$ form a solution over $[0, \alpha_2(1))$ of the differential system (B.3) considered in domain $\mathcal{D}_2$ satisfying the initial conditions $\beta_2(0) = \eta$ and $\phi_{12}(0) = 0$. Since $\beta_2(t_2) - V_2(\phi_{12}(t_2), t_2) = 1 - V_2(\phi_{12}(t_2), t_2) > 0$ at $t_2 = \alpha_2(1)$, the system (B.3) is well-behaved (in the sense that its right-hand sides are continuous and locally Lipschitz) at the point $(t_2, \beta_2, \phi_{12}) = (\alpha_2(1), 1, \alpha_1(1))$ and the solution can be continued beyond $\alpha_2(1)$ over an interval $[0, \gamma)$ with $\gamma > \alpha_2(1)$.

Take a positive $\epsilon < \min(\gamma, 1) - \alpha_2(1)$ such that $0 < \phi_{12}\big(\alpha_2(1) + \epsilon\big) < 1$. Consider the values of solution (β_2, ϕ_{12}), at $t_2 = \alpha_2(1) + \epsilon$, of system (B.3) with various parameters η's.

Since $\beta_2(\alpha_2(1)) = 1$ and β_2 is strictly increasing, we have $\beta_2(\alpha_2(1)+\epsilon) > 1$. From the theory of ordinary differential equations (see, for example, Pontriagin (1962), pp. 198), there exists $\delta > 0$ such that for any $\tilde{\eta} \in (\eta-\delta, \eta+\delta)$, the solution $(\tilde{\beta}_2, \tilde{\phi}_{12})$ corresponding to $\tilde{\eta}$ is defined on interval $[0, \alpha_2(1)+\epsilon)$ and satisfies $\tilde{\beta}_2(\alpha_2(1)+\epsilon) > 1$ and $0 < \tilde{\phi}_{12}(\alpha_2(1)+\epsilon) < 1$. From Lemma B2, $\tilde{\alpha}_1 = \tilde{\phi}_{12}\tilde{\beta}_2^{-1}$ and $\alpha_2 = \beta_2^{-1}$ form a solution of system (4.1) for the parameter $\tilde{\eta}$ at $\tilde{\beta}_2(\alpha_2(1)+\epsilon) > 1$. Since $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are strictly increasing, we have $0 < \tilde{\alpha}_1(1), \tilde{\alpha}_2(1) < 1$ and therefore $\tilde{\eta} \in \Lambda_I$ for all $\tilde{\eta} \in (\eta-\delta, \eta+\delta)$. ∎

The following lemma contains an important result of the paper. It states that the bidder with the larger toehold is always more aggressive in the sense that he posts a higher bid for the same type $t \in (0,1)$.

LEMMA B4. *Let Assumptions 1–2 be satisfied. Let (α_1, α_2) be a solution over an interval $[\eta, \gamma)$, with $0 < \eta < \gamma \leq 1$, of the differential system (4.1) considered in domain $\mathcal{D}$. If $\theta_1 = \theta_2$, then $\alpha_1(b) = \alpha_2(b)$ for all $b \in [\eta, \gamma]$. If $\theta_1 < \theta_2$ (or $\theta_1 > \theta_2$), then $\alpha_1(b) > \alpha_2(b)$ (or $\alpha_1(b) < \alpha_2(b)$) for all $b \in (\eta, \gamma)$.*

PROOF: We first prove for the case of $\theta_1 = \theta_2$ by contradiction. We again use the change of variables in (B.1) and (B.2). Suppose, without loss of generality, that $x_1(\hat{b}) > x_2(\hat{b})$ for some $\hat{b} \in [\eta, \gamma)$. The boundary condition $x_1(\eta) = x_2(\eta) = 0$ implies that $\hat{b} > \eta$ and that we can define, from the continuity of x_1 and x_2, $\underline{b} = \max\{b : b < \hat{b} \;\&\; x_1(b) = x_2(b)\}$, with $\eta \leq \underline{b} < \hat{b}$. For any $b \in (\underline{b}, \hat{b}]$, we have $x_1(b) > x_2(b)$ and thus, from (B.2) and the monotonicity of V_1 and V_2, $\frac{d(x_1(b)-x_2(b))}{db} \leq 0$. This implies that $x_1(\underline{b}) - x_2(\underline{b}) \geq x_1(\hat{b}) - x_2(\hat{b}) > 0$, a contradiction. We have thus shown that $x_1(b) = x_2(b)$ for all $b \in [\eta, \gamma)$. The equivalence at $b = \gamma$ follows by taking the limit.

We next prove for the case of $\theta_1 < \theta_2$ by contradiction. (The case of $\theta_1 > \theta_2$ can be proved similarly.) We again use (B.1) and (B.2). Suppose that the set $B = \{b \in (\eta, \gamma) : x_1(b) \leq x_2(b)\}$ is not empty, then we can define $\hat{b} = \inf B$. Since $x_1(\eta) = x_2(\eta) = 0$ and, from (B.2), $\frac{dx_1}{db} > \frac{dx_2}{db}$, there exists a small enough positive $\epsilon > 0$ such that $x_1(b) > x_2(b)$ for any $b \in (\eta, \eta+\epsilon)$, and we thus have $\hat{b} > \eta$. Since x_1 and x_2 are continuous, we have $x_1(\hat{b}) = x_2(\hat{b})$ and thus, from (B.2), $\frac{dx_1}{db} > \frac{dx_2}{db}$ at $\hat{b}$. So for a small enough $\delta > 0$, we have $x_1(\hat{b}-\delta) < x_2(\hat{b}-\delta)$, a contradiction to the definition of $\hat{b}$. We have thus shown that $x_1(b) > x_2(b)$ for all $b \in (\eta, \gamma)$. ∎

The following lemma proves an important monotonicity property of the solutions with respect to η: the solutions (α_1, α_2) of system (4.1) within $\mathcal{D}$ are strictly decreasing in the starting point η and the upper bound of the corresponding maximal interval is non-decreasing in η.

LEMMA B5. *Let Assumptions 1–4 be satisfied. Let (α_1, α_2) be the maximal solution of the problem (4.1) and (4.2), over interval $[\eta, \gamma)$, in domain $\mathcal{D}$ for a parameter $\eta \in (0,1)$ and let $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ be the maximal solution of problem (4.1) and (4.2), over interval $[\tilde{\eta}, \tilde{\gamma})$, in domain $\mathcal{D}$ for a parameter $\tilde{\eta} \in (0,1)$. Let $\eta < \tilde{\eta}$. Then $\gamma \leq \tilde{\gamma}$ and, moreover, if $\tilde{\eta} < \gamma$, then $\alpha_i(b) > \tilde{\alpha}_i(b)$ for all $b \in [\tilde{\eta}, \gamma)$ and for $i = 1, 2$.*

PROOF: We know that $\eta < \gamma$ and $\tilde{\eta} < \tilde{\gamma}$. If $\gamma \leq \tilde{\eta}$, then the lemma is trivially true. We therefore assume $\gamma > \tilde{\eta}$. Since α_1 and α_2 are strictly increasing, we have $\alpha_i(\tilde{\eta}) > \tilde{\alpha}_i(\tilde{\eta}) = 0$ for $i = 1, 2$. Let $d = \sup\{b \in [\tilde{\eta}, \min(\gamma, \tilde{\gamma})] \,|\, \alpha_i(b') > \tilde{\alpha}_i(b'),$ for all $b' \in [\tilde{\eta}, b)$, and for $i = 1, 2\}$. We have to show that $d = \min(\gamma, \tilde{\gamma})$. Note that $\alpha_i(\tilde{\eta}) > \tilde{\alpha}_i(\tilde{\eta}) = 0$ for $i = 1, 2$ and thus continuity implies that $d > \tilde{\eta}$. Suppose $d < \min(\gamma, \tilde{\gamma})$. By continuity, we can assume, without loss of generality, that $\alpha_1(d) = \tilde{\alpha}_1(d)$ and $\alpha_2(d) \geq \tilde{\alpha}_2(d)$. If $\alpha_2(d) = \tilde{\alpha}_2(d)$, then solutions (α_1, α_2) and $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ of system (4.1) would be equal at d and would thus be equal over $[\tilde{\eta}, \min(\gamma, \tilde{\gamma}))$ (from Assumption 4), which is impossible since $\alpha_i(\tilde{\eta}) > \tilde{\alpha}_i(\tilde{\eta}) = 0$ for $i = 1, 2$. So we have $\alpha_2(d) > \tilde{\alpha}_2(d)$. But equation (4.1) and the monotonicity property of V_2 imply that $\frac{d\alpha_1(b)}{db}|_{b=d} > \frac{d\tilde{\alpha}_1(b)}{db}|_{b=d}$. Thus there exists $\delta > 0$ such that $\alpha_1(b) < \tilde{\alpha}_1(b)$ for all $b \in (d - \delta, d)$. This, however, contradicts the definition of d and we thus have $\alpha_i(b) > \tilde{\alpha}_i(b)$ for all $b \in \big[\tilde{\eta}, \min(\gamma, \tilde{\gamma})\big)$ and $\alpha_i\big(\min(\gamma, \tilde{\gamma})\big) \geq \tilde{\alpha}_i\big(\min(\gamma, \tilde{\gamma})\big)$, for $i = 1, 2$.

Next, we prove $\gamma \leq \tilde{\gamma}$. Suppose $\gamma > \tilde{\gamma}$ instead, then, as proved above, $\alpha_i(\tilde{\gamma}) \geq \tilde{\alpha}_i(\tilde{\gamma})$, for $i = 1, 2$, and the point $\big(\tilde{\gamma}, \tilde{\alpha}_1(\tilde{\gamma}), \tilde{\alpha}_2(\tilde{\gamma})\big)$ lies in the interior of $\mathcal{D}$, which contradicts the definition of $\tilde{\gamma}$. The lemma is thus proved. ∎

Next, we show that, if there is a point $(\hat{b}, \hat{x}_1, \hat{x}_2) \in \mathcal{D}$ such that $\hat{b} = \bar{V}_1(\hat{x}_1, \hat{x}_2) = \bar{V}_2(\hat{x}_1, \hat{x}_2)$, then the solution of system (B.2) in domain $\mathcal{D}$ corresponding to any initial point (b_0, x_{10}, x_{20}) in $\mathcal{D}$ that is close enough to the point $(\hat{b}, \hat{x}_1, \hat{x}_2)$ will be type-*II*. Intuitively, this is due to the fact that, at points close to $(\hat{b}, \hat{x}_1, \hat{x}_2)$, the right hand sides of (B.2) are very large, resulting in a very large increase in x_1 and x_2 for a small increase in b and leading the trajectory of the solution to the surface of $II \cup II'$. This result is used in the proofs of Lemma B7 and Lemma B8.

LEMMA B6. *Let Assumptions 1–3 be satisfied. Let $(\hat{b}, \hat{x}_1, \hat{x}_2) \in [0, 1) \times [0, \infty) \times [0, \infty)$ be such that $\hat{b} = \bar{V}_1(\hat{x}_1, \hat{x}_2) = \bar{V}_2(\hat{x}_1, \hat{x}_2)$. Then there exist positive ϵ_0, ϵ_1, and ϵ_2 such that, for any point (b_0, x_{10}, x_{20}) within $\mathcal{D}$ satisfying $b_0 \in (\hat{b} - \epsilon_0, \hat{b} + \epsilon_0)$, $x_{10} \in (\hat{x}_1 - \epsilon_1, \hat{x}_1 + \epsilon_1)$, and $x_{20} \in (\hat{x}_2 - \epsilon_2, \hat{x}_2 + \epsilon_2)$, the solution of (B.2) in domain $\mathcal{D}$ corresponding to the initial conditions of $x_1(b_0) = x_{10}$ and $x_2(b_0) = x_{20}$ is type-II.*

PROOF: We first choose ϵ_0, ϵ_1, and ϵ_2 and then show, by contradiction, that the lemma holds under these choices. We take a positive

$$\epsilon_0 < \frac{1}{2} \min\left(1 - \hat{b}, \bar{V}_1(\hat{x}_1 + \frac{1}{6}\theta_2, \hat{x}_2 + \frac{1}{6}\theta_1) - \bar{V}_1(\hat{x}_1, \hat{x}_2)\right)$$

and take positive ϵ_1 and ϵ_2 that satisfy

$$\epsilon_1 < \frac{1}{6}\theta_2 \quad \text{and} \quad \epsilon_2 < \frac{1}{6}\theta_1, \tag{B.4}$$

and

$$\begin{aligned} &\bar{V}_1(\hat{x}_1, \hat{x}_2) - \bar{V}_1(\hat{x}_1 - \epsilon_1, \hat{x}_2 - \epsilon_2) < \epsilon_0; \\ &\bar{V}_2(\hat{x}_1, \hat{x}_2) - \bar{V}_2(\hat{x}_1 - \epsilon_1, \hat{x}_2 - \epsilon_2) < \epsilon_0. \end{aligned} \tag{B.5}$$

Suppose that the lemma doesn't hold under these choices; that is, the solution of system (B.2) defined over interval $[b_0, \gamma)$ in $\mathcal{D}$ corresponding to initial conditions $x_1(b_0) = x_{10}$ and $x_2(b_0) = x_{20}$, for a certain $b_0 \in (\hat{b} - \epsilon_0, \hat{b} + \epsilon_0)$, $x_{10} \in (\hat{x}_1 - \epsilon_1, \hat{x}_1 + \epsilon_1)$, and $x_{20} \in (\hat{x}_2 - \epsilon_2, \hat{x}_2 + \epsilon_2)$, is not type-*II*. Then the solution is either type-*B* or type-*I* with $\gamma = 1$ and that x_1 and x_2 should be increasing over $[b_0, 1)$ and well-defined at $b = b_0 + \epsilon_0 < \hat{b} + 2\epsilon_0 < 1$ within domain $\mathcal{D}$, and thus

$$
\begin{aligned}
x_1(b_0 + \epsilon_0) &= x_1(b_0) + \int_{b_0}^{b_0+\epsilon_0} \frac{\theta_2}{b - \bar{V}_2\big(x_1(b), x_2(b)\big)}\, db \\
&> x_{10} + \int_{b_0}^{b_0+\epsilon_0} \frac{\theta_2}{b_0 + \epsilon_0 - \bar{V}_2(x_{10}, x_{20})}\, db \\
&> \hat{x}_1 - \epsilon_1 + \frac{\theta_2 \epsilon_0}{\hat{b} + 2\epsilon_0 - \bar{V}_2(\hat{x}_1 - \epsilon_1, \hat{x}_2 - \epsilon_2)} \\
&= \hat{x}_1 - \epsilon_1 + \frac{\theta_2 \epsilon_0}{2\epsilon_0 + \bar{V}_2(\hat{x}_1, \hat{x}_2) - \bar{V}_2(\hat{x}_1 - \epsilon_1, \hat{x}_2 - \epsilon_2)} \\
&> \hat{x}_1 - \epsilon_1 + \frac{\theta_2 \epsilon_0}{2\epsilon_0 + \epsilon_0} \quad \text{(from (B.5))} \\
&> \hat{x}_1 + \frac{1}{6}\theta_2. \quad \text{(from (B.4))}
\end{aligned}
$$

Similarly, we have $x_2(b_0 + \epsilon_0) > \hat{x}_2 + \frac{1}{6}\theta_1$ and thus

$$
\begin{aligned}
\bar{V}_1\big(x_1(b_0 + \epsilon_0), x_2(b_0 + \epsilon_0)\big) &> \bar{V}_1\big(\hat{x}_1 + \frac{1}{6}\theta_2, \hat{x}_2 + \frac{1}{6}\theta_1\big) \\
&> \bar{V}_1\big(\hat{x}_1, \hat{x}_2\big) + 2\epsilon_0 \quad \text{(from definition of } \epsilon_0) \\
&= \hat{b} + 2\epsilon_0 \\
&> b_0 + \epsilon_0
\end{aligned}
$$

which contradicts the assumption that x_1 and x_2 are defined at $b_0 + \epsilon_0$ inside domain $\mathcal{D}$. We have thus finished the proof. ∎

Lemma B7. *Let Assumptions 1–4 be satisfied. The set Λ_{II} of all parameters $\eta \in (0, 1)$ for which the maximal solution in domain $\mathcal{D}$ is type-II is an open set if any one of the following conditions is met:*

(i) Symmetric toehold;

(ii) Pure common valuation;

(iii) Pure private valuation;

(iv) Assumption 5 holds.

Proof: Let $\eta \in (0, 1)$ be such that the maximal solution (α_1, α_2) of system (4.1) in $\mathcal{D}$ with initial conditions $\alpha_1(\eta) = \alpha_2(\eta) = 0$ is type-*II* and defined over the maximal interval $[\eta, \gamma)$

with $\eta < \gamma < 1$ and, without loss of generality, let $V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) = \gamma$. As discussed before, the hitting point $\big(\gamma, \alpha_1(\gamma), \alpha_2(\gamma)\big)$ is in the interior of $II \cup II'$. We need to show that there exists $\delta > 0$ such that the maximal solution $(\tilde{\alpha}_1, \tilde{\alpha}_2)$, corresponding to any $\tilde{\eta} \in (\eta - \delta, \eta + \delta)$, of system (4.1) in domain $\mathcal{D}$ is also type-II over interval $[\tilde{\eta}, \tilde{\gamma})$ with $\tilde{\eta} < \tilde{\gamma} < 1$ and $\max\big(V_1\big(\tilde{\alpha}_1(\tilde{\gamma}), \tilde{\alpha}_2(\tilde{\gamma})\big), V_2\big(\tilde{\alpha}_1(\tilde{\gamma}), \tilde{\alpha}_2(\tilde{\gamma})\big)\big) = \tilde{\gamma}$. This can be done easily for $\tilde{\eta} \in (\eta - \delta_1, \eta)$, for any $\delta_1 \in (0, \eta)$, since Lemma B5 implies that the solution of (4.1) within $\mathcal{D}$ corresponding to any such $\tilde{\eta}$ has its upper bound of definition interval $\tilde{\gamma} \le \gamma$ and is also type-II. The following proves that the solution is also type-II for any $\tilde{\eta}$ that is larger than but close enough to η. The proof is rather lengthy, and we outline the steps first before going into the details of the proof.

We discuss the only two possible cases. The first case is when $\gamma = V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) = V_2\big(\alpha_1(\gamma), \alpha_2(\gamma)\big)$, and the second case is when $\gamma = V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) > V_2\big(\alpha_1(\gamma), \alpha_2(\gamma)\big)$. The rest of the proof goes as follows. We first show that the cases of symmetric toehold, pure common valuation, and pure private valuation all correspond to the first case. We next show that, for the first case, there exists $\delta_2 > 0$ such that $(\eta, \eta + \delta_2) \in \Lambda_{II}$. Finally, we show that, for the second case, under Assumption 5, there exists $\delta_3 > 0$ such that $(\eta, \eta + \delta_3) \in \Lambda_{II}$.

For the case of symmetric toehold, since $\alpha_1(b) = \alpha_2(b)$ for any $b \in [\eta, \gamma]$, we have $\gamma = V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) = V_2\big(\alpha_1(\gamma), \alpha_2(\gamma)\big)$.[17] For the pure common valuation case, since V_1 and V_2 are the same, we also have $\gamma = V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) = V_2\big(\alpha_1(\gamma), \alpha_2(\gamma)\big)$. We next show that the same is true for the case of pure private valuation. We prove this by contradiction. Suppose that the solution is such that $\gamma = v\big(\alpha_1(\gamma)\big) > v\big(\alpha_2(\gamma)\big)$, with $1 > \alpha_1(\gamma) > \alpha_2(\gamma)$, then (B.2) and the continuity and Lipschitz property of v imply that there exists a positive $\epsilon > 0$, $K > 0$, and a positive $k > \frac{dx_1(\gamma)}{db} = \frac{\theta_2}{\gamma - \bar{v}\left(x_2(\gamma)\right)}$ such that, for any $b \in [\gamma - \epsilon, \gamma)$, we have $|x_1(b) - x_1(\gamma)| \le k|b - \gamma|$, and

$$\begin{aligned}
\frac{dx_2(b)}{db} &= \frac{\theta_1}{b - \bar{v}\big(x_1(b)\big)} \\
&\ge \frac{\theta_1}{|b - \gamma| + |\bar{v}\big(x_1(b)\big) - \bar{v}\big(x_1(\gamma)\big)|} \\
&\ge \frac{\theta_1}{|b - \gamma| + K|x_1(b) - x_1(\gamma)|} \\
&\ge \frac{\theta_1}{(1 + Kk)|b - \gamma|},
\end{aligned}$$

which, by integration, implies that $x_2(\gamma) = \infty$, a contradiction. So a type-II solution for a case of pure private valuation must have $\gamma = V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) = V_2\big(\alpha_1(\gamma), \alpha_2(\gamma)\big)$.

We next show, for the case of $\gamma = V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) = V_2\big(\alpha_1(\gamma), \alpha_2(\gamma)\big)$, that there exists $\delta_2 > 0$ such that $(\eta, \eta + \delta_2) \in \Lambda_{II}$. Note that, applying Lemma B6 to the case of $(\hat{b}, \hat{x}_1, \hat{x}_2) = (\gamma, x_1(\gamma), x_2(\gamma))$ here with $b_0 = \gamma$, we conclude that there exist positive ϵ_1 and ϵ_2 such that, if the solution $(\tilde{x}_1, \tilde{x}_2)$ of (B.2) in $\mathcal{D}$ corresponding to some $\tilde{\eta}$ is well-defined at γ and satisfies $\tilde{x}_1(\gamma) \in (\hat{x}_1 - \epsilon_1, \hat{x}_1]$ and $\tilde{x}_2(\gamma) \in (\hat{x}_2 - \epsilon_2, \hat{x}_2]$, then the solution will be type-II. So we only need to prove that there exists a positive δ_2 such that the solution of (B.2) in $\mathcal{D}$ corresponding to any $\tilde{\eta} \in (\eta, \eta + \delta_2)$ is well-defined at γ and satisfies $\tilde{x}_1(\gamma) \in (\hat{x}_1 - \epsilon_1, \hat{x}_1]$

[17] Assumption 3(i) and 3(iv) imply that $V_1(t, t) = V_2(t, t)$ for any $t \in (0, 1)$.

and $\tilde{x}_2(\gamma) \in (\hat{x}_2 - \epsilon_2, \hat{x}_2]$. First, the monotonicity property in Lemma B5 implies that the solution corresponding to any $\tilde{\eta} > \eta$ is well-defined at $b = \gamma$ (at least in the limit). Secondly, the continuity of the solution of an ordinary differential equation with respect to its initial conditions (see Pontriagin (1962)) implies that $\tilde{x}_i(b) \to x_i(b)$ for any $b \in (\eta, \gamma)$ as $\tilde{\eta} \downarrow \eta$. So we have

$$\lim_{\tilde{\eta}\downarrow\eta} \tilde{x}_i(\gamma) = \lim_{\tilde{\eta}\downarrow\eta} \lim_{b\uparrow\gamma} \tilde{x}_i(b) = \lim_{b\uparrow\gamma} \lim_{\tilde{\eta}\downarrow\eta} \tilde{x}_i(b) = \lim_{b\uparrow\gamma} x_i(b) = x_i(\gamma),$$

where the exchange of limits is correct since $\tilde{x}_i(b)$ increases as b increases and as $\tilde{\eta}$ decreases. The monotonicity of the solution with respect to the starting point η (Lemma B5) then implies that there exists a positive δ_2 that satisfies the desired property.

Finally, we need to show, for the second case with $\gamma = V_1\big(\alpha_1(\gamma), \alpha_2(\gamma)\big) > V_2\big(\alpha_1(\gamma), \alpha_2(\gamma)\big)$, that there exists $\delta_3 > 0$ such that $(\eta, \eta+\delta_3) \in \Lambda_{II}$. We can assume that Assumption 5 holds here because this is the only remaining case for which the lemma has not been proven yet. From Lemma B2, we know that $\beta_2 = \alpha_2^{-1}$ and $\phi_{12} = \alpha_1\alpha_2^{-1}$ form a solution over $[0, \alpha_2(\gamma))$ of the differential system (B.3) considered in domain $\mathcal{D}_2$ satisfying the initial conditions $\beta_2(0) = \eta$ and $\phi_{12}(0) = 0$. Since $\beta_2(t_2) - V_2(\phi_{12}(t_2), t_2) > 0$ at $t_2 = \alpha_2(\gamma)$ for the second case, the system (B.3) is well-behaved (in the sense that its right-hand sides are continuous and locally Lipschitz) at the point $(t_2, \beta_2, \phi_{12}) = (\alpha_2(\gamma), \gamma, \alpha_1(\gamma))$ and the solution can be continued beyond $t_2 = \alpha_2(\gamma)$ over an interval $[0, \nu)$ with $\alpha_2(\gamma) < \nu < 1$. Continuity implies that there exists a small enough $\xi \in \big(0, \nu - \alpha_2(\gamma)\big)$ such that $0 < \beta_2, \phi_{12} < 1$ for all t_2 in the extended interval $\big(\alpha_2(\gamma), \alpha_2(\gamma)+\epsilon\big)$. Furthermore, since $\frac{d\beta_2(t_2)}{dt_2} = \frac{d\phi_{12}(t_2)}{dt_2} = 0$ at $t_2 = \alpha_2(\gamma)$, Assumption 5 implies that, by making ξ smaller if necessary, the solution (β_2, ϕ_{12}) over the extended interval $\big(\alpha_2(\gamma), \alpha_2(\gamma) + \xi\big)$ lies strictly outside of domain $\mathcal{D}$. Otherwise, we can find a sequence of $\xi_n \downarrow 0$ such that $\beta_2(\alpha_2(\gamma) + \xi_n) \geq V_1\big(\phi_{12}\big(\alpha_2(\gamma) + \xi_n\big), \alpha_2(\gamma) + \xi_n\big)$ and we have

$$\begin{aligned}
0 &= \frac{d\beta_2(t_2)}{dt_2}\Big|_{t_2=\alpha_2(\gamma)} \\
&= \liminf_{\xi_n\downarrow 0} \frac{\beta_2(\alpha_2(\gamma) + \xi_n) - \beta_2(\alpha_2(\gamma))}{\xi_n} \\
&\geq \liminf_{\xi_n\downarrow 0} \frac{V_1\big(\phi_{12}\big(\alpha_2(\gamma) + \xi_n\big), \alpha_2(\gamma) + \xi_n\big) - V_1\big(\phi_{12}\big(\alpha_2(\gamma)\big), \alpha_2(\gamma)\big)}{\xi_n} \\
&= \liminf_{\xi_n\downarrow 0} \frac{V_1\big(\phi_{12}\big(\alpha_2(\gamma) + \xi_n\big), \alpha_2(\gamma) + \xi_n\big) - V_1\big(\phi_{12}\big(\alpha_2(\gamma)\big), \alpha_2(\gamma) + \xi_n\big)}{\xi_n} \\
&\quad + \liminf_{\xi_n\downarrow 0} \frac{V_1\big(\phi_{12}\big(\alpha_2(\gamma)\big), \alpha_2(\gamma) + \xi_n\big) - V_1\big(\phi_{12}\big(\alpha_2(\gamma)\big), \alpha_2(\gamma)\big)}{\xi_n}.
\end{aligned}$$

The first term of the last line of the above equation can be shown to be zero by using $\frac{d\phi_{12}(t_2)}{dt_2}|_{t_2=\alpha_2(\gamma)} = 0$ and the local Lipschitz property of V_1. We then have

$$0 \geq \liminf_{\xi\to 0} \frac{V_1\big(\phi_{12}\big(\alpha_2(\gamma)\big), \alpha_2(\gamma) + \xi\big) - V_1\big(\phi_{12}\big(\alpha_2(\gamma)\big), \alpha_2(\gamma)\big)}{\xi},$$

a contradiction to Assumption 5.

To proceed, we consider the values of solution (β_2, ϕ_{12}), over interval $[0, \alpha_2(\gamma) + \xi)$, of system (B.3) with various parameters η's. We can take a small enough $\epsilon > 0$ such that: (1)

The intersection of II with the cylinder of radius ϵ around the trajectory of solution (β_2, ϕ_{12}), defined as $\mathcal{C} = \{(t_2, b, t_1) | t_2 \in [0, \alpha_2(\gamma) + \xi] \ \& \ \|(t_2, b, t_1) - (t_2, \beta_2(t_2), \phi_{12}(t_2))\| < \epsilon\}$, divides the cylinder into two parts and lies in the interior of II; (2) The end surface of the cylinder $\mathcal{C}$ at $t_2 = \alpha_2(\gamma) + \xi$ lies strictly outside of domain $\mathcal{D}$. From the theory of ordinary differential equations, there exists $\delta_3 > 0$ such that for any $\tilde{\eta} \in (\eta, \eta + \delta_3)$, the solution $(\tilde{\beta}_2, \tilde{\phi}_{12})$ corresponding to $\tilde{\eta}$ is defined on interval $[0, \alpha_2(\gamma) + \xi)$ and lies within the cylinder $\mathcal{C}$. The solution then has to cross the interior of II and, again using Lemma B2, we know that the solution $\tilde{\alpha}_1 = \tilde{\phi}_{12}\tilde{\beta}_2^{-1}$ and $\tilde{\beta}_2^{-1}$ in the domain $\mathcal{D}$ is the solution of system (4.1) under initial conditions $\tilde{\alpha}_1(\tilde{\eta}) = \tilde{\alpha}_2(\tilde{\eta}) = 0$ and it is a type-II solution. We have thus finished the proof. ∎

LEMMA B8. *Let Assumptions 1–3 be satisfied. There exists an* $\eta_1 \in (0, 1)$ *such that the solution for any* $\eta \in (\eta_1, 1)$ *is type-I. There also exists an* $\eta_2 \in (0, 1)$ *such that the solution for any* $\eta \in (0, \eta_2)$ *is type-II.*

PROOF: To prove the existence of a type-I solution, we first prove that if there exist η_1 and M that satisfy

$$0 < \eta_1 < 1, \quad 0 < M < \infty; \tag{B.6}$$

$$\eta_1 > \bar{V}_2(M, M); \tag{B.7}$$

$$M > \frac{\max(\theta_1, \theta_2)}{\eta_1 - \bar{V}_2(M, M)}(1 - \eta_1), \tag{B.8}$$

then $(\eta_1, 1) \subseteq \Lambda_I$. Take $\eta \in (\eta_1, 1)$. We use the change of variables in (B.1) and (B.2). Suppose that the solution corresponding to η for system (4.1) in domain $\mathcal{D}$ defined over the maximal interval $[\eta, \gamma)$ is not type-I, then it's either type-B or type-II. Since $x_i(1) = \infty$ $(i = 1, 2)$ for a type-B solution and $\bar{V}_1(M, M) = \bar{V}_2(M, M) < \eta_1 < \eta < \gamma = \max\big(\bar{V}_1(x_1(\gamma), x_2(\gamma)), \bar{V}_2(x_1(\gamma), x_2(\gamma))\big)$, in either case, from the fact that x_1 and x_2 are continuous and strictly increasing, we know that there exists $\hat{b} \in [\eta, 1)$ such that, without loss of generality, $x_1(\hat{b}) = M$ and $0 < x_1, x_2 < M$ for all $b \in (\eta, \hat{b})$. But equation (B.2) implies that

$$\begin{aligned}
x_1(\hat{b}) &= \int_\eta^{\hat{b}} \frac{\theta_2}{b - \bar{V}_2\big(x_1(b), x_2(b)\big)} \, db \\
&< \int_\eta^{\hat{b}} \frac{\theta_2}{\eta - \bar{V}_2\big(M, M\big)} \, db \\
&= \frac{\theta_2}{\eta - \bar{V}_2\big(M, M\big)}(\hat{b} - \eta) \\
&< \frac{\max(\theta_1, \theta_2)}{\eta_1 - \bar{V}_2\big(M, M\big)}(1 - \eta) \\
&< M,
\end{aligned}$$

a contradiction. We thus have $(\eta_1, 1) \subseteq \Lambda_I$.

To finish the proof for the existence of a type-I solution, we need to show that there exists η_1 and M satisfying (B.6)—(B.8). We take M and η_1 such that

$$
\begin{aligned}
&V_2(M, M) = \frac{1}{2}; \\
&\frac{1}{2}\frac{M + 2\max(\theta_1, \theta_2)}{M + \max(\theta_1, \theta_2)} < \eta_1 < 1.
\end{aligned}
$$

Then M and η_1 satisfy (B.6)—(B.8).

We next prove the existence of a type-*II* solution. This can be done by applying Lemma B6 to the case of $(\hat{b}, \hat{x}_1, \hat{x}_2) = (0, 0, 0)$ here with $x_{10} = x_{20} = 0$. Then Lemma B6 implies that there exists a positive η_2 such that the solution of (B.2) in $\mathcal{D}$ corresponding to any $\eta \in (0, \eta_2)$ is type-*II*. We have thus finished the proof. ∎

Appendix C: Proof of Equilibrium Uniqueness

In this appendix, we prove the uniqueness of any given equilibrium. For the proof in this section, it is useful to work with the following set of differential equations which, as shown in the following corollary, must be satisfied by the equilibrium strategies. This new set of differential equations makes a change of variable $u_2 = -\ln[1 - F(t_2)]$ and treats $u_2 \in [0, \infty]$ as the independent variable.

COROLLARY 2. *Under Assumptions 1–3, a pair of strategies* (β_1, β_2) *is a Nash equilibrium if and only if the strategies are pure, the bidding functions are strictly increasing, and there exists* $\eta \in (0, 1)$ *such that the two functions* $\varphi_{12} : [0, \infty] \to [0, \infty]$ *and* $p_2 : [0, \infty] \to [0, 1]$, *defined by*

$$\begin{cases} \varphi_{12}(u_2) = -\ln\left[1 - F\left(\beta_1^{-1}\beta_2(t_2)\right)\right], & \text{for all } t_2 \in [0, 1]; \\ p_2(u_2) = \beta_2(t_2), & \text{for all } t_2 \in [0, 1], \end{cases}$$

with a change of variable $u_2 = -\ln[1 - F(t_2)]$, *are solutions, over the interval* $[\eta, 1)$ *and considered in domain*

$$\mathcal{D}_3 = \{(u_2, \varphi_{12}, p) \in \mathbb{R}^3 |\, 0 \le u_2, \varphi_{12} < \infty;\ \max(\bar{V}_1(\varphi_{12}, u_2), \bar{V}_2(\varphi_{12}, u_2)) < p_2 < 1\}$$

of the differential system

$$\frac{d\varphi_{12}(u_2)}{du_2} = \frac{\theta_2}{\theta_1}\frac{p_2(u_2) - \bar{V}_1(\varphi_{12}(u_2), u_2)}{p_2(u_2) - \bar{V}_2(\varphi_{12}(u_2), u_2)}; \tag{C.1}$$

$$\frac{dp_2(u_2)}{du_2} = \frac{1}{\theta_1}\left[p_2(u_2) - \bar{V}_1(\varphi_{12}(u_2), u_2)\right], \tag{C.2}$$

under the boundary conditions

$$\varphi_{12}(0) = 0, \quad p_2(0) = \eta; \tag{C.3}$$

$$\varphi_{12}(\infty) = \infty, \quad p_2(\infty) = \infty. \tag{C.4}$$

PROOF: The corollary follows from Proposition 1, Corollary 1, and Lemma B2 with an appropriate change of variables. ∎

Proof of Proposition 3

PROOF: Corollary 2 implies that we only need to show the uniqueness of a solution for system (C.1)–(C.4). From the theory of ordinary differential equations, we know that if two solutions of the system correspond to the same η, then they must be identical. So we only need to rule out the existence of two solutions, with each corresponding to a different η, for system (C.1)–(C.4). We prove it by contradiction. Without loss of generality, we assume $\theta_1 \ge \theta_2$. (The case of $\theta_1 \le \theta_2$ can be proved similarly by exchanging the index of the two parties throughout, including the system (C.1)–(C.4).) Suppose two solutions, (φ_{12}, p_2) and $(\tilde{\varphi}_{12}, \tilde{p}_2)$, of system (C.1)–(C.4) in domain $\mathcal{D}_3$ correspond to, respectively, η and $\tilde{\eta}$, with $0 < \eta < \tilde{\eta} < 1$. Then we first prove $p_2(u_2) > \tilde{p}_2(u_2)$ and $\varphi_{12}(u_2) \geq \tilde{\varphi}_{12}(u_2)$ for all $u_2 \in [0, \infty)$ and then show that there exists a contradiction.

We first prove for the simpler cases of symmetric toehold and pure common valuation. For the case of symmetric toehold, with $\theta_1 = \theta_2$, Lemma B4 shows that $\varphi_{12}(u_2) = u_2 = \tilde{\varphi}_{12}(u_2)$ for all $u_2 \in [0, \infty]$. For the case of pure common valuation, equation (C.1) is reduced to $\frac{d\varphi_{12}(u_2)}{du_2} = \frac{\theta_2}{\theta_1}$, which, coupled with boundary condition (C.3), implies that $\varphi_{12}(u_2) = \tilde{\varphi}_{12}(u_2) = \frac{\theta_2}{\theta_1} u_2$. In both cases, we have $\tilde{p}_2(0) - p_2(0) = \tilde{\eta} - \eta > 0$ and $\frac{d}{du_2}\left[\tilde{p}_2(u_2) - p_2(u_2)\right] = \frac{1}{\theta_1}\left[\tilde{p}_2(u_2) - p_2(u_2)\right]$. It is then obvious that $\tilde{p}_2(\infty) - p_2(\infty) > \tilde{\eta} - \eta > 0$, contradicting to $\tilde{p}_2(\infty) = p_2(\infty) = 1$. We have thus finished the proof for the case of symmetric toehold and pure common valuation. We can then assume $\theta_1 > \theta_2$ for the rest of the proof.

We next prove for the case of pure private valuation. First, note that, by definition, $p_2(u_2) = \beta_2(t_2) = \alpha_2^{-1}(t_2)$. Since $\alpha_2(b)$, for any given $b \in [0, 1)$, is strictly decreasing in the starting point η (Lemma B5), we have, for any given $u_2 \in [0, \infty)$, $p_2(u_2) < \tilde{p}_2(u_2)$. We next prove $\varphi_{12}(u_2) \geq \tilde{\varphi}_{12}(u_2)$ for any $u_2 \in (0, \infty)$. Suppose that $\varphi_{12}(\hat{u}) < \tilde{\varphi}_{12}(\hat{u})$ for some $\hat{u} \in (0, \infty)$. Since $\varphi_{12}(0) = \tilde{\varphi}_{12}(0)$, and since φ_{12} and $\tilde{\varphi}_{12}$ are both continuous, we can define $\underline{u} = \max\{u : u < \hat{u}; \varphi_{12}(u) = \tilde{\varphi}_{12}(u)\}$, and have $0 \leq \underline{u} < \hat{u}$. Then, for any $u_2 \in (\underline{u}, \hat{u}]$, we have $\varphi_{12}(u_2) < \tilde{\varphi}_{12}(u_2) < u_2$ (the last inequality is from the assumption of $\theta_1 > \theta_2$ and from Lemma B5). From (C.2) and $p_2(u_2) < \tilde{p}_2(u_2)$, we have

$$
\begin{aligned}
\frac{d\varphi_{12}(u_2)}{du_2} &= \frac{\theta_2}{\theta_1} \frac{p_2(u_2) - \bar{v}(\varphi_{12}(u_2))}{p_2(u_2) - \bar{v}(u_2)} \\
&> \frac{\theta_2}{\theta_1} \frac{p_2(u_2) - \bar{v}(\tilde{\varphi}_{12}(u_2))}{p_2(u_2) - \bar{v}(u_2)} \\
&> \frac{\theta_2}{\theta_1} \frac{\tilde{p}_2(u_2) - \bar{v}(\tilde{\varphi}_{12}(u_2))}{\tilde{p}_2(u_2) - \bar{v}(u_2)} \\
&= \frac{d\tilde{\varphi}_{12}(u_2)}{du_2}.
\end{aligned}
$$

Since $\varphi_{12}(\hat{u}) < \tilde{\varphi}_{12}(\hat{u})$, we must have $\varphi_{12}(\underline{u}) < \tilde{\varphi}_{12}(\underline{u})$, a contradiction. So $\varphi_{12}(u_2) \geq \tilde{\varphi}_{12}(u_2)$ for any $u_2 \in (0, \infty)$. Combine this with equation (C.2), we have $\frac{d}{du_2}\left[\tilde{p}_2(u_2) - p_2(u_2)\right] \geq \frac{1}{\theta_1}\left[\tilde{p}_2(u_2) - p_2(u_2)\right]$. Combining this with $\tilde{p}_2(0) - p_2(0) = \tilde{\eta} - \eta > 0$, we conclude, while ignoring the detail here, that $\tilde{p}_2(\infty) - p_2(\infty) > \tilde{\eta} - \eta > 0$, contradicting to $\tilde{p}_2(\infty) = p_2(\infty) = 1$.

Finally we turn to the general case under Assumption 6. As in the case of pure private valuation, we again have $p_2(u_2) < \tilde{p}_2(u_2)$ for all $u_2 \in [0, \infty)$. We next prove $\varphi_{12}(u_2) \geq \tilde{\varphi}_{12}(u_2)$ for all $u_2 \in (0, \infty)$. We first show that there exists a positive $\epsilon > 0$ such that $\varphi_{12}(u_2) > \tilde{\varphi}_{12}(u_2)$ for all $u_2 \in (0, \epsilon]$. Equation (C.1) implies that, for small u_2, we have $\varphi_{12}(u_2) = \frac{\theta_2}{\theta_1} u_2 + o(u_2)$ and $\tilde{\varphi}_{12}(u_2) = \frac{\theta_2}{\theta_1} u_2 + o(u_2)$. Let $v_{11} = \frac{\partial \bar{V}_1}{\partial u_1}(0,0) = \frac{\partial \bar{V}_2}{\partial u_2}(0,0)$ and $v_{12} = \frac{\partial \bar{V}_1}{\partial u_2}(0,0) = \frac{\partial \bar{V}_2}{\partial u_1}(0,0)$, Assumption 6 implies that $v_{11} > v_{12} \geq 0$. We have, for small u_2,

$$\begin{aligned}
&\frac{d}{du_2}\left[\varphi_{12}(u_2) - \tilde{\varphi}_{12}(u_2)\right] \\
=&\frac{\theta_2}{\theta_1}\left[\frac{p_2(u_2) - \bar{V}_1(\varphi_{12}(u_2), u_2)}{p_2(u_2) - \bar{V}_2(\varphi_{12}(u_2), u_2)} - \frac{\tilde{p}_2(u_2) - \bar{V}_1(\tilde{\varphi}_{12}(u_2), u_2)}{\tilde{p}_2(u_2) - \bar{V}_2(\tilde{\varphi}_{12}(u_2), u_2)}\right] \\
=&\frac{\theta_2}{\theta_1}\left[\frac{p_2(u_2) - \left(v_{11}\frac{\theta_2}{\theta_1} + v_{12}\right)u_2}{p_2(u_2) - \left(v_{12}\frac{\theta_2}{\theta_1} + v_{11}\right)u_2} - \frac{\tilde{p}_2(u_2) - \left(v_{11}\frac{\theta_2}{\theta_1} + v_{12}\right)u_2}{\tilde{p}_2(u_2) - \left(v_{12}\frac{\theta_2}{\theta_1} + v_{11}\right)u_2}\right] + o(u_2).
\end{aligned}$$

So there exists a small enough $\epsilon > 0$ such that, for any $u_2 \in (0, \epsilon]$, we have $\frac{d}{du_2}\left[\varphi_{12}(u_2) - \tilde{\varphi}_{12}(u_2)\right] > 0$. This, coupled with $\varphi_{12}(0) = \tilde{\varphi}_{12}(0) = 0$, implies that $\varphi_{12}(u_2) > \tilde{\varphi}_{12}(u_2)$ for all $u_2 \in (0, \epsilon]$.

Next, we show that $\varphi_{12}(u_2) > \tilde{\varphi}_{12}(u_2)$ for all $u_2 \in (0, \infty)$. Suppose otherwise, then we define $\hat{u} = \inf\{u : \epsilon < u < \infty;\ \varphi_{12}(u) \le \tilde{\varphi}_{12}(u)\}$. Continuity implies that $\epsilon < \hat{u} < \infty$ and $\varphi_{12}(\hat{u}) = \tilde{\varphi}_{12}(\hat{u})$. But since $\hat{u} > \varphi_{12}(\hat{u})$ (from $\theta_1 > \theta_2$), Assumption 6 implies that $\bar{V}_1(\varphi_{12}(\hat{u}), \hat{u}) < \bar{V}_1(\varphi_{12}(\hat{u}), \hat{u})$, equation (C.1) and $p_2(\hat{u}) < \tilde{p}_2(\hat{u})$ then imply that $\frac{d\varphi_{12}}{du_2}(\hat{u}) > \frac{d\tilde{\varphi}_{12}}{du_2}(\hat{u})$, and thus $\varphi_{12}(u_2) < \tilde{\varphi}_{12}(u_2)$ for a u_2 less than but close enough to $\hat{u}$, which contradicts with the definition of $\hat{u}$. So we have shown $\varphi_{12}(u_2) > \tilde{\varphi}_{12}(u_2)$ for all $u_2 \in (0, \infty)$. As in the case of pure private valuation, combining this result with equation (C.2), we have $\frac{d}{du_2}\left[\tilde{p}_2(u_2) - p_2(u_2)\right] \ge \frac{1}{\theta_1}\left[\tilde{p}_2(u_2) - p_2(u_2)\right]$. Combining this with $\tilde{p}_2(0) - p_2(0) = \tilde{\eta} - \eta > 0$, we conclude that $\tilde{p}_2(\infty) - p_2(\infty) > \tilde{\eta} - \eta > 0$, contradicting to $\tilde{p}_2(\infty) = p_2(\infty) = 1$. We have thus finished the proof for this general case and for the proposition. ∎

References

Bradley, M., A. Desai, and E. H. Kim (1988) "Synergistic Gains From Corporate Acquisitions and Their Divisions Between the Stockholders of Target and Acquirer Firms," *Journal of Financial Economics* **11:** 3-40.

Bulow, Jeremy, Ming Huang, and Paul Klemperer (1995) " Toeholds and Takeovers," Working Paper, Stanford University.

Burkart, Mike (1995) "Initial Shareholdings and Overbidding In Takeover Contests," Forthcoming in *Journal of Finance*.

Chowdhry, Bhagwan and Narasimhan Jegadeesh (1994) "Pre-Tender Offer Share Acquisition Strategy in Takeovers," *Journal of Financial and Quantitative Analysis* **29:** 117-129.

Fishman, Michael J. (1988) "A Theory of Preemptive Takeover Bidding," *Rand Journal of Economics* **19:** 88-101.

Giammarino, Ronald M. and Robert L. Heinkel (1986) "A Model of Dynamic Takeover Behavior," *Journal of Finance* **41:** 465-480.

Grossman, Sanford J. and Oliver D. Hart (1980) "Takeover Bids, the Free-Rider Problem, and the Theory of the Corporation," *Bell Journal of Economics* **11:** 42-64.

Hirshleifer, David and I. P. L. Png (1990) "The Information Conveyed by a Tender Offer and the Takeover Price of a Target Firm," *Review of Financial Studies* **2:** 587-606.

Hirshleifer, David and Sheridan Titman (1990) "Share Tendering Strategies and the Success of Hostile Takeover Bids," *Journal of Political Economy* **98:** 295-324.

Lebrun, Bernard (1993) "The Existence of an Equilibrium in First Price Auctions," forthcoming in *Economic Theory.*

Lebrun, Bernard (1994) "First Price Auction: Properties of the Equilibria in the Asymmetric N Bidder Case," Working Paper, V.P.I. & S.U..

Marshall, Robert C., Michael J. Meurer, Jean-Francois Richard, and Walter Stromquist (1994) "Numerical Analysis of Asymmetric First Price Auctions," *Games and Economic Behavior* **7:** 193-220.

Maskin, Eric and John Riley (1994a) "Existence in Sealed High Bid Auctions," revised version of UCLA Working Paper 407.

Maskin, Eric and John Riley (1994b) "Uniqueness in Sealed High Bid Auctions," revised version of UCLA Working Paper 407.

Pontriagin, Lev Semenovich (1962) *Ordinary Differential Equations.* Reading, Mass., Addison-Wesley.

Shleifer, Andrei and Robert W. Vishny (1986) "Large Shareholders and Corporate Control," *Journal of Political Economy* **94:** 223-249.

Singh, Rajdeep (1994) "Takeover Bidding with Toeholds: The Case of the Owner's Curse," Working Paper, Washington University.