UNEMPLOYMENT, PARTICIPATION AND MARKET SIZE

Godfrey Keller, Kevin Roberts and Margaret Stevens*
Department of Economics, University of Oxford
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Abstract

Without strong empirical support, random matching models of the labour market typically assume constant returns to scale in matching, resulting in a focus on steady states rather than dynamics. We construct an equilibrium random matching model with a general matching technology and endogenous market participation, introducing market size effects: the job-finding rate and participation incentives change with the level of unemployment. Agent behaviour is more complex than under constant returns, with heterogeneity in search behaviour for workers with different degrees of attachment to the labour market. Heterogeneity makes the model inherently multi-dimensional, so techniques are developed to reduce the dimensionality to establish local and global stability; a complicating factor is the possibility of multiple equilibria, welfare-ranked by market size. Locally decreasing returns in matching generate plausible joint dynamics of employment, unemployment and participation. An extension of the Hosios condition internalises search externalities.

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1 Introduction

Random matching models capture the implications of labour market frictions by specifying a matching function: the rate at which workers and firms meet is a function of the

*Corresponding author.
numbers of agents on each side of the market. A natural property of the matching function is that the rate at which an individual agent matches depends on market tightness – the relative number of agents on each side of the market. It may also depend on the absolute number of agents: a market could be too small to support effective mechanisms for exchanging information about opportunities for trade, or so large that participants suffer from information overload. Market size effects were the focus of Diamond’s (1982) model of multiple equilibria and co-ordination failure, but have since received little attention in the extensive matching literature. The dominant equilibrium random matching model of the labour market, most fully described by Pissarides (2000), rules them out by assuming a matching function with everywhere-constant returns to scale, usually justified by an appeal to empirical evidence.

But the evidence for constant returns is not compelling: of the eight aggregate studies surveyed by Petrongolo and Pissarides (2001), only three – Pissarides (1986) and Layard et al (1991) for Britain, and van Ours (1991) for the Netherlands – support constant returns. Two studies (Burda and Wyplosz, 1995, for France, Germany, Spain and the UK; Berman, 1997, for Israel) obtain decreasing returns, and three (Blanchard and Diamond, 1990, and Warren, 1996, for the US; Yashiv, 2000, for Israel) find increasing returns. Elsbury (2010) estimates the US aggregate matching function using the new measures of the unemployment outflow rate provided by Shimer (2005b) and finds decreasing returns. The evidence for disaggregated markets is similarly mixed: substantial departures from constant returns have been found for regional and occupational markets.

Assuming constant returns has had important implications for the search and matching literature, because it focuses attention on steady states rather than dynamic adjustment. The dynamic behaviour of the Pissarides model is very simple: away from the steady state unemployment evolves slowly, but since market size does not matter this has no effect on other variables; workers are indifferent to the rate of unemployment, and reservation wages, participation, market tightness and the job-finding rate all remain constant at their equilibrium values. To reconcile the constant-returns model with the observed dynamic variation in job-finding rates (Shimer, 2005a) it is therefore necessary to assume high-frequency shocks to productivity or other parameters. Hall (2005) argues explicitly that

\footnote{For travel-to-work areas in Britain, the results of Coles and Smith (1996) support constant returns, but Burgess and Profit (2001) allow for area fixed effects and find decreasing returns. Kano and Ohta (2005) also obtain decreasing returns for Japanese regions. Several studies of local labour markets have found increasing returns: Kangasharju, Pekkonen and Pekkala (2005) for Finland; Baker, Hogan and Ragan (1996) for Canada; and Munich, Svejnar and Terrell (1999) for the Czech Republic. Fahr and Sunde (2004) find increasing returns for craft and technical occupations, and substantially decreasing returns for white collar occupations and for highly educated workers.}
turnover dynamics are irrelevant, showing that unemployment closely tracks its steady-state level. But as we will show in this paper, his argument relies on constant returns; an alternative explanation is provided by dynamic adjustment under decreasing returns.

We present an equilibrium matching model of the labour market with a general form for the matching function, which takes account of market size effects and endogenous participation. Steady state equilibria have simple properties, but the model provides a richer dynamic environment. Since the job-finding rate changes with the level of unemployment, workers’ search strategies depend on the current and expected future state of the market. Those with higher outside options are less attached to the market, and set higher reservation wages. When the market is expected to improve in the future, participating workers may withdraw temporarily to wait for more favourable conditions.

After specifying the model in section 2, we obtain conditions for existence and local stability of steady-state equilibria in section 3. Multiple equilibria are welfare-ranked according to market size, irrespective of the shape of the matching function and the elasticity of labour supply. The heterogeneity of individual responses is analysed in section 4, and local and global dynamics are investigated in sections 5 and 6 respectively. Equilibria with decreasing returns are locally saddle-path stable; those with increasing returns can be stable provided that returns to scale do not increase steeply and labour supply is sufficiently inelastic. Following a shock that increases unemployment, reservation wages either rise or fall depending on the direction of the market size effect, and participation, reservation wages and market tightness adjust along the saddle-path. The rate of convergence depends on both returns to scale and the elasticity of labour supply.

Finally in section 7 we compare decentralised equilibria with the efficient allocation, and derive a generalisation of the well-known Hosios condition for efficiency. As for the constant returns case, the decentralised equilibrium can be efficient if the shares of the surplus obtained by workers and firms are equal to the elasticities of the matching function with respect to unemployment and vacancies. Thus efficiency requires an employment tax with decreasing returns, or a subsidy with increasing returns.

1.1 Labour Market Matching Models with Constant Returns

Equilibrium random matching models are now well-established as a framework for understanding unemployment. In the textbook random matching model of Pissarides (2000), ex-ante identical workers and firms search in the labour market and meet according to a matching technology that is linearly homogeneous in the stocks of unemployed workers and vacancies. Each match produces a flow of output, with constant returns in production, and the wage is determined through bilateral bargaining. When a match is destroyed
as a result of an exogenous productivity shock the worker re-enters the market. The size of the labour force is fixed, and a zero-profit condition determines the stock of vacancies.

The search-theoretic approach is consistent at a microeconomic level with many features of observed labour market behaviour (Rogerson, Shimer and Wright, 2005). At the aggregate level, the model provides a theory of equilibrium unemployment, predicting labour market flows broadly consistent with business cycle evidence (Mortensen and Pissarides, 1994 and 1999; Cole and Rogerson, 1999). Incorporating labour market matching improves the ability of RBC models to reproduce macroeconomic stylised facts (Merz 1995, 1999; Andolfatto, 1996; and Den Haan, Ramey and Watson, 2000).

A number of problems have been highlighted, however. Shimer (2005a) shows that the Mortensen-Pissarides model cannot account for the strong procyclicality of the job-finding rate and market tightness (the $v/u$ ratio). Cole and Rogerson (1999) argue that the model needs to allow for heterogeneity in search intensity, since consistency with the evidence requires workers classified as out of the labour force to be treated as low-intensity searchers. Veracierto (2004) points out that neither standard RBC models (in which all non-employed workers are non-participants), nor those augmented by matching frictions (which have assumed a fixed labour force) can explain the joint dynamics of employment, unemployment and labour force participation.

The efficiency properties of matching models with constant returns are well-understood. In addition to the inevitable losses due to the existence of frictions, search externalities arise because each agent affects the matching rate for other agents. When wages are determined ex-post, the market equilibrium will not in general be constrained-efficient. For the standard model with constant returns and homogeneous agents, Hosios (1990) showed that externalities are internalised if the bargaining share of workers is equal to the elasticity of matching with respect to unemployment. In this environment, all external effects act through market tightness, and a single condition for relative surplus shares is sufficient to ensure that agents make efficient search and matching decisions.

1.2 Non-Constant Returns, Multiple Equilibria and Dynamics

In the elegant stylised model of Diamond (1982) there is only one type of searching agent. Trading opportunities occur when they meet each other, and their meeting rate increases with the number of searchers generating multiple equilibria and potential coordination failure. Diamond and Fudenberg (1989) analyse the dynamics of the Diamond model, emphasising self-fulfilling expectations. For some initial conditions there are rational expectations equilibrium paths to both high and low activity equilibria, and there may be cyclical equilibrium paths. Boldrin, Kiyotaki and Wright (1993) provide a full analysis
of dynamics in a generalisation of the Diamond model with differentiated commodities, focusing on the existence of limit cycles.

The assumption of a single type of agent limits the applicability of the Diamond model, although Camera (2000) uses this approach to obtain multiple equilibria in a monetary search model. There are rather few analyses of the implications of non-constant returns to scale when the matching function depends separately on the numbers of buyers and sellers. Pissarides (1984) shows that with a general matching technology equilibria are inefficient – the Hosios (1990) condition cannot hold without constant returns. Howitt and McAfee (1987) use a specific increasing returns technology to introduce Diamond’s thin-market externality and hence multiple equilibria into a labour market model, again emphasising inefficiency. In Hyde’s (1997) bargaining model with homogeneous buyers and heterogeneous sellers, an increasing returns matching technology gives rise to high and low participation equilibria. Burdett and Wright (1998) model a labour market with fixed and equal numbers of agents on both sides, and idiosyncratic match productivity; multiple equilibria may occur either as a result of increasing returns in matching or non-transferable utility, but are ruled out when the productivity distribution is log-concave.²

None of the analyses of labour market models with non-constant returns to scale in matching considers dynamics, although Mortensen (1989) suggests that they will be similar to those of matching models with aggregate increasing returns in production (Drazen, 1988; Mortensen, 1999). The purpose of this paper is to show that escaping the straitjacket of constant returns permits a richer dynamic structure to emerge: importantly, it generates plausible out-of-steady-state dynamics which (in contrast to the constant-returns case) can contribute to explaining the temporal behaviour of labour market variables.

2 The Model

2.1 The Matching Technology

Unemployed workers and potential employers meet randomly according to a matching technology that determines the meeting rate, $M$, as a function of the mass of unemployed workers, $u$, and the mass of vacant jobs, $v$:³ $M = M(u, v)$. We assume that $M(0, v) = M(u, 0) = 0$, and that $M$ is strictly increasing in both arguments, homothetic,

²Ex-ante heterogeneity or ex-ante investments can also lead to multiple equilibria in models with a constant-returns matching technology (Acemoglu, 1997, 2001; van den Berg, 2003).

³In an earlier version of this paper we allowed for variable and heterogeneous search intensity. Here, search intensity is exogenous and the same for all participating workers. Results are essentially the same; we will note any significant differences where they arise.
and quasi-concave.\textsuperscript{4} Hence it can be expressed:

\[ M = \Phi(m(u,v)) \]

where \( \Phi \) is a strictly increasing function, \( \Phi(0) = 0 \), and \( m(u,v) \) is homogeneous of degree one with strictly convex isoquants (so \( m \) is concave). We can normalise this decomposition by assuming that \( m(1,1) = 1 \).

It is helpful to think of \( m(u,v) \) as the level of aggregate search activity in the market, and \( \Phi(m) \) as converting activity into matching. Activity increases linearly with the stocks of searching workers and firms. We define the elasticity of matching with respect to activity, \( \eta(m) \), and the average matching rate, \( \phi(m) \), by:

\[
\eta(m) \equiv \frac{m\Phi'(m)}{\Phi(m)} \quad \text{and} \quad \phi(m) \equiv \frac{\Phi(m)}{m}
\]

The matching function has locally decreasing, constant, or increasing returns to scale when the elasticity \( \eta \) is, respectively, less than, equal to, or greater than 1. The average matching rate is a measure of the effectiveness of matching. For a constant returns matching function it is constant; otherwise \( \phi(m) \) rises or falls where returns to scale are increasing or decreasing respectively. In the example in Figure 1a (see section 3.1), there are locally constant returns at \( m^* \), where \( \eta(m) = 1 \) and the average matching rate \( \phi(m) \) is maximised; if \( m < m^* \) there are increasing returns, the marginal matching rate is greater than the average matching rate, and \( \eta > 1 \); above \( m^* \) the converse conditions hold. Intuitively, \( m^* \) is the point where search activity leads most effectively to matching.

Since activity \( m(u,v) \) is homogeneous of degree 1 its elasticities with respect to unemployment and vacancies, denoted by \( 1 - \alpha \) and \( \alpha \) respectively, are functions only of market tightness, \( \theta \equiv v/u \), (as are average and marginal activity with respect to \( u \) and \( v \)). The elasticities of the matching rate \( M = \Phi(m) \) with respect to \( u \) and \( v \) are:

\[
\eta_u = (1 - \alpha)\eta \quad \text{and} \quad \eta_v = \alpha\eta
\]

The elasticity of substitution between unemployment and vacancies also depends on \( \theta \) only, and is the same for \( M \) and \( m \). It is convenient to define \( \mu(\theta) \) as activity per worker:

\[ \mu(\theta) \equiv m(1,\theta) \]

where \( \mu(\theta) \) also has elasticity \( \alpha \). Then the job-finding rate for workers, \( \lambda \), which in the standard model depends on \( \theta \) only, is proportional to the average matching rate:

\[ \lambda \equiv M/u = \phi(m)\mu(\theta) \]

\textsuperscript{4}Homotheticity is an unimportant assumption for steady state analysis but provides a mild simplification of the dynamic analysis. In particular, it admits a separation of scale and composition effects.
2.2 Participation, Unemployment, and Employment

All agents are infinitely lived with common discount rate $\rho$. A worker is in one of three states at any instant: employed, unemployed and searching for a job, or “inactive” – that is to say, outside the market.

Unemployment income is normalised to zero, and workers can move instantaneously between unemployment and inactivity. They differ in their alternative opportunities: a worker of type $q$ obtains a constant flow income $q$ while inactive that he must forgo while participating. Outside income $q$ can be interpreted as the value of leisure, or home production, or of participation in a different market. Labour supply to the market is described by an increasing function $L(q)$, the number of workers with outside income less than or equal to $q$; the elasticity of labour supply is denoted by $\gamma$, and $L(0) = 0$.

There are many firms, each with a single potential job. A firm with no employee will create and maintain a vacancy whenever the present value of doing so is greater than zero. Thus the supply of vacancies is perfectly elastic at zero profit. While maintaining a vacancy the firm incurs a constant flow search cost $c$.

Match productivity is stochastic: an employed worker produces a constant flow of output $x$, which is a random variable realised when the worker and firm meet. A potential match is not necessarily consummated since if productivity is low the agents may prefer to search for a better one. Match formation entails an instantaneous cost $K$, which is the same for every match and can be interpreted as a specific training cost. Matches are destroyed at constant rate $\delta$, in which case the agents may search for a new match, or leave the market.

2.3 The Match Surplus, and Surplus Sharing

Consider a match with productivity $x$, between a firm and a worker of type $q$. The match generates income $x$ while it lasts; if it is destroyed at time $t$ the worker obtains $V_u(t; q)$, the expected present value of his income while not employed, and the firm obtains zero (by the vacancy creation condition).

Let $Y(x, t; q)$ be the expected present value of the match at time $t$. We assume that matching decisions are privately efficient, so a potential match will be consummated if the initial surplus $Y - V_u$ exceeds the cost of match formation, $K$, and maintained thereafter until either $Y = V_u$, or it is exogenously destroyed. However, we will focus on equilibria in which matches are never endogenously destroyed: if the surplus is initially high enough.

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$^5$Equivalently, firms are of indeterminate size and have constant returns in production.
that the match is worth forming, it will never fall to zero.\footnote{This could be guaranteed by restricting the range of the distribution of match productivity $x$ to be less than $(\rho + \delta)K$. In fact, however, we need only that the change in the reservation wage on any equilibrium path is less than this bound – a much less stringent condition.} The match value $Y$ satisfies:

$$\rho Y = x + \delta(V_u - Y) + \frac{\partial Y}{\partial t} \quad (1)$$

Since an increase in productivity, $dx$, would raise the income flow by this amount for the duration of the match, its effect on the match value is given by:

$$\frac{\partial Y}{\partial x} = \frac{1}{\rho + \delta} \quad (2)$$

A match will be consummated if and only if its initial value $Y(x, t_0) - K > V_u(t_0)$. Let $y$ be net productivity: $y \equiv x - (\rho + \delta)K$. Since $Y(x, t) - K \equiv Y(y, t)$, and $Y$ increases with productivity, it is acceptable if and only if net productivity $y$ exceeds the worker’s reservation wage $z(q, t)$, which satisfies:

$$Y(z, t; q) = V_u(t; q) \quad (3)$$

Let net productivity $y$ have distribution function $G(y)$ and supremum $\bar{y}$. With reservation productivity $z(q, t)$, the probability that a match is acceptable is:

$$\pi(z) \equiv P(y \geq z) = 1 - G(z)$$

The expected productivity of an acceptable match is given by:

$$E(y \mid y \geq z) = z + h(z) \text{ where } h(z) = \frac{1}{\pi(z)} \int_z^\infty (1 - G(y)) \, dy$$

and the expected surplus from a meeting is:\footnote{We suppress arguments where it can be done without confusion, here writing $Y(y)$ for $Y(y, t; q)$.}

$$E[\max(Y(y) - Y(z), 0)] = \int_z^\infty (Y(y) - Y(z)) \, dG(y) = \frac{h(z)\pi(z)}{\rho + \delta} \equiv S(z)$$

(where (2) is used to evaluate the integral). Note that $S(z)$ decreases with reservation productivity $z$. It is common in search models to assume that the distribution of productivity is log-concave, which guarantees that the flow surplus $h(z)$ also decreases with $z$. Here we impose only the weaker condition that $h(z)$ has elasticity less than one.\footnote{We do not rely heavily on this condition, but it reduces the number of cases to be considered.}
We do not model wage determination explicitly, but simply assume that if a match is consummated the worker and firm receive non-negative shares $\beta_1$ and $\beta_2$ respectively of the initial net surplus $Y(y) - V_u$. To allow for the possibility of an employment tax or subsidy, we do not assume that $\beta_1$ and $\beta_2$ sum to one.\footnote{For example, if the surplus is taxed at rate $\tau$ and the remaining surplus is shared by Nash bargaining in which the worker has bargaining power $\beta$, we have $\beta_1 = (1 - \tau)\beta$ and $\beta_2 = (1 - \tau)(1 - \beta)$.}

### 2.4 The Decisions of Individual Agents

We now determine the reservation wage $z(q,t)$ and participation decision for a worker of type $q$, and the vacancy creation condition for an individual firm, taking the behaviour of other agents as given. For a worker who is not currently employed, market opportunities depend on the job-finding rate $\lambda$. In a time period of length $dt$, he can take the outside income $qdt$, or participate in the market, in which case he will encounter a potential match with probability $\lambda dt$. Hence $V_u(q,t)$ satisfies:

$$\rho V_u = \max\{q, \lambda \beta_1 S(z)\} + \frac{dV_u}{dt}$$

(4)

The reservation wage $z(q,t)$ satisfies (3) continuously, so:

$$\frac{1}{\rho + \delta} \frac{dz}{dt} + \frac{\partial Y}{\partial t} = \frac{dV_u}{dt}$$

(5)

Using (3), (5), and (1) evaluated at $z$, we can eliminate $V_u$ from (4) to obtain:

$$z = \max\{q, a\} + \frac{\dot{z}}{\rho + \delta}$$

(6)

where $a(z, \lambda) \equiv \lambda \beta_1 S(z)$

(7)

and $\dot{z}$ denotes the derivative with respect to time. Recalling that unemployment income is normalised to zero, note that $a(z, \lambda)$ can be interpreted as the current expected market income available to the worker; he participates when this is greater than his alternative income $q$. Current income $a$ increases with the job-finding rate $\lambda$, but it decreases with $z$ because with a higher reservation wage the worker is less likely to find an acceptable job.

A firm with a vacancy meets workers at rate $\lambda/\theta$; its expected surplus from a meeting with a worker with reservation wage $z$ is $\beta_2 S(z)$, and the expected surplus from a random meeting is $\beta_2 \bar{S}$, where $\bar{S}$ is the expectation of $S(z)$ across all participating workers. Free entry implies that the expected return to a vacancy is equal to the cost, so we have:

$$\theta c = \lambda \beta_2 \bar{S}$$

(8)
2.5 Aggregate Unemployment and Participation

To determine aggregate variables we need to keep track of the distribution of worker-types across employment, unemployment and inactivity. Let $N(q,t)$ be the current number of non-employed workers with outside income less than $q$. $N(q,t)$ is the effective labour supply, instantaneously available to the matching market at time $t$. The number of employed workers with outside income less than $q$ is $L(q) - N(q)$ (again suppressing the time argument). From (6) we can describe participation by a function $p(q,t)$:

$$p(q) = \begin{cases} 
1 & \text{if } a(z(q), \lambda) \geq q \\
0 & \text{otherwise}
\end{cases} \quad (9)$$

Aggregate variables can then be expressed in terms of $N$ and $p$. Assuming that we can write $N(q) = \int q N(q) \, dq$ for some non-negative function $n(q)$, we have:

$$u = \int p(q) \, dN(q) \quad (10)$$

$$\bar{S} = \frac{1}{u} \int S(z(q))p(q) \, dN(q) \quad (11)$$

The distribution of non-employed workers $N(q)$ evolves according to the difference between the inflow from employment as jobs are destroyed, and the outflow to employment, allowing for the reservation wages of each type:

$$\frac{\partial N(q)}{\partial t} = \delta(L(q) - N(q)) - \lambda \int \pi(z(q))p(q) \, dN(q) \quad (12)$$

This equation, together with (6) to (8) for individual behaviour and (9) to (11) for aggregate variables, completes the description of an evolving market.

3 The Steady State

In a steady state, the numbers of participants are constant, as are their reservation values. From (6) we can see that all participating workers will have the same reservation wage $z^*$ satisfying $z = a(z, \lambda)$, so all matches have the same expected surplus $S(z^*)$. The active labour force is $L(z^*)$ and unemployment is $u = N(z^*)$. From equations (6) to (8) and (12), a steady-state equilibrium $(z^*, \theta^*, u^*)$ satisfies:

$$z = a(z, \lambda) \quad (13)$$

$$\theta = \left( \frac{\partial z}{\partial \lambda} \right) a(z, \lambda) \quad (14)$$

$$\Phi(m) \pi(z) = \delta(L(z) - u) \quad (15)$$
where $\lambda \equiv \phi(m)\mu(\theta)$ and $m \equiv u\mu(\theta)$ \hfill (16)

and the distribution of non-employed workers is:

$$
N(q) = \begin{cases} 
  u + L(q) - L(z) & \text{if } q \geq z \\
  \frac{u}{L(z)}L(q) & \text{if } q \leq z
\end{cases} \hfill (17)
$$

3.1 Existence of Equilibria

To establish existence, we can eliminate $\theta$ from the steady-state equations to obtain a pair of equations in the reservation wage $z$ and market activity $m$:

**Lemma 1** In a steady state, $\theta = \left(\frac{\beta_2}{\beta_1}\right)z$. Then

(i) For all $m$ such that $\phi(m) \geq \phi_0$ (a positive constant) (13) and (14) have a unique solution $z = \zeta_1(m) \in [0, \bar{y}]$, where $\zeta_1(m) = 0$ if $\phi(m) = \phi_0$, $\zeta'_1 \equiv \eta(m) - 1$.

(ii) For all $m \geq 0$, (15) has a unique solution $z = \zeta_2(m)$, where $\zeta_2(0) = 0$, $\zeta'_2 > 0$, and $\zeta_2(\bar{m}) = \bar{y}$ for some $\bar{m} < \infty$.

**Proof:** See Appendix A.

The reservation wage of unemployed workers varies with market activity: $z = \zeta_1(m)$. What matters is the average matching rate, $\phi(m)$. If this is low – below a constant $\phi_0$ that depends on the parameters – there is no solution to (13) and (14): the return to search is too low to induce anyone to participate. Above $\phi_0$, the reservation wage $z$ increases with the average matching rate, so $z = \zeta_1(m)$ rises or falls with market activity $m$ according to whether returns to scale in matching are increasing or decreasing. In the constant returns case there is a unique reservation wage, irrespective of the level of activity in the market.

Equilibria are found at levels of market activity where the reservation wage that is optimal for individual agents, $z = \zeta_1(m)$, is consistent with constant unemployment: that is, where $\zeta_1(m) = \zeta_2(m)$. Both functions are continuous, and $\zeta_2(m)$ is strictly increasing. If the matching function has everywhere non-increasing returns to scale $\zeta_1(m)$ is non-increasing and there is at most one equilibrium; otherwise there may be several equilibria. By considering the properties of $\zeta_1$ and $\zeta_2$ we can show that:

**Proposition 1** A steady-state equilibrium $(z, \theta, u)$ with activity $m > 0$ exists for any $\beta_1, \beta_2 > 0$, provided that $M_v(1,0)$ is sufficiently large. It is unique if $\eta(m) \leq \bar{\eta}(\gamma) \forall m$, where $\bar{\eta}(\gamma) \equiv \frac{1 + \gamma}{\alpha + \gamma}$.

**Proof:** See Appendix A.
Figure 1: (a) A matching function with initially increasing and then decreasing returns (b) corresponding equilibria at the intersection of $\zeta_1$ and $\zeta_2$

Figure 1b illustrates possible equilibria for the matching function in Figure 1a, which has initially increasing and subsequently decreasing returns to scale. For this form of matching function, at most one equilibrium – the one with the highest activity level – can occur in the region of decreasing returns. More generally, the equilibrium with highest activity always occurs at a point where $\zeta'_2 > \zeta'_1$. Note that, since the function $\zeta_2(m)$ is increasing, equilibria with higher activity levels have higher average matching rates and
higher levels of worker income.

As in the standard model a sufficient (but not necessary) condition for existence is that there are incentives for vacancy creation even when the market is small. The sufficient condition for uniqueness is obtained from a single-crossing condition: \( \zeta'_2 > \zeta'_1 \) at any equilibrium where \( \eta(m) \leq \bar{\eta}(\gamma) \). The uniqueness condition may be compared with the result of Burdett and Wright (1998) that with fixed and identical numbers of workers and firms, log-concavity of the productivity distribution is sufficient to guarantee uniqueness with increasing returns. Here log-concavity is not enough: when agents are able to respond to market conditions we need an upper bound on the degree of returns to scale, which becomes more restrictive with increasing elasticity of labour supply, \( \gamma \) (and even more so if workers can vary their search intensity).

Later we will consider the dynamic stability of equilibria, but we can show immediately that equilibria like (2) where \( \zeta'_2 < \zeta'_1 \) are implausible, in that they are not robust to any variation in the number of vacancies.

### 3.2 Incentives for Vacancy Creation

We have used the conventional assumption that the number of vacancies is determined by a zero-profit condition. Behind this is an implicit assumption that profits would fall if more firms were to enter, and rise if firms left. However, there is nothing in the model to ensure that this holds. It is possible that at a solution of the steady-state equilibrium conditions with increasing returns to market activity, creation of additional vacancies would lead to a rise in profit and further incentives to create vacancies. This would be an implausible equilibrium – effectively a profit-minimising point for firms.

A minimal requirement for a market equilibrium is therefore that it should be stable with respect to vacancy creation – that a rise in vacancies should lead to a fall in the profit flow of firms (and vice-versa). This requires a bound on the rate of increasing returns at the equilibrium, and the condition fails at alternate equilibria:

**Proposition 2** With one or more steady-state equilibria, numbered in order of decreasing market size, even-numbered equilibria are unstable with respect to vacancy creation. Odd-numbered equilibria are stable with respect to vacancies if and only if \( \eta < \bar{\eta}(L\gamma/u) \).

**Proof:** In a steady-state equilibrium satisfying (13) to (15), the profit flow, \( \Pi = -c + \frac{1}{2} \beta_2 S(z) \), is zero. Suppose that \( v \) increases instantaneously by \( dv \). This has no effect on reservation wages because agents expect to be in equilibrium in future when matches are formed. However the change in \( v \) changes the current activity level \( m \), and hence also the current income of a of workers, so some workers enter or leave the market temporarily.
The changes in activity level $m(u,v)$ and profit $\Pi$ are:

$$\frac{dm}{m} = (1 - \alpha)\frac{du}{u} + \alpha \frac{dv}{v} \quad \text{and} \quad \frac{d\Pi}{c} = \eta \frac{dm}{m} - \frac{dv}{v}$$

The instantaneous change in unemployment is $du = n(z)da$. Calculating the change in current income $a$ using (7) gives:

$$\frac{du}{u} = n(z)\left( \frac{\eta \frac{dm}{m} - \frac{du}{u}}{u} \right)$$

Eliminating $dm$ and $du$ from these three equations gives the overall effect on profit:

$$\frac{v}{c} \frac{d\Pi}{dv} \equiv \eta \left( \alpha + \frac{n(z)z}{u} \right) - \left( 1 + \frac{n(z)z}{u} \right)$$

Stability requires this expression to be negative. From (17), $n(q)$ is discontinuous at $z$: if $z$ rises the change in labour supply is $n(z) = L'(z)$ whereas if $z$ falls $n(z) = uL'(z)/L$. We have stability in both directions if and only if $\frac{d\Pi}{dv}$ is negative when $n = L'$; that is, if and only if $\eta < \bar{\eta}(L\gamma/u)$. Since $\bar{\eta}(L\gamma/u) < \bar{\eta}(\gamma)$, this condition implies that $\zeta_2' > \zeta_1'$ (see the proof of Proposition 1). Hence it cannot hold at equilibria where $\zeta_2' < \zeta_1'$ – that is, at even-numbered equilibria – but may do so at odd-numbered equilibria.

The interpretation is straightforward: stability requires decreasing returns to vacancy creation, allowing for the responses of workers and firms. With fixed participation of workers ($\gamma = 0$) the condition is simply that the elasticity of matching with respect to vacancies, $\alpha \eta$, is less than one. The upper bound $\bar{\eta}(L\gamma/u)$ falls as $\gamma$ rises. In the limiting case of a perfectly elastic supply of workers stability requires decreasing returns to scale at the equilibrium: $\bar{\eta}(L\gamma/u) \to 1$. In addition $\bar{\eta}(L\gamma/u)$ depends on the unemployment rate $u/L$. If it is low, the number of workers who enter in response to a rise in market income is high relative to the number searching in the steady state. Such a steady-state is therefore less stable \textit{ceteris paribus}, and $\bar{\eta}(L\gamma/u)$ is lower.

### 3.3 Steady State Welfare

In a steady-state equilibrium social welfare $\Omega$, which we take to be the present value of aggregate net income, is given by:

$$\rho \Omega = (L(z) - u)(z + h(z)) + \int_z q dL(q) - u\theta c$$

the flow of output, plus the outside income of non-participants, less the cost of vacancies.
Using the equilibrium conditions this can be written as:

\[ \rho \Omega = \int \max\{z, q\} dL(q) + \Phi(m) \pi(z) h(z) \left( \frac{1}{\delta} - \frac{\beta_1 + \beta_2}{\rho + \delta} \right) \]

Note that the original expression for \( \Omega \) does not depend on \( \beta_1 \) or \( \beta_2 \). Although we allow \( \beta_1 + \beta_2 \neq 1 \), the corresponding tax or subsidy is assumed to be implemented as a transfer between agents within the model: for example, since workers earn rents in equilibrium, an employment subsidy could be financed by a lump-sum tax on workers. \( \beta_1 + \beta_2 \) appears in the second expression for \( \Omega \) only because it determines the equilibrium values of \( z, u \) and \( \theta \), and hence affects welfare indirectly.

If there are several equilibria, we know that since they satisfy \( z = \zeta_2(m) \), equilibria with higher market size provide higher income \( z \) for workers. They also satisfy \( z = \zeta_1(m) \), which (as shown in the Appendix) implies that \( \Phi(m) \pi(z) h(z) \) increases with \( m \). Hence a comparison of welfare gives an unambiguous ranking:

**Proposition 3** When multiple equilibria exist, steady-state welfare increases with equilibrium market size if \( \beta_1 + \beta_2 \leq 1 + \rho/\delta \).

Thus if the employment subsidy is not too high,\(^{10}\) the equilibrium with the highest market size is welfare-superior: the average matching rate, the expected income \( z \) for unemployed workers, labour supply \( L(z) \), and the aggregate surplus from employment \( (L(z) - u) h(z) \) are all higher than at other equilibria.

### 4 Dynamics: Participation and Reservation Wages

With non-constant returns to market activity the job-finding rate, \( \lambda \), varies over time when the market is not in steady state. The strategies of workers on a dynamic equilibrium path are determined by their expectation of the future path of the job-finding rate. Assume that \( \lambda \) is continuously differentiable with respect to time,\(^{11}\) with a steady-state value \( \lambda^* \). The reservation wage of a worker with alternative income \( q \) satisfies:

\[ z = \max\{q, a(z, \lambda)\} + \frac{\dot{z}}{\rho + \delta} \quad (18) \]

\[ \lim_{t \to \infty} z(q, t) = \max\{q, z^*\} \quad \text{where} \quad z^* = a(z^*, \lambda^*) \quad (19) \]

\(^{10}\)If \( \beta_1 + \beta_2 > 1 + \rho/\delta \), workers and firms have high incentives to search, and as activity rises search costs may increase faster than output.

\(^{11}\)This will be so in equilibrium with a well-behaved matching function and continuous labour supply.
We can integrate (18) to obtain:

\[
z(q,t) = (\rho + \delta) \int_t^\infty \exp[-(\rho + \delta)(\tau - t)] \max\{q, a(z, \lambda)\} \, d\tau
\]  

(20)

From inspection of these equations it is clear that when \( \lambda(t) \neq \lambda^* \), the behaviour of a worker of type \( q \) depends on whether or not he expects to participate in future. There are three possible cases:

- **Permanent Inactivity.** For a worker who expects never to participate, \( z(q,t) = q \).

- **Permanent Attachment.** A worker who expects always to participate has a reservation wage independent of his outside income: \( z(q,t) = z(t) \), where:

\[
z = a(z, \lambda) + \frac{\dot{z}}{\rho + \delta} \quad z \to z^*
\]  

(21)

- **Intermittent Participation.** The current reservation wage is strictly above \( q \), and at the time of future entry (exit):

\[
a(z(q), \lambda) = q, \quad \dot{a} \equiv a_z \dot{z} + a_\lambda \dot{\lambda} \geq 0 \, (< 0), \quad \text{and} \quad \dot{z} > 0 \, (\geq 0)
\]  

(22)

As we would expect, workers with higher outside options participate less, and have higher reservation wages:

**Proposition 4** If \( q_1 > q_2 \):

(i) if type \( q_1 \) participates at time \( t \), then type \( q_2 \) also participates;

(ii) \( z(q_1, t) \geq z(q_2, t) \) with strict inequality unless they are both permanent participants.

**Proof:** See Appendix A.  

Thus, workers' behaviour varies with their degree of attachment to the labour market, which is determined by their alternative income, relative to evolving market opportunities. Those with low outside incomes are permanently attached, and their outside income is irrelevant: they all set the same reservation wage. Workers with higher levels of \( q \) expect to leave the market either permanently or temporarily in the future, and the higher their outside income the lower their attachment – that is, the less they expect to participate. Less attached workers have higher reservation wages; with endogenous search intensity they would also search less intensively. Those with outside income above the steady-state level of market income, \( z^* \), will eventually leave the market for permanent inactivity, with reservation wage equal to outside income.
4.1 Who Participates?

From Proposition 4(i) it follows that there is a critical value \( \hat{q}(t) \) such that all workers with \( q \leq \hat{q}(t) \) participate at time \( t \). The determination of the critical participant \( \hat{q}(t) \) depends in general on the whole evolution of \( \lambda \). For some precise results we focus on the case when \( \lambda \) evolves monotonically towards a steady state. Differentiating (21) with respect to time, noting that \( \dot{z} \to 0 \), and applying the results (22) for entry and exit, it is straightforward to verify:

**Proposition 5** (i) In an improving market (\( \dot{\lambda} > 0 \) for all future \( t \)), \( \dot{z} > 0 \). Sufficiently close to the steady state that \( a(\hat{z}, \lambda) \) is increasing, all participants have reservation wage \( \hat{z} \) and a worker participates if and only if \( a(\hat{z}, \lambda) \geq q \). (ii) In a falling market (\( \dot{\lambda} < 0 \) for all future \( t \)), \( \dot{z} < 0 \), and a worker participates if and only if \( a(q, \lambda) \geq q \).

**Corollary 1** In a neighbourhood of a steady-state, the marginal market participant \( \hat{q} \) is determined by:

\[
\hat{q} = \begin{cases} 
  a(\hat{q}, \lambda) & \text{if } \dot{\lambda} < 0 \\
  a(z, \lambda) & \text{if } \dot{\lambda} > 0
\end{cases}
\]

(23)

where \( z = \hat{z}(t) \), the reservation wage of a permanently attached worker.

These results rely on the convergence of \( \lambda \) and \( \hat{z} \) close to the steady state. Away from the steady state there is no simple characterisation of the marginal participant, because quite complex behaviour can arise. Consider, for example, the case of an improving market, when the job-finding rate is increasing and will continue to do so. Then the reservation wage \( z \) of permanent participants (who have low outside income) is also rising. Agents whose outside income is just below \( \hat{z} \) wait outside the market until it improves further – they would not want to accept a job at only just above \( q \). But if the market enters a period of more rapid improvement, some agents who are already participating may withdraw, enjoy their outside income for a short time, and then re-enter the much-improved market.

5 Stability of Steady-State Equilibria

In section 3 it was shown that if the matching function has increasing returns at some levels of market activity there may be several steady-state equilibria, although for appropriate incentives for vacancy creation there is an upper bound \( \bar{\eta}(L\gamma/u) \) on the degree of increasing returns at equilibrium. In this section we will obtain conditions for dynamic stability. Heterogeneity in reservation wages makes the problem inherently multidimensional. However, we can reduce the dimensionality if we focus on the neighbourhood of a steady state.
From section 4 we know that, close to a steady-state equilibrium when market conditions are improving ($\lambda$ is rising), all participating workers have the same reservation wage. So $z$, $u$, and $\theta$ satisfy:

\begin{align*}
    z &= a(z, \lambda) + \frac{\dot{z}}{\rho + \delta} \quad (24) \\
    \theta &= \left( \frac{\dot{z}}{m_c} \right) a(z, \lambda) \quad (25) \\
    \text{where } \lambda &= \phi(m)\mu(\theta) \quad \text{and } m = u \mu(\theta) \quad (26)
\end{align*}

When $\lambda$ is falling, there are some workers who will leave before the steady state is reached and who therefore have different reservation wages. The presence of these workers affects the average surplus $\bar{S}(z)$ which determines $\theta$ through the incentive to create vacancies (equation (8)). But in the neighbourhood of the steady state, the difference between $\bar{S}(z)$ and the corresponding value $S(z)$ for permanently attached workers is of second order. Ignoring this difference, the equations above are also satisfied in a falling market by $u$, $\theta$, and the value of $z$ for permanently attached workers.

Non-employed workers participate if and only if they have alternative income less than $\hat{q}$ determined by (23). Unemployment is the number of non-employed workers with outside income less than $\hat{q}$:

$$u = N(\hat{q}(t), t) \quad \Rightarrow \quad \dot{u} = n(\hat{q})\dot{\hat{q}} + \frac{\partial N}{\partial t} \bigg|_{\hat{q}}$$

Evaluating $\partial N/\partial t$ using (12) gives:

$$\dot{u} = n(\hat{q})\dot{\hat{q}} + \delta(L(\hat{q}) - u) - \Phi(m)\pi(z) \quad (27)$$

In the neighbourhood of a steady state, market variables $z$, $\theta$ and $u$ satisfy equations (24), (25) and (27). Note from (17) that the partial derivative $n(\hat{q})$ is discontinuous at $z$. But in any case it is bounded: $0 \leq n(\hat{q}, t) \leq L'(\hat{q})$, since the number of non-employed workers with outside income in any interval cannot exceed the total number of workers in this interval.

### 5.1 Local Dynamics

To analyse local dynamics, the system of equations (23) to (27) can be reduced to a 2-dimensional system in the reservation wage $z$ of (almost all) participating workers, and market activity $m$.

**Lemma 2** In the neighbourhood of a steady state, equations (23), (25) and (26) determine
\( \theta, u, \lambda \) and \( \hat{q} \) as implicit functions of \( z \) and \( m \):

\[
\begin{align*}
\theta &= \Theta(z, m), \quad \Theta_z < 0, \quad \Theta_m \overset{\text{sgn}}{=} \eta - 1; \\
u &= U(z, m), \quad U_z > 0, \quad U_m \overset{\text{sgn}}{=} 1 - \eta \alpha; \\
\lambda &= \Lambda(z, m), \quad \Lambda_z < 0, \quad \Lambda_m \overset{\text{sgn}}{=} \eta - 1; \\
\hat{q} &= Q(z, m), \quad Q_z < 0, \quad Q_m \overset{\text{sgn}}{=} \eta - 1.
\end{align*}
\]

The proof, including expressions for the elasticities, is given in Appendix A. Equations (24) and (27), for the dynamics of \( z \) and \( u \), can now be written in terms of \( z \) and \( m \):

\[
\begin{align*}
\dot{z} &= F_1(z, m) \\
(U_m - n(\hat{q})Q_m) \dot{m} &= (n(\hat{q})Q_z - U_z) \dot{z} + F_2(z, m)
\end{align*}
\]

where

\[
\begin{align*}
F_1 &= (\rho + \delta)(z - a(z, \lambda)), \\
F_2 &= \delta(L(\hat{q}) - u) - \Phi(m)\pi(z),
\end{align*}
\]

and \( \hat{q} = Q(z, m), \quad \lambda = \Lambda(z, m), \quad u = U(z, m) \)

Linearising in the neighbourhood of the steady state, we can see that the stability of the system depends on the matrix:

\[
\begin{pmatrix}
F_{1z} & F_{1m} \\
F_{2z} + (nQ_z - U_z)F_{1z} & F_{2m} + (nQ_z - U_z)F_{1m} \\
U_m - nQ_m & U_m - nQ_m
\end{pmatrix}
\]

in which all functions are evaluated at the steady state (where \( \hat{q} = z \)). The derivatives of \( F_1 \) and \( F_2 \) can be evaluated using the elasticities in the proof of Lemma 2; in particular, note that \( F_{1z} > 0 \) and \( F_{1m} \overset{\text{sgn}}{=} 1 - \eta \). The system is saddle-path stable if and only if the determinant:

\[\Delta = \frac{F_{1z}F_{2m} - F_{1m}F_{2z}}{U_m - nQ_m}\]

is negative. It can be shown that if \( \eta < \bar{\eta}(L_\gamma/u) \) (stability with respect to vacancy creation) the denominator of \( \Delta \) is positive, and the numerator is negative,\(^{12}\) so the equilibrium is saddle-path stable. Moreover the direction of slope of the saddle path depends on the slope of \( F_1 = 0 \), and therefore on returns to scale. In summary:

**Proposition 6** A steady-state equilibrium that is stable with respect to vacancy creation is locally saddle-path stable. The saddle path is downward-sloping in \( m \)-\( z \) space if \( \eta < 1 \),

\(^{12}\)The numerator has the same sign as \( \zeta_1' - \zeta_2' \); this is not surprising, since \( F_1 = 0 \) and \( F_2 = 0 \) are locally identical to \( z = \zeta_1 \) and \( z = \zeta_2 \).
flat if \( \eta = 1 \), and upward-sloping if \( \eta > 1 \).

**Proof:** See Appendix A.

Combining this result with those of section 3, we can summarise the properties of the equilibria that can arise in a market where the matching function does not have everywhere constant returns to scale. Numbering them in order of decreasing market size, we can rule out even-numbered equilibria because they are not robust (unstable) with respect to incentives for vacancy creation. Odd-numbered equilibria are stable in this sense provided that returns to scale are not too high and labour supply is not too elastic \( (\eta < \bar{\eta}(L\gamma/u)) \), in which case they are also dynamically saddle-path stable. The equilibrium with the largest market is welfare-superior to others, and is stable if \( \eta < \bar{\eta}(L\gamma/u) \). Figure 2 shows the same multiple equilibria that were illustrated in Figure 1. The welfare-superior equilibrium (1) has decreasing returns to scale, so it is stable, with a downward-sloping saddle path; (2) is unstable; (3) has increasing returns, so it is stable if and only if \( \eta < \bar{\eta}(L\gamma/u) \), in which case the saddle path slopes upwards.

In the neighbourhood of a stable steady state, the market converges along the saddle path at rate \( \nu \), where \( \nu \) is the negative eigenvalue of the matrix above, satisfying:

\[
\nu^2 - \nu X + \Delta = 0 \quad \text{where} \quad X = F_{1z} + \frac{F_{2m} + (nQ_z - U_z)F_{1m}}{U_m - nQ_m}
\]

(28)
Compared with the standard constant returns case, the speed of adjustment $|\nu|$ is higher under increasing returns and lower under decreasing returns:

**Proposition 7** In the neighbourhood of a steady state $(z, \theta, u)$, the speed of market adjustment $|\nu|$ increases with the returns to scale parameter $\eta$: \( \frac{\partial |\nu|}{\partial \eta} > 0 \), and:

\[
|\nu| \begin{cases} 
\geq \delta + \lambda \pi & \text{for } \eta \geq \delta \\
< \delta + \lambda \pi & \text{for } \eta < \delta 
\end{cases}
\]

**Proof:** See Appendix A.

5.2 Decreasing returns and the response to shocks

In this section we demonstrate that decreasing returns in matching imply plausible dynamic variation in labour market variables in the response of the economy to a shock. Market size matters: as the level of activity $m$ adjusts towards the steady state the efficiency of matching changes, and workers respond by changing their reservation value $z$, and may enter or leave that market. Following a shock, both $z$ and $m$ may jump. The expectational variable $z$ can jump to any value, to ensure that the market moves onto the saddle path. Market activity $m$ can change instantaneously as workers move between unemployment and non-participation, but the size of the jump is constrained because some workers are employed.

Suppose, for example, that the market is at equilibrium (1) in Figure 2, and an unexpected productivity shock destroys some jobs so that unemployment and hence market activity are too high, and matching is less efficient. At the steady-state reservation wage the job-finding rate $\lambda$ falls. Workers lower their reservation wages instantaneously, and those who have the highest outside income leave the market, reducing $u$ instantaneously. The more elastic is labour supply, the greater this initial jump. But (unless labour supply is perfectly elastic) unemployment is still above its steady-state level, so firms create more vacancies and activity $m$ is also high. Then market activity falls gradually as unemployed workers find jobs, raising the effectiveness of matching; workers gradually raise their reservation wages, and inactive workers re-enter, until the steady state is restored. Using Lemma 2 it can be verified that as the market returns to the steady state along the saddle path:

- Unemployment $u$ falls.
- Vacancies fall, but market tightness rises.
- Participation $L(\hat{q})$ rises, and the unemployment rate $u/L$ falls.
These dynamic effects provide an alternative interpretation of the dynamic behaviour of unemployment and the job-finding rate documented by Shimer (2005) for US data. With constant returns, movements in the job-finding rate can only be interpreted as shifts of the steady state, but with decreasing returns these two variables move in opposite directions during periods of dynamic adjustment. They also suggest a different perspective on the results of Hall (2005). He estimates the “equilibrium” unemployment rate at time $t$ by $u_t^* = \delta / (\delta + \lambda_t)$ where $\lambda_t$ is the current job-finding rate (assuming inelastic labour supply and $\pi = 1$). With constant returns to scale in matching, $u_t^*$ is the current steady state, so his finding that actual unemployment $u_t$ closely tracks $u_t^*$ can be interpreted as suggesting that – because the inflow and outflow rates are high – dynamic adjustment is irrelevant. But with decreasing returns, $\lambda$ is not constant, and $u_t^*$ is not a steady state. If a shock raises unemployment above its steady-state level, it also reduces $\lambda$ and raises $u_t^*$. In the return to the steady-state, $u_t$ and $u_t^*$ fall together. This provides some intuition for the result in Proposition 7, that decreasing returns slow convergence: it does so because the “short-run equilibrium” $u_t^*$ rises and falls with level of unemployment.\textsuperscript{13}

The speed of adjustment is also affected by elastic labour supply. The effect is asymmetric – it depends on whether the market is rising (that is, reservation values $z$ are increasing) or falling towards the steady state, because workers behave differently in these two cases:

**Proposition 8**  
In the neighbourhood of a steady state $(z, \theta, u)$ with decreasing returns to scale ($\eta < 1$), the speed of market adjustment $|\nu|$ lies between $\delta$ and $\delta + \lambda \pi$.

1. When the market is rising ($\dot{z} > 0$) $\frac{\partial |\nu|}{\partial \gamma} > 0$ and $\lim_{\gamma \to \infty} |\nu| = \delta + \lambda \pi$

2. When the market is falling ($\dot{z} < 0$) $\frac{\partial |\nu|}{\partial \gamma} < 0$ and $\lim_{\gamma \to \infty} |\nu| = \delta$

**PROOF:** See Appendix A.

The intuition for this result is that with decreasing returns, market activity in the rising market is higher than at the steady state, and hence the average matching rate is lower. The withdrawal of some workers to inactivity mitigates this problem and accelerates the return to the steady state. Conversely in the case of a falling market, the average matching rate is high but adjustment is hindered by the participation of additional workers who will eventually withdraw.

\textsuperscript{13}Note, however, that a problem highlighted by Shimer and Hall remains: although market tightness and unemployment move in opposite directions, short-run dynamics cannot explain the strong negative correlation of unemployment and vacancies. Their suggested explanation of sticky wages would also help here. If wages are sticky, a rise in unemployment does not raise the firm’s share of the surplus, so firms do not respond by creating more vacancies.
Lastly, the fact that participation changes has implications for the speed of adjustment of employment, which may now be different from that of unemployment. Employment $e$ satisfies:

$$\dot{e} + \delta e = \Phi(m)\pi(z)$$

Both $m$ and $z$ converge at rate $\nu$. Linearising the right-hand side in the neighbourhood of the steady state, and integrating, we obtain:

$$e = e^* + (e_0 - e^*) \left[ k \exp(-|\nu|t) + (1 - k) \exp(-\delta t) \right]$$

where $e_0$ is employment at $t=0$, and $e^*$ is steady-state employment. It can be verified that $0 \leq k \leq 1$. When labour supply is fixed, $k = 1$ and employment converges at rate $\nu$. Conversely if labour supply is perfectly elastic, $k = 0$ and employment converges only at rate $\delta$ – this can be seen directly from the differential equation, since in this case $m$ and $z$ jump back instantaneously to their steady-state values following a shock, so the right-hand side is constant.

![Figure 3: Adjustment of employment and unemployment following a negative employment shock](image)

In general when workers respond to the changing conditions in the labour market by moving between unemployment and inactivity, this has the effect of slowing the recovery of employment. Suppose employment is below its steady state level. If non-employed workers remain in the market, firms create additional vacancies and the aggregate matching rate rises until employment recovers. But if workers withdraw, this doesn’t happen, and non-employment remains high for longer.
Figure 3 shows the effect of decreasing returns and inactivity on the convergence of employment and unemployment to their steady state levels following a shock which instantaneously destroys some jobs. The initial rise in unemployment is lower, because some workers withdraw; convergence of unemployment is slower because of decreasing returns, and employment recovers still more slowly because firms create fewer vacancies while workers are not participating.

5.3 Global Dynamics

With multiple rational expectations equilibria, agents face a coordination problem. In addition to the local stability condition, stability of a steady-state equilibrium requires that all agents expect the market to return to this steady state. Where there are several stable steady states, it is possible that any one of them can be reached from a given initial position, if agents believe that it will be. We demonstrate this property by considering a simple special case of the model.

Suppose that every match has the same productivity \( y > 0 \). There is a mass normalised to one of workers with outside income zero, who always participate, and an equal mass with outside income \( q_H \in (0, y) \):

\[
L(q) = \begin{cases} 
1 & \text{if } 0 \leq q < q_H \\
2 & \text{if } q \geq q_H
\end{cases}
\]

Matches are always consummated and the expected surplus for a worker with reservation wage \( z \) is \( S(z) = y - z \). Suppose that the average matching rate is either high or low depending on the level of market activity:

\[
\Phi(m) = \begin{cases} 
\phi_\ell m & \text{if } m \leq \tilde{m} \\
\phi_h m & \text{otherwise}
\end{cases}
\]

where \( \phi_\ell < \phi_h \)

It is straightforward to show that for a given value of the average matching rate \( \phi \), the steady-state equations (13) to (16) have a solution:

\[
z = z(\phi), \quad \theta = \theta(\phi), \quad \lambda = \phi \mu(\theta), \quad u = \frac{L(z)\delta}{\delta + \lambda}, \quad m = u\mu(\theta)
\]

So we have two possible stable steady states \((z_i, \theta_i, u_i)\) corresponding to the two levels of the average matching rate \( \phi_i \); an equilibrium with a high (low) level of \( \phi \) exists if the corresponding activity level \( m \) is above (below) \( \tilde{m} \). The equilibrium with higher average matching rate has higher income, tightness, participation and activity, and lower
unemployment rate \( u/L \). If:

\[
0 < z_t < q_H < z_h < y \quad \text{and} \quad \frac{\delta \mu(\theta_t)}{\delta + \lambda_t} < \tilde{m} < \frac{2\delta \mu(\theta_h)}{\delta + \lambda_h}
\]

we have multiple equilibria: a Pareto superior one in which all workers participate, and a second with low activity in which only low-types (with low outside income) participate.

![Figure 4: High and low activity equilibria and saddle paths](image)

Since the matching function has constant returns to scale within the regions of high and low activity, the saddle paths for both equilibria are horizontal in \( m-z \) space (see Figure 4). The market may be able to jump straight onto either saddle path, with constant worker income and market tightness, and converge to the steady state as employment and hence market activity evolves. Whether this is possible from an arbitrary starting point depends on how many workers are initially employed – which determines whether market activity can instantaneously adjust to a level within the same region as the steady state.

The path of the economy from any initial position depends entirely on the expectations of workers. It can reach either of the two steady states (even if the horizontal saddle path is not reached immediately) and there are many equilibrium paths. To illustrate some of the possibilities, consider a “new” market, starting from zero employment. The following equilibrium paths are shown in Figure 5:

a) All workers enter immediately, and the market converges to equilibrium on the high saddle path.
b) Only the low-types enter, and the market converges to the low equilibrium along the low saddle path. This can only happen if $\phi_\ell$ is sufficiently low that $\mu(\theta_\ell) \leq \tilde{m}$, so that even when all the low-types are searching the activity level is below $\tilde{m}$.

c) All workers enter, and the average matching rate is initially high, but the market converges to the low equilibrium. Since workers know that the average matching rate will fall their reservation wages are lower than $z_h$ (so market tightness is initially higher than $\theta_h$). Unemployment falls, and eventually all the high-types leave the market together, permanently, and the low saddle path is reached. Exit of the high-types cannot occur until the number of employed low-types is sufficiently high (above $\tilde{e} \equiv 1 - \tilde{m}/\mu(\theta_\ell)$) that market activity below $\tilde{m}$ falls when the high-types leave.

![Equilibrium paths for a new market](image)

Figure 5: Equilibrium paths for a new market

On the paths shown in Figure 5 the high types leave the market at most once; there are others on which the market alternates between high and low activity – possibly indefinitely.
6 Efficiency

It is well known that with constant returns to scale in matching, the decentralised matching market is efficient (subject to matching frictions) under the Hosios condition: when the surplus shares $\beta_1$ and $\beta_2$ are equal to the elasticities of matching with respect to unemployment and vacancies. This gives individual agents the right incentives to ensure that search externalities are internalised, and hence maximise total output net of search costs. Thus it is possible, in principle, for a market in which wages are determined by ex-post bargaining between worker and firm to achieve efficiency, although in practice there is no reason to expect the condition to be satisfied.

In this section we determine the social planner’s optimum for the matching model with non-constant returns and derive a simple generalisation of the Hosios condition.

6.1 The Social Planner’s Problem

Consider a social planner who wishes to maximise the present value of aggregate net income, continuously determining the number of vacancies (or equivalently market tightness), and for each type of worker the participation decision and reservation productivity level for match consummation. The planner is subject to the frictions imposed by the matching technology, which determine how the distribution of workers across the three labour market states evolves over time. The objective is to maximise:

$$
\int_0^\infty \exp(-\rho t) \left[ Y - \theta c \int pn dq + \int (1 - p)qn dq \right] dt
$$

where $Y$ is the total instantaneous output of current matches. The second and third terms are total vacancy costs, and the income of inactive workers, respectively. The current states of workers and total output cannot be directly controlled by the planner; from (12) $n(q)$ evolves according to:

$$
\dot{n} = \delta(L' - n) - \lambda \rho \pi(z)n
$$

and the path of current output is given by:

$$
\dot{Y} = \lambda \int p\pi(z) [z + h(z)] n dq - \delta Y
$$
6.2 First-order Necessary Conditions for Optimality

Proposition 9  On a socially optimal dynamic path, participation \( p(\cdot) \), reservation productivity \( z(\cdot) \) and market tightness \( \theta \) satisfy:

\[
p(q) = \begin{cases} 
1 & \text{if } a^*(z(q), \lambda) \geq q \\
0 & \text{otherwise}
\end{cases} \tag{29}
\]

\[
\frac{\dot{z}}{\rho + \delta} = \max \{q, a^*(z, \lambda)\} \tag{30}
\]

where \( a^*(z, \lambda) = \lambda \left( \eta_u S(z) + (1 - \eta_u)[S(z) - \bar{S}] \right) \tag{31} \)

\[
\theta_c = \eta_v \lambda \tilde{S} \tag{32}
\]

PROOF: By the Principal of Optimality, the social planner’s value function \( \Omega(Y, n(\cdot)) \) satisfies the Bellman equation:

\[
\rho \Omega(Y, n(\cdot)) = \max_{\theta, p(\cdot), z(\cdot)} \left\{ Y - \theta c \int p n \, dq + \int (1 - p) q n \, dq \right. \\
+ \left. \Omega_Y \dot{Y} + \int \Omega_{n(q)} \dot{n} \, dq \right\}
\]

We can write down first-order conditions with respect to \( \theta, p(\cdot) \) and \( z(\cdot) \), together with envelope conditions for \( Y \) and \( n(\cdot) \). Starting with the first-order condition for \( z(\cdot) \) and the envelope condition for \( Y \) we have:

\[
0 = p(q) \left( \Omega_Y z(q) - \Omega_{n(q)} \right) \tag{33}
\]

\[
\rho \Omega_Y = 1 - \delta \Omega_Y + \dot{\Omega}_Y \tag{34}
\]

Since \( \Omega_Y \) is bounded,\(^{16} \) the solutions are:\(^{17} \)

\[
\Omega_Y = \frac{1}{\rho + \delta} \quad \text{and} \quad \Omega_{n(q)} = \frac{z(q)}{\rho + \delta} \tag{35}
\]

---

\(^{14}\)Since \( \Omega \) is a function of \( Y \), \( \Omega_Y \) denotes \( \partial \Omega / \partial Y \) as usual; however, \( \Omega \) is a functional of \( n(\cdot) \), and we write \( \Omega_{n(q)} \) for the partial derivative of \( \Omega \) with respect to \( n \) at \( q \) in the sense of Volterra. Furthermore, since we are maximising the objective with respect to the functions \( p(\cdot) \) and \( z(\cdot) \), the first-order conditions for them will also be in terms of Volterra derivatives. (See Appendix B.)

\(^{15}\)Observe that \( \frac{\partial}{\partial Y} \Omega_Y \dot{Y} + \frac{\partial}{\partial n} \int \Omega_{n(q)} \dot{n} \, dq = \Omega_Y Y \dot{Y} + \int \Omega_{n(q)} \dot{n} \, dq = \dot{\Omega}_Y. \)

\(^{16}\)The additional value, \( d\Omega \), of an increase in output, \( dY \), lies between 0 and \( dY / \rho \), i.e. what the SP gets if the match that produced the extra output is destroyed immediately, and what she gets if it is never destroyed; so \( 0 \leq \Omega_Y \leq 1/\rho. \)

\(^{17}\)The reservation value \( z(q) \) is indeterminate when \( p(q) = 0 \); however in that case we can simply set it equal to \( (\rho + \delta) \Omega_{n(q)}. \)
These are the social marginal values of a unit of output, and an unemployed worker of type \( q \), respectively. The first-order condition with respect to \( \theta \) is:

\[
u \theta c = \eta_m \lambda \int p \pi(z) \left[ \Omega_Y[z + h(z)] - \Omega_{n(q)} \right] n \, dq \]

which, using (33) and (11), gives us condition (32).

The envelope condition for \( n(\cdot) \) is:

\[
\rho \Omega_{n(q)} = (1 - p(q))q + p(q) \left\{ -\theta c + \lambda[S(z) - \bar{S}] + \eta \lambda \bar{S} \right\} - \delta \Omega_{n(q)} + \dot{\Omega}_{n(q)}
\]

and eliminating \( \Omega_{n(q)} \) and \( \theta \) using (35) and (32), we obtain:

\[
z = (1 - p) q + p a^*(z, \lambda) + \frac{\dot{z}}{\rho + \delta}
\]

where \( a^*(z, \lambda) \) is given by (31).

Finally, consider the choice of \( p(\cdot) \). The derivative of the maximand in the Bellman equation with respect to \( p \) at \( q \), evaluated using the first-order conditions above, is:

\[
n(q) \left( a^*(z(q), \lambda) - q \right)
\]

Hence the social planner chooses \( p(q) \) according to (29), and equation (36) for \( z(q) \) can be written in the form (30).

\section*{6.3 The Steady State and Second-order Conditions}

Inspection of the optimality conditions (29) to (32) shows, by the same argument as for the decentralised equilibrium, that in a steady state all participating workers have the same reservation productivity \( z \). Thus in an optimal steady state \( z, \theta \) and \( u \) satisfy:

\[
\begin{align*}
z &= \lambda \eta_w S(z) \\
\theta c &= \lambda \eta_w S(z) \\
\Phi(m) \pi(z) &= \delta(L(z) - u)
\end{align*}
\]

These equations are clearly very similar to the equations for the decentralised steady state: they differ only in that the surplus shares are replaced by the elasticities of matching with respect to unemployment and vacancies. It appears that a generalisation of the
Hosios condition holds. But we should first consider whether a solution to the first-order conditions exists, and whether the first-order conditions are sufficient for a social optimum.

### 6.3.1 Existence

It can be shown in the same way as for the decentralised case that a solution to the social planner’s first-order conditions exists provided that $M_v(1,0)$ is sufficiently large. The only substantive difference is that in the analogue of Lemma 1 $z$ depends on market size through the marginal matching rate $\Phi'(m)$, rather than the average $\Phi(m)/m$. It is also useful to note that the crossing condition in this analysis implies that there can be at most one solution in any concave region of the matching function $\Phi(m)$.

### 6.3.2 Second-order Conditions

As well as satisfying the first-order conditions, an optimal choice of $\theta$, $p(\cdot)$ and $z(\cdot)$ must correspond to a local maximum of instantaneous welfare.

With $\delta f$ denoting a differential change in a function $f$, we write $E[\delta f]$ for $\frac{1}{\delta} \int p m \delta f(q) dq$ and similarly for $\text{Var}[\delta f]$. We decompose the relevant differential changes in $\theta$, $p(\cdot)$ (whose changes are felt via changes in $u$) and $z(\cdot)$ into the vector $d_E$, and $d_V$, given by:

$$
\begin{align*}
  d_E' &= \begin{pmatrix} d\theta & du & E[\delta z] \end{pmatrix} \\
  d_V &= \text{Var}[\delta z]
\end{align*}
$$

The second-order differential change in the objective $\Omega$, at a steady-state solution of the first-order conditions, is then directly proportional to $d_E' H_E d_E - d_V^2$, where the symmetric matrix $H_E$ is given by:

$$
H_E = \begin{pmatrix}
  \frac{\alpha}{\sigma^2} \left( \frac{\alpha m \Phi''}{\Phi'} - \frac{1-\alpha}{\sigma} \right) & \frac{\alpha}{\sigma u} m \Phi'' & 0 \\
  \frac{1}{\sigma^2} & \frac{1}{\eta^2} \frac{m \Phi''}{\Phi'} & 0 \\
  \frac{1}{\eta} \frac{\pi'}{\pi h} & \frac{1}{\eta} \frac{\pi'}{\pi h} & \frac{1}{\eta} \frac{\pi'}{\pi h}
\end{pmatrix}
$$

and

$$
H_V = \frac{1}{\eta} \left| \frac{\pi'}{\pi h} \right|
$$

respectively, and $\sigma$ is the elasticity of substitution between unemployment and vacancies.

If $H_E$ is negative definite, then this change is negative and we have a local maximum. On the other hand, if $H_E$ is not negative definite, then there is a vector of changes that makes $d_E' H_E d_E$ non-negative and at the same time has $d_V = 0$, and so the second-order effect on $\Omega$ is non-negative.

---

18 $du = dN(\hat{q})$
If $L'(z) > 0$, it can be verified that $H_E$ is negative definite if and only if $\Phi''(m) < 0$, giving necessary and sufficient conditions for a strict local maximum.\(^{19}\) With an upward-sloping supply of workers to the market, it can never be optimal to be on a convex part of the matching function – since by bringing another worker into the market the social planner raises activity and the marginal matching rate, and hence increases income $z$ for all the workers in the market.

When labour supply is inelastic, $du = 0$. In this special case, the relevant matrix (the sub-matrix corresponding to changes in $\theta$ and $z$) is negative definite if and only if $m\Phi''/\Phi' < (1 - \alpha)/(\alpha\sigma)$. This condition can be interpreted as saying that if, at an optimum, raising activity would increase the matching rate, the effect must be small enough that the gain for workers is outweighed by the increased vacancy costs.

### 6.3.3 Steady State Welfare

Evaluating total welfare at a steady-state solution of the first-order conditions gives:

$$
\rho \Omega = \int \max\{z, q\} dL(q) + (L - u)h(z) \left(1 - \eta \frac{\delta}{\rho + \delta}\right)
$$

Equivalently the net social surplus from the market – the difference between $\rho \Omega$ and total income when all workers are inactive – is:

$$
\Delta(\rho \Omega) = \int z(z - q) dL(q) + (L(z) - u)h(z) \left(1 - \eta \frac{\delta}{\rho + \delta}\right)
$$

(40)

It follows that if there is a solution to the first-order conditions in a region of non-increasing returns, the social planner can achieve a positive social surplus. A positive surplus can occur at a solution with increasing returns, but if $\eta > 1 + \rho/\delta$ this requires the total rents to outweigh losses from employment.

### 6.4 Necessary and Sufficient Conditions for Optimality

Putting together the results above we can now state our main efficiency result:

**Proposition 10** A steady-state allocation is a local welfare optimum if and only if

(i) $z, \theta$ and $u$ satisfy the social planner’s first-order conditions (30) to (32), and

(ii) $\Phi''(m) < 0$ (or $\frac{m\Phi''}{\Phi'} < \frac{1 - \alpha}{\alpha\sigma}$ if labour supply is inelastic).

\(^{19}\)If $\Phi'' = 0$, we have a maximum with respect to $\theta$ and $z$, but the second-order effect of changing $u$ is zero.
Corollary 2 (Generalised Hosios Condition) A decentralised steady state equilibrium with variable labour supply is a local welfare optimum if and only if \( \Phi''(m) < 0 \) and \( \beta_1 = \eta_u, \beta_2 = \eta_v \).

We can also conclude that if the matching function \( \Phi(m) \) has only one concave region, a local optimum is the unique global optimum provided that the net surplus (40) is positive.

Corollary 2 means that efficiency can in principle be achieved by a policy maker who sets the surplus shares of workers and firms, as in the standard constant returns case. The difference is that with non-constant returns a tax or subsidy is required: the surplus shares do not sum to one. If the optimum has decreasing returns to scale, the surplus must be taxed in order to give agents appropriate search incentives. If the optimum has increasing returns the policy maker must use a subsidy. As in section 3.3 note that this can be financed within the model: at a social optimum with increasing returns and a positive net social surplus (40), it will always be possible to finance the required subsidy from the rents earned by workers – using a lump-sum tax.

The generalisation of the Hosios condition is quite intuitive. With constant returns to scale in matching, search externalities arise when the relative number of agents – market tightness – is suboptimal, and efficiency can be achieved by manipulating the relative return to search. With non-constant returns there are also market size effects: additional search externalities that arise when the absolute number of agents is not optimal. To achieve efficiency in this case, it is necessary to control both the relative and absolute return to search.

6.4.1 Efficiency away from the steady state

With everywhere-constant returns, the Hosios condition is necessary and sufficient for efficiency not only at the steady-state, but everywhere on the equilibrium path. With a more general matching function this is not the case. Comparing the social planner’s first-order conditions in Proposition 9 with the dynamic equations for individuals in section 2.4, we can see that, while setting \( \beta_2 = \eta_v \) allows firms to behave optimally, \( \beta_1 = \eta_u \) is not sufficient for optimal worker behaviour when workers are heterogeneous. There is an additional external effect from a worker who has a different reservation wage from the average in the market. His reservation wage changes the average surplus \( \bar{S} \); so it changes the incentives facing firms, affecting market tightness and hence the matching rate for other workers. For example, consider a worker who has high enough outside income that he will leave the market in future. His reservation wage is above average, so his expected surplus is below average. From (31) we can see that the social planner would lower his
reservation wage, to mitigate his negative effect on market tightness and the aggregate matching rate. This result is similar to that of Shimer and Smith (2001) who show that the Hosios condition is not sufficient for efficiency with ex-ante heterogeneous agents who have different reservation productivities.

6.5 Interpretation: Search Externalities

Our efficiency analysis, as well as providing a neat generalisation of the Hosios condition, gives some insight into the different sources of inefficiency in random matching models. Agents’ decisions to search affect the returns to other agents by changing market tightness or market size (both of which can affect individual matching rates) or the expected surplus from a match. In a model with constant returns and homogeneous agents only the first of these has an external effect, and it can be offset by an appropriate sharing rule. Non-constant returns introduces a market size externality, which can be internalised using a uniform employment tax or subsidy. Heterogeneity of reservation values means that an agent’s decision to search changes the expected surplus for agents on the other side of the market; this externality could be offset only with agent-specific taxes or subsidies.

Conclusions

The widespread theoretical assumption of constant returns to scale in labour market matching is convenient, but is not well supported by empirical evidence and has caused the dynamics of matching models to be ignored. Without it, market size matters – the average matching rate increases or decreases with market size depending on returns to scale. Dynamic effects can then be taken seriously in the search for explanations of the evolution of labour market variables.

The generalisation of the random matching model developed here demonstrates the implications of market size effects. Matching markets can have stable decentralised equilibria with either decreasing or increasing returns. When there are multiple equilibria, larger markets deliver higher welfare, but pessimistic expectations can lead to a low activity equilibrium. Decreasing returns induce slow adjustment and plausible dynamic variation in participation, unemployment and job-finding. Finally, a generalisation of the Hosios condition for efficiency applies: setting the surplus shares of workers and firms equal to the elasticities of matching with respect to unemployment and vacancies achieves not only optimal market tightness, but also optimal market size.
Appendix

A Various Proofs

Proof of Lemma 1: \[ \theta = \theta_1(z) \equiv \left( \frac{\beta_2}{\beta_1} \right) z \] follows immediately from (13) and (14). Substituting for \( \theta \), equation (14) can be written in terms of \( z \) and \( m \) only:

\[ \phi(m) = \frac{c}{\beta_2 S(z)} \frac{\theta_1(z)}{\mu(\theta_1(z))} \]  

(A.1)

The right-hand side is a continuous, differentiable and increasing function of \( z \), with limits \( \phi_0 \equiv c/\beta_2 S(0) \mu'(0) \) at 0 and \( \infty \) at \( \bar{y} \). Hence \( \forall m \) such that \( \phi(m) \geq \phi_0 \) this equation has a solution \( z = \zeta_1(m) \in [0, \bar{y}] \) as required. Differentiating (A.1):

\[ \frac{m}{z} \frac{d\zeta_1}{dm} \left( \frac{z}{h} + 1 - \alpha \right) = \eta(m) - 1 \]  

(A.2)

Now consider the flow condition (15). Substituting \( u = m/\mu \) and \( \theta = \theta_1(z) \):

\[ \left( \delta L(z) - \Phi(m) \pi(z) \right) \mu(\theta_1(z)) = \delta m \]  

(A.3)

This determines \( z = \zeta_2(m) \), with \( \zeta_2(0) = 0 \), and derivative:

\[ \frac{m}{z} \frac{d\zeta_2}{dm} \left( \gamma L + u\alpha - (L - u)z \mu'(z) \right) = u + (L - u)\eta \]  

(A.4)

\( \zeta_2(m) \) is strictly increasing, reaching a maximum value \( \bar{y} \) at a finite value \( \bar{m} \).

Proof of Proposition 1: An equilibrium is a solution of \( \zeta_2(m) = \zeta_1(m) \). If we define \( \zeta_1(m) = 0 \) when \( \phi(m) < \phi_0 \), then both \( \zeta_2 \) and \( \zeta_1 \) are continuous on \([0, \bar{m}]\), and \( \zeta_1(\bar{m}) < \zeta_2(\bar{m}) = \bar{y} \), so a sufficient condition for a solution to exist is \( \zeta_1(0) > \zeta_2(0) = 0 \); that is, the average matching rate at 0 is greater than \( \phi_0 \):

\[ \lim_{m \to 0} \phi(m) > \phi_0 \]

\[ \Rightarrow M_\nu(1,0) = \Phi'(0) \mu'(0) > \frac{c}{\beta_2 S(0)} \]

If this condition holds, we have at least one solution \((z, m)\) with \( z > 0 \) and \( m > 0 \); corresponding equilibrium values of the other variables are \( \theta = \theta_1(z) \) and \( u = m/\mu(\theta) \). Write \( Y \equiv 1 + \gamma - \eta(\alpha + \gamma) \). If \( \eta(m) \leq \bar{\eta}(\gamma) \), then \( Y \geq 0 \). At any equilibrium, from (A.4)
Suppose that

$$
\frac{d\zeta_2}{dm} \frac{d\zeta_1}{dm} = u \left( \frac{z}{h} + Y \right) + (L - u) \left( \eta \left( \frac{z}{h} + 1 - \alpha \right) + (1 - \eta) \left( \gamma - \frac{z\pi'}{\pi} \right) \right) \text{ (A.5)}
$$

$$
\frac{d\zeta_1}{dm} = L \left( \frac{z}{h} + Y \right) - (L - u)(1 - \eta) \left( 1 - \frac{zh'}{h} \right) \text{ (A.6)}
$$

Since \( \pi' < 0 \), and \( zh'/h < 1 \) by assumption, this is positive if \( \eta \leq 1 \) from (A.5), and if

$$
1 < \eta < \bar{\eta}(\gamma) \text{ from (A.6)}. \quad \text{So if } \eta(m) \leq \bar{\eta}(\gamma) \text{ for all } m, \zeta_1 \text{ and } \zeta_2 \text{ can only cross once, and the equilibrium is unique.}
$$

**Proof of Proposition 3:** Most of the argument is given in the text; it remains to prove that \( \Phi(m)\pi(z)h(z) \) is increasing in \( m \) between equilibria. From the proof of Proposition 1, equilibrium values of \( z \) and \( m \) satisfy \( z = \zeta_1(m) \) and:

$$
\frac{d}{dm} \left( \Phi(m)\pi(z)h(z) \right) \overset{\text{sgn}}{=} \frac{d\zeta_1}{dm} = \frac{m\Phi'}{\Phi} + \frac{(\pi h')'}{\pi h} \frac{d\zeta_1}{dm} = \eta(m) - \frac{m d\zeta_1}{h dm}
$$

which is strictly positive from (A.2).

**Proof of Proposition 4:** From (18):

$$
z(q_1) - z(q_2) = Q(t) + \frac{1}{\rho z} (z(q_1) - \dot{z}(q_2)) \text{ (A.7)}
$$

where \( Q \equiv \max\{q_1, a(z(q_1))\} - \max\{q_2, a(z(q_2))\} \).

1. Suppose that \( z(q_1) < z(q_2) \) for some \( t \). Then \( Q > 0 \), so \( \dot{z}(q_1) - \dot{z}(q_2) < 0 \), and \( z(q_1) < z(q_2) \) for all future \( t \). But this is impossible since from (19) \( z(q_1) \geq z(q_2) \) in the limit. Hence we must have \( z(q_1) \geq z(q_2) \) for all \( t \).

2. If worker \( q_1 \) is participating at time \( t \), \( q_1 \leq a(z(q_1)) \). Then from 1 (above), \( a(z(q_2)) \geq a(z(q_1)) \geq q_1 > q_2 \), so \( q_2 \) strictly prefers to participate.

3. Compare \( z(q_1) \) and \( z(q_2) \) at time \( t_0 \) in the following cases:

   (a) \( a(z(q_1)) < q_1 \) in an interval \([t_0, t_1]\) (so \( q_1 \) does not participate).

   If \( z(q_1) = z(q_2) \) at \( t_0 \), \( Q(t_0) = q_1 - \max\{q_2, a(z(q_1))\} > 0 \) so \( \dot{z}(q_1) - \dot{z}(q_2) < 0 \).

   Then \( z(q_1) < z(q_2) \) immediately after \( t_0 \), which is impossible. Hence \( z(q_1) > z(q_2) \) at \( t_0 \).

   (b) \( a(z(q_1, t_0)) = q_1 \), and \( \dot{a} < 0 \) (so \( q_1 \) exits at \( t_0 \)).

   From 2 (above), \( q_2 \) strictly prefers to participate at \( t_0 \). Then \( Q(t) = q_1 - a(z(q_2)) \) for some interval \( t \in [t_0, t_1] \). If \( z(q_1) = z(q_2) \) at \( t_0 \), \( Q(t_0) = 0 \) and \( \dot{z}(q_1) = \dot{z}(q_2) \). Since \( a(z(q_1)) \) is strictly decreasing at \( t_0 \), so is \( a(z(q_2)) \), and \( \dot{Q}(t_0) > 0 \). Differentiating (A.7), the second derivative of \( z(q_1) - z(q_2) \) is strictly
negative at $t_0$. Again this implies $z(q_1) < z(q_2)$ after $t_0$, which is impossible, so $z(q_1) > z(q_2)$ at $t_0$.

(c) $a(z(q_1)) \geq q_1$ in an interval $[t_0, t_1)$.

Then both $q_1$ and $q_2$ participate, and $z(q_1)$ and $z(q_2)$ satisfy the same equation $z = a(z, t) + \frac{1}{\rho + \delta} \dot{z}$ in this interval, which has a unique solution through any point $(z, t)$. If $q_1$ will exit in future, $z(q_1) > z(q_2)$ at that time, and hence $z(q_1) > z(q_2)$ at $t_0$. Otherwise both participate permanently with $z(q_1) = z(q_2) = z$ for all $t \geq t_0$.

\textbf{Proof of Lemma 2:} From (25):

$$\theta = \Theta(z, m), \quad \frac{h(z)\Theta_z}{\theta} = \frac{-1}{1 - \alpha}, \quad \frac{m\Theta_m}{\theta} = \frac{\eta - 1}{1 - \alpha}.$$ 

Then from $u = m/\mu(\theta)$ and $\lambda = \mu(\theta)\phi(m)$:

$$u = U(z, m), \quad \frac{h(z)U_z}{u} = \frac{\alpha}{1 - \alpha}, \quad \frac{mU_m}{u} = \frac{1 - \eta\alpha}{1 - \alpha};$$

$$\lambda = \Lambda(z, m), \quad \frac{h(z)\Lambda_z}{\lambda} = \frac{-\alpha}{1 - \alpha}, \quad \frac{m\Lambda_m}{\lambda} = \frac{\eta - 1}{1 - \alpha}.$$ 

Finally, using (23):

$$\hat{q} = Q(z, m)$$

where in a rising market: $Q_z = a_z + a_\lambda\Lambda_z$, $Q_m = a_\lambda\Lambda_m$

and in a falling market: $(1 - a_z)Q_z = a_\lambda\Lambda_z$, $(1 - a_z)Q_m = a_\lambda\Lambda_m$

which can be evaluated using the results above together with, from (7):

$$\frac{\lambda a_\lambda}{a} = 1 \quad \text{and} \quad \frac{h(z)a_z}{a} = -1$$

\textbf{Proof of Proposition 6:} If the equilibrium is stable with respect to vacancies, $\eta(m) < \bar{\eta}(L\gamma/u)$. To determine the sign of $\Delta$, the derivatives of $F_1$ and $F_2$ are:

$$F_{1z} = (\rho + \delta) (1 - a_z - a_\lambda\Lambda_z) = (\rho + \delta) \left(1 + \frac{z}{h(1 - \alpha)}\right),$$

$$F_{1m} = -(\rho + \delta) a_\lambda\Lambda_m = (\rho + \delta) \frac{(1 - \eta) z}{(1 - \alpha)m};$$

$$F_{2z} = \delta(L'(z)Q_z - U_z) - \Phi(m)\pi'(z)$$

$$F_{2m} = \delta(L'(z)Q_m - U_m) - \Phi'(m)\pi(z)$$

$F_{2z}$ and $F_{2m}$ can be evaluated using the elasticities in the proof of Lemma 2; they differ in rising and falling markets. After a little rearrangement, using the fact that $F_{1z} Q_m +$
\[(1 - Q_z)F_{1m} = 0,\] the numerator of \(\Delta\) can be written in both rising and falling markets:

\[
F_{1z}F_{2m} - F_{1m}F_{2z} = F_{1m}\Phi\pi' - F_{1z}\Phi'\pi - \delta [F_{1m}L' + F_{1z}U_m - F_{1m}U_z]
\]

\[-\frac{\delta(\rho + \delta)}{(1 - \alpha)m} \left( L\left(\frac{z}{h} + Y(\gamma)\right) - (L - u)(1 - \eta)\left(\gamma - \frac{z\pi'}{\pi}\right) \right)\]

where \(Y(\gamma) \equiv 1 + \gamma - \eta(\gamma + \alpha)\). The denominator of \(\Delta\) is:

\[U_m - nQ_m = \begin{cases} \frac{u}{m(1-\alpha)}Y(\gamma) & \text{(rising)} \\ \frac{u}{m(1-\alpha)}Y(\gamma_1) & \text{(falling)} \end{cases} > \bar{\eta}(L\gamma/u) - \eta \quad \text{where} \quad \gamma_1 = \frac{\gamma L}{u(1 + \frac{h}{\pi})}\]

So when \(\eta < \bar{\eta}(L\gamma/u)\), the denominator is positive, and from (A.5) and (A.6) the numerator is negative. Hence \(\Delta < 0\) and the equilibrium is saddle-path stable. The slope of the saddle-path is determined by the sign of \(F_{1m}\), which is equal to the sign of \(1 - \eta\). So it is downward-sloping, upward-sloping or flat when \(\eta\) is, respectively, less than, greater than, or equal to 1.

**Proof of Proposition 7:** \(\nu\) is the negative root of (28). Evaluating \(X\) and \(\Delta\) using the results from the proof of Proposition 6:

\[X = \begin{cases} (\rho + \delta) - (\delta + \lambda\pi) + \frac{(\rho + \delta)\frac{z}{h} + (1 - \eta)\lambda\pi}{Y(\gamma)} & \text{(rising)} \\ (\rho + \delta)\left(1 + \frac{z}{h}\right) - \delta + \eta\frac{(\rho + \delta)\alpha\frac{z}{h} - (1 - \alpha)\lambda\pi}{Y(\gamma_1)} & \text{(falling)} \end{cases}\]

\[\Delta = \begin{cases} \frac{\rho + \delta}{Y(\gamma)} \left((\delta + \lambda\pi)Y(\gamma) + (\delta + \lambda\pi)\frac{z}{h} - (1 - \eta)\lambda\pi\left(1 - \frac{z\pi'}{\pi}\right)\right) & \text{(rising)} \\ -\frac{\rho + \delta}{Y(\gamma_1)} \left((\delta + \lambda\pi)Y(\gamma_1) + (\delta + \lambda\pi)\frac{z}{h} - (1 - \eta)\lambda\pi\left(1 - \frac{z\pi'}{\pi}\right)\right) & \text{(falling)} \end{cases}\]

When \(\eta = 1\) the expressions for the rising and falling cases are identical, and independent of \(\gamma\), and it is straightforward to verify that \(\nu = \frac{-1}{2}(X - \sqrt{X^2 - 4\Delta}) = -(\delta + \lambda\pi)\).

Differentiating (28) we have:

\[
\frac{\partial \nu}{\partial \eta}(X - 2\nu) = \frac{\partial \Delta}{\partial \eta} - \nu \frac{\partial X}{\partial \eta}
\]

Then, differentiating the expressions above for \(X\) and \(\Delta\), and noting that \(X - 2\nu =

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\[20\text{In general the saddle-path of a system: } \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{with } a > 0 \text{ is downward-sloping if } b > 0, \text{ flat if } b = 0, \text{ and upward-sloping if } b < 0.\]
\[
\sqrt{X^2 - 4\Delta} > 0 \text{ we have, for the rising case:}
\]
\[
\frac{\partial \nu}{\partial \eta} \overset{\text{sgn}}{=} - (\rho + \delta) \left[ \left( \delta + \lambda \pi \right) \left( \alpha + \gamma \right) \frac{\delta}{h} + \left( 1 - \frac{\delta}{h} \right) \left( 1 - \alpha \right) \lambda \pi \right] - \nu \left[ \left( \rho + \delta \right) \left( \alpha + \gamma \right) \frac{\delta}{h} - \left( 1 - \alpha \right) \lambda \pi \right]
\]
\[
\overset{\text{sgn}}{=} - \left( \rho + \delta \right) \left( \alpha + \gamma \right) \frac{\delta}{h} \left( \delta + \lambda \pi + \nu \right) - \left( 1 - \alpha \right) \lambda \pi \left( \left( \rho + \delta \right) \left( 1 - \frac{\delta}{h} \right) - \nu \right)
\]

Now note that \( \frac{\partial \nu}{\partial \eta} \) has the same sign as a linear function of \( \nu \), in which the coefficients do not depend on \( \eta \). Irrespective of the signs of the coefficients, this implies that \( \frac{\partial \nu}{\partial \eta} \) can never change sign. When \( \eta = 1 \), \( \nu = - \left( \delta + \lambda \pi \right) \), and from the second expression \( \frac{\partial \nu}{\partial \eta} < 0 \).

Hence \( \frac{\partial \nu}{\partial \eta} < 0 \) for all \( \eta \); furthermore \( \nu \gtrless - \left( \delta + \lambda \pi \right) \) when \( \eta \lneq 1 \) and similarly in the falling case:
\[
\frac{\partial \nu}{\partial \eta} \overset{\text{sgn}}{=} - (\rho + \delta) \left[ \alpha \left( 1 + \gamma_1 \right) \frac{\delta}{h} \left( \delta + \lambda \pi + \nu \right) + \left( 1 - \alpha \right) \delta + \lambda \pi \left( \gamma_1 \left( 1 + \frac{\delta}{h} \right) - \gamma \right) \right]
\]
\[
- \left( 1 - \alpha \right) \lambda \pi \left( \left( \rho + \delta \right) \left( 1 - \frac{\delta}{h} \right) - \left( 1 + \gamma_1 \right) \nu \right)
\]

Again this expression is a linear function of \( \nu \) that doesn’t depend on \( \eta \), and since \( \gamma_1 \left( 1 + \frac{\delta}{h} \right) > \gamma \) it is negative at \( \eta = 1 \). The same argument as in the rising case applies.

**Proof of Proposition 8:** When \( \eta < 1 \) it can be verified using the expressions for \( X \) and \( \Delta \) in the proof of Proposition 7 that as \( \gamma \to \infty \), \( \nu \to - \left( \delta + \lambda \pi \right) \) in the rising case, and \( \nu \to - \delta \) in the falling case. As before we have:
\[
\frac{\partial \nu}{\partial \gamma} \left( X - 2\nu \right) = \frac{\partial \Delta}{\partial \gamma} - \nu \frac{\partial X}{\partial \gamma}
\]

and hence, for the rising case:
\[
\frac{\partial \nu}{\partial \gamma} \overset{\text{sgn}}{=} (\rho + \delta) \frac{\delta}{h} \left( \delta + \lambda \pi + \nu \right) - \left( 1 - \eta \right) \lambda \pi \left( \left( \rho + \delta \right) \left( 1 - \frac{\delta}{h} \right) - \nu \right)
\]

By the same argument as before \( \nu \) is strictly monotonic. Furthermore, for \( \gamma \) sufficiently large \( \delta + \lambda \pi + \nu \) is small and the derivative is negative. Hence it is negative for all \( \gamma \).

In the falling case:
\[
\frac{\partial \nu}{\partial \gamma} \overset{\text{sgn}}{=} (\rho + \delta) \frac{\delta}{h} \left( \delta + \lambda \pi + \nu \right) \alpha \eta + \left( \rho + \delta \right) \lambda \pi \left[ \left( 1 - \eta \alpha \right) \left( 1 + \frac{\delta}{h} \right) - \left( 1 - \eta \right) \left( 1 - \frac{\delta}{h} \right) \right] - \nu \eta \left( 1 - \alpha \right) \lambda \pi
\]

All three terms in this expression are positive (for the expression in square brackets this follows from \( \frac{\delta}{h} + \frac{\delta}{h} > 0 \)). Hence the derivative is positive for all \( \gamma \).

Finally, for the bounds on \( \nu \), we already know (from Proposition 7) that \( |\nu| < \delta + \lambda \pi \).
In the falling case $|\nu| > \delta$ follows from $\frac{\partial \nu}{\partial \gamma} > 0$ and $\lim_{\gamma \to \infty} \nu = -\delta$. In the rising case $|\nu| > \delta$ can be verified directly: $\nu + \delta = \frac{1}{2} (X + \delta - \sqrt{(X + \delta)^2 - 4(\Delta + \delta X + \delta^2)})$, and using the expressions for $X$ and $\Delta$ from Proposition 7 we find that $\Delta + \delta X + \delta^2 < 0$. ■

B  Notes on Volterra derivatives

Let $f$ be a function and let $I$ be a functional that takes $f$ as its argument. We wish to determine the differential change in $I[f]$ resulting from a differential change in $f$. Following Volterra (1930), we fix a point $q_1$ in the domain of $f$ and define a particular small change in $f$, denoted by $\Delta f$, as follows: (a) $\Delta f(q) \neq 0$ iff $q \in (q_1 - w/2, q_1 + w/2)$; (b) either $\Delta f \geq 0$ or $\Delta f \leq 0$; (c) $|\Delta f(q)| \leq h$. Consider the fraction

$$\frac{I[f + \Delta f] - I[f]}{\int \Delta f(q) dq}$$

as $w \downarrow 0, h \downarrow 0$. If the limit exists (uniformly in $f$ and in $q_1$), then it is the first derivative of $I$ w.r.t. $f$ at $q_1$ in the sense of Volterra, and is variously denoted by $I'[f(\cdot); q_1]$, $I'[f; q_1]$, or $[\delta I/\delta f]_{q_1}$.

With $\delta f$ denoting a general differential change in the function $f$, $\delta I$ denotes the total differential of $I$ (w.r.t. $f$), and is given by

$$\delta I = \int I'[f; q] \delta f(q) dq,$$

which is seen as the continuous analogue of $dI = \sum_{i=1}^k \frac{\partial I}{\partial f_i} df_i$ when $I$ is a function of the $k$-vector $(f_1, \ldots, f_k)$.

Partial derivatives of a more general functional are defined similarly.\(^{21}\) Also, since $I'$ is again a functional, second- and higher order derivatives can be defined in the obvious way, and are denoted by $I''[f; q_1, q_2]$, etc., the total second-order differential of $I$ (w.r.t. $f$) being given by

$$\delta^2 I = \iint I''[f; q_1, q_2] \delta f(q_1) \delta f(q_2) dq_1 dq_2.$$

Of particular interest here, consider $I$ being defined by an integral: $I[f] \equiv \int \psi(f(q)) dq$ for some function $\psi$. In this case, $I'[f; q]$ is simply given by $\psi'(f(q))$, and so optimisation of $I$ w.r.t. $f$ is in effect pointwise. Further, when $I'$ is not defined by an integral, the total

\(^{21}\)There seems to be no agreed notation for partial derivatives; when $I$ is a functional of $f$ and $g$ we use $I_f[f, g; q]$ or, more simply, $I_{f(q)}$. 39
second-order differential of $I$ (w.r.t. $f$) is given by

$$
\delta^2 I = \int \psi''(f(q)) (\delta f(q))^2 \, dq
$$

which is the continuous analogue of cross-partial derivatives being zero.

References


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