SIGNALING AND REPUTATION IN REPEATED GAMES, I: FINITE GAMES

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Abstract In a repeated interaction, if the actions of an agent signal a private characteristic, or type, they will influence how he is expected to act in future, giving reputational incentives. If the signaler’s type can change over time, these incentives can persist.

A general model of repeated signaling is presented. The type space is finite or a continuum, and players have arbitrary supermodular payoffs. There is a unique equilibrium with continual (minimal) separation of types. It is selected by a recursive version of the equilibrium refinement D1. The equilibrium is calculable, showing a quantitative dependence of reputation on patience, the length of the game, and the random process on types.

KEYWORDS: repeated signaling, reputation, supermodular games.

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1. INTRODUCTION

1.1. Signaling in repeated games and a directional form of reputation

How do an agent’s current actions affect how he is expected to act in future? This question is important to many situations of repeated interaction and is basic to the question of reputation.

Signaling models have provided a tractable answer. In these, a signaler’s action in a first stage signals his “type”, which affects how he is expected to act in a second stage. While some signaling models have involved stages that are fundamentally different (e.g. education and work stages), many have set signaling in the context of a repeated game with 2 or more stages. Table I gives a selection of existing and potential applications.

<table>
<thead>
<tr>
<th>Signaler</th>
<th>Type</th>
<th>Action</th>
<th>Respondent</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm</td>
<td>Quality</td>
<td>Product quality</td>
<td>Consumers</td>
<td>Demand</td>
</tr>
<tr>
<td>Incumbent</td>
<td>Cost</td>
<td>Price/quantity</td>
<td>Competitors</td>
<td>Entry/quantity</td>
</tr>
<tr>
<td>Worker</td>
<td>Ability</td>
<td>Output</td>
<td>Market</td>
<td>Wage</td>
</tr>
<tr>
<td>Central bank</td>
<td>Toughness</td>
<td>Interest rate</td>
<td>Economy</td>
<td>Inflation/Empl.</td>
</tr>
<tr>
<td>Buyer</td>
<td>Value</td>
<td>Quantity</td>
<td>Seller</td>
<td>Price</td>
</tr>
<tr>
<td>Citizen</td>
<td>Pro-social</td>
<td>Social/criminal</td>
<td>Society</td>
<td>Participate/prison</td>
</tr>
</tbody>
</table>

The effect of signaling in a repeated game is to generate reputational effects. Suppose actions are ordered so that higher actions are taken by higher types and generate a more favorable response by the respondent. If the signaler takes a higher action now, he is seen as a higher type and so is expected to take higher actions in the future, if types are correlated over time. If players move simultaneously in the stage game or the respondent moves first\(^1\), this gives a reputational reason for taking higher actions than would be myopically optimal.

For this reason in applied models studying signaling in repeated games, reputation effects have been central. Most of these have been two stage games, with signaling in the first stage. Early examples include Milgrom and Roberts (1982a), the original model of limit pricing; and Backus and Driffield (1986), studying monetary policy un-

\(^1\)“Respondent” will denote the uninformed player even when he is moving first. In this case it is expectations of types and actions that he is responding to.
der inflation expectations. In the first stage lower prices and “drier” monetary policy are observed, respectively, than in a static Nash equilibrium benchmark.

Building on this approach, a few papers have developed repeated signaling models, with more than one signaling stage. These are often a better representation of the situations being studied, since interactions are typically not limited to two stages, and signaling can therefore occur at more than one point in time. Repeated signaling models include Mester (1992), a three-stage joint-signaling model of Cournot competition; Vincent (1998), a trading model where the buyer’s value is private information; and Mailath and Samuelson (2001), a model of firm investment in product quality. These exhibit reputational effects in all non-terminal stages: higher production quantities, lower demand, and higher quality respectively, compared to static Nash equilibrium.\(^2\)

These models involve a type which varies according to a random process; this is connected to there being a continual incentive to signal. In most applications, the possibility of change cannot be excluded and so a changeable type may be the right assumption. Mester (1992) and Vincent (1998) look at equilibria in which there is full separation of types at each stage, which implies a continual motive to signal. Vincent makes a precise connection between a varying type and continual signaling: even though a continually separating equilibrium still exists if type is fixed with probability 1, it can be justified formally by an equilibrium refinement only when type is changeable. This line of argument is taken here.\(^3\)

In these signaling models, the signaler is better informed than the respondent. A model with some similar effects and something like signaling is Holmstrom (1999), but there the “signaler” does not know his own type, but the choice of effort affects the outcome and therefore the inferred type. This simplifies the analysis, and under linear-quadratic functional forms, the model finds reputational incentives for effort.

1.1.1. **Contribution**

This paper presents a canonical form of a repeated signaling game, with simultaneous moves in the stage game and types correlated over time. It is shown that contin-

\(^2\)Kaya (2008), also studying repeated signaling, is an exception. There the signaler moves first in the stage game, so there is no question of reputation.

\(^3\)An alternative approach to repeated signaling games is to have a type that is constant, and never believed to have changed. Then not only do the above arguments for continual signaling fail, but there is some reason to expect signaling to happen all in the first period. This occurs in Toxvaerd (2011), a model of dynamic limit pricing. I have found similar results in my own investigations.
ually separating equilibria exist and have a clean and calculable recursive structure. So the tractability of signaling games, with their separating equilibria, extends to repeated signaling games and their continually separating equilibria.

The setting generalizes the applied literature studying signaling in repeated games along a number of dimensions.

1. The length of the game is arbitrary, finite or infinite. This allows for long games, as Vincent (1998) and Mailath and Samuelson (2001) do, which is perhaps the most important case for a repeated signaling game, while short games incorporate existing 2-stage signaling models.

2. Specific functional forms on payoffs are replaced with supermodularity and monotonicity properties. This makes the model applicable across a range of settings, and clarifies the analysis of the game. The assumptions allow the respondent to care not only about the signaler’s actions but also directly about his type.4

3. Types are allowed to be any ordered finite set or a continuum. Aside from Mester (1992), which uses a continuum of types, repeated signaling games have only considered 2 types. For most settings a continuum is the appropriate specification. Allowing for arbitrary finite types in addition to the continuum allows for computation of equilibria. Allowing for two types allows comparison to existing models. Any monotonic Markov process on types is allowed.

These generalizations are individually significant, but also combine to give substantively new properties. Previous studies have been ambivalent about whether the signaler is better off than in the static Nash equilibrium benchmark. Allowing for long games and a continuum of types changes the picture. In a long game, we can have a stable level of signaling, and only with a continuum of types can there be an interval of actions that get taken by the signaler, with the signaler able to latch on to any of these. If the signaler’s type is highly persistent, then this is like commitment, and a Stackelberg property emerges. We will see this in computed results, and it is formally stated and proved in Roddie (2011b).

The recursive structure of the game involves preservation of supermodular value functions. This paper adds signaling to the small but important set of game-theoretic structures that preserve supermodularity in dynamic games. See Vives (2009) for a

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4See Section 2.4.2 for an application to work incentives.
survey of existing work. It also demonstrates functional methods for these games, decomposing the workings of a game into maps between spaces of economically relevant functions that have monotonic or supermodular properties.

1.2. Equilibria of repeated signaling games: Properties and methods

1.2.1. Setup and basic properties

We shall study a finitely repeated game, with $k$ stages.\(^5\) In each stage game, the signaler chooses a signal $x$ and the respondent an action $y$ simultaneously. The signaler’s type $\theta$ varies according to Markov process $\psi$. The signaler has stage game payoff $u_{P1}(\theta, x, y)$, with discount factor $\delta_{P1}$. The respondent does not have private information, so his actions have no effect on the future, so he will best-respond in the stage game, responding in a monotonic way to a distribution of possible types and actions of the signaler.\(^6\)

We find a continually separating equilibrium, in which at each stage, different types of the signaler separate from each other. We shall be able to justify this by equilibrium refinement when $\psi$ has full support. In such an equilibrium, there is continual signaling because of the continual need to separate. Signaling a higher type in period $t$ improves beliefs in period $t + 1$, but since separation will happen at $t + 1$, there is no effect on $t + 2$ on. So the benefit of signaling a higher type in period $t$ is to improve the respondent’s action in period $t + 1$.

1.2.2. Value functions and signaling incentives

Signaling incentives in period $t$ are given by the value function for period $t + 1$. Let $v_{t+1}^- (\theta', \theta_{t+1})$ be the value in period $t + 1$ of having signaled $\theta'$ in period $t$ rather than the lowest type $\theta_{\min}$. This just takes into account discounting and the possibility of type change, this function generates a value $v_t' (\theta', \theta_t)$ of signaling $\theta'$ (over $\theta_{\min}$) in period $t$. Then in period $t$, given beliefs $\hat{\theta}_t$ at the start of the period, suppose the respondent’s expected action is $y^*$.

Then if the signaler is of type $\theta_t$ and takes action $x$, he receives payoff $u_{P1}(\theta_t, x, y^*) + v_t' (\theta', \theta_t)$ for signaling $\theta'$ (rather than $\theta_{\min}$). This describes signaling incentives in period $t$.

\(^5\)See Roddie (2011b) for analysis of the infinite horizon.

\(^6\)To satisfy the directional assumptions, it may be necessary to reorder some variables.
1.2.3. *A simple case (Section 2)*

The simplest interesting form of the model is the case \( u_{p1}(\theta, x, y) = u_X(\theta, x) + y \), where \( u_X \) has increasing differences. This simplifies things in two ways. Firstly, there are no strategic interactions in \( u_{p1} \) between \( x \) and \( y \), and therefore signaling incentives are independent of \( y^* \). Secondly, different types value increases in subsequent \( y \) identically. This makes the value function \( v_t(\theta) \) independent of \( \theta_t \).

Now signaling incentives are given by \( u_X(\theta, x) + v_t'(\theta') \). There is a dominant separating equilibrium, which is then the strategy of the signaler in period \( t \). Given this strategy, the respondent reacts in a monotonic way to beliefs \( \hat{\theta}_t \), and this generates the value of signaling higher \( \hat{\theta}_t \), giving \( v_t^- \). So value function iteration consists of mapping a monotonic value function \( v_t^- \) into the signaler’s strategy in period \( t \), and then into a monotonic function \( v_t^- \).

1.2.4. *The general case (Sections 3 to 4)*

In general we allow for strategic interactions between \( x \) and \( y \), and consider supermodular \( u_{p1} \). Supermodularity is a productive framework for multi-stage games with strategic interactions (see for example Vives (2009)), and for signaling games, as shown in Roddie (2011a). Roddie (2011a) finds that supermodularity is sufficient for the single crossing needed to analyze signaling games productively, that a dominant separating equilibrium then exists, and that a parametrized supermodular signaling game gives a payoff in this equilibrium that is supermodular in type and the parameter.

Now the value functions \( v_t'(\theta', \theta_t) \) we need to consider will be supermodular, and increasing in \( \theta' \). This is then true of \( v_t' \) and so for each \( y^* \) the signaling incentive \( u_{p1}(\theta_t, x, y^*) + v_t'(\theta', \theta_t) \) is supermodular, so satisfies single crossing, and so there exists a dominant separating equilibrium (Roddie (2011a)), which depends on \( y^* \). Payoffs in this equilibrium are a supermodular function of \( (y^*, \theta_t) \) (Roddie (2011a)).

It remains to find \( y^* \). Given \( y^* \) and an initial type distribution \( \hat{\theta}_t \), we know the signaler’s strategy, and this gives a distribution over types and actions to which the respondent responds - with \( y^* \). \( y^* \) is then a fixed point, whose unique existence is assured by making Lipschitz assumptions on \( u_{p1} \) and \( u_{p2} \), and which is increasing in \( \hat{\theta}_t \).

Taking \( \hat{\theta}_t = \psi(\theta') \), where \( \theta' \) was the previously signaled type, we then have a value
function $v_t^* (\theta', \theta_t)$, which is supermodular and increasing in $\theta'$. This solves the game recursively.

1.2.5. Refinement

In the above analysis, for each signaling game we chose the dominant separating equilibrium as the signaler’s strategy. This is commonly done in analysis of static signaling games. In these games there are a multiplicity of separating and pooling equilibria. With a continuum of types, there is only one separating equilibrium, so the most important thing is to justify separating over non-separating equilibria. A formal argument is presented in terms equilibrium refinement $D_1$, which models beliefs off the equilibrium path and selects the dominant separating equilibrium Cho and Sobel (1997). The same logic applies for repeated signaling games, and we apply a recursive form of $D_1$, effectively using $D_1$ at each stage to select the continually separating equilibrium described above.

2. A MODEL OF WORK INCENTIVES

A worker interacts repeatedly with the job market, for $k < \infty$ periods. In each stage, the market determines a wage $y \in \mathbb{R}$, and once employed the worker decides how much work to do, which is a choice of productivity $x \in [x_{min}, x_{max}]$. The worker has ability $\theta$, taken from a set $\Theta \subseteq \mathbb{R}$ that is finite or an interval, with least and greatest elements $\theta_{min}, \theta_{max}$. The worker has discount factor $\delta_{P1}$ and stage game payoff $u_{P1} (\theta, x, y) = u_X (\theta, x) + y$. Type is a Markov process over time, so that if the worker has ability $\theta$ in a given period, his ability will have distribution $\psi (\theta) \in \Delta \Theta$ in the next period, where $\psi$ is strictly increasing.  

Higher ability workers find it easier to be more productive. This will be expressed in an increasing difference assumption.

**Definition** (Standard) If $A, B$ are partially ordered sets, $f : A \times B \rightarrow \mathbb{R}$ has strictly increasing (weakly increasing / constant) differences if: for $a' > a$ and $b' > b$, $f (a', b') - f (a', b) > (\geq / =) f (a, b') - f (a, b)$. Then $f (a, b, c...)$ has weakly increasing differences (etc.) in $(a, b)$ if $f (c, ...) (a, b)$ has weakly/strictly increasing differences (etc.) as a

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7 In this section there is no difference between simultaneous moves and the signaler moving second.
8 I.e. weakly increasing and injective. $\Delta Z$ denotes the set of probability measures on a measurable space $Z$. If $Z \subseteq \mathbb{R}^n$ then $\Delta Z$ is partially ordered by stochastic dominance (Muller and Stoyan (2002)). Here $\Theta \in \mathbb{R}$ and this is FOSD.
function of \((a, b)\). \(f\) is weakly supermodular if it has weakly increasing differences in all pairs of variables.

**Assumption** \(u_X(\theta, x)\) is continuous, strictly quasi-concave in \(x\), and has strictly increasing differences.\(^9\)

In any period the wage received is based on the expectation of productivity and possibly also ability. Suppose the worker is expected to use strategy \(s_{P_1}(\theta)\), increasing in \(\theta\). If the market believes the signaler’s type is has the distribution \(\hat{\theta}\), it gives a competitive wage \(\rho'(\hat{\theta}) (s_{P_1})\), increasing in \(\hat{\theta}\). For example wage could equal expected productivity: \(\rho'(\hat{\theta}) (s_{P_1}) = \int s_{P_1}(\theta) d\hat{\theta}(\theta)\).

We shall look for a continually separating equilibrium, where at each point the strategy of the signaler is a strictly increasing function of type. Let \(\theta_{\text{min}}, \theta_{\text{max}}\) and \(x_{\text{min}}, x_{\text{max}}\) be the lowest and highest elements of \(\Theta\) and \(X\). To ensure there are costly enough signals to separate, the highest signal must be sufficiently undesirable:

**Definition** Let \(\underline{y}\) and \(\overline{y}\) be the lowest and highest values of \(\rho'\), and \(\bar{Y} := [\underline{y}, \overline{y}]\).

**Assumption** \(\overline{y} - \underline{y} < \max_x u_X(\theta_{\text{max}}, x) - u_X(\theta_{\text{max}}, x_{\text{max}})\).

This is a weak assumption: if a worker has commitment payoff as a function of \(x\), and if this has a maximum and falls at at least some rate after a certain point, then the assumption must hold.

### 2.1. Value function iteration

We shall find an equilibrium with Markov strategies \(s_{P_1}(t)(\hat{\theta}, \theta)\) for the signaler, where \(t\) is the period, \(\hat{\theta}\) the initial beliefs at \(t\). This gives a value function \(V_t(\hat{\theta}_t, \theta_t)\), which is the signaler’s utility from period \(t\) on, discounted to period \(t\), if the signaler is initially believed to be \(\hat{\theta}_t\). As discussed in the introduction, since beliefs \(\hat{\theta}_t\) in a continually separating equilibrium only affect outcomes in period \(t\), \(V_t(\hat{\theta}, \theta) - V_t(\hat{\theta}_0, \theta)\) is just given by the change in payoffs in period \(t\).

As mentioned in the introduction, there are two simplifications made in the form of \(u_{P_1}\). The first is that there are constant differences in \((x, y)\): \(u_{P_1}(\theta, x, y) = u_X(\theta, x) + \)

\(^9\)This is implied by the differential conditions \(\left(\frac{\partial}{\partial \theta}\right)^2 u_X < 0\) and \(\frac{\partial^2}{\partial \theta^2} u_X > 0\).

\(^{10}\)Explicitly \(\underline{y} = \rho'([\theta_{\text{min}}]) (\theta \mapsto \theta_{\text{min}})\) and \(\overline{y} = \rho'([\theta_{\text{max}}]) (\theta \mapsto \theta_{\text{max}})\).
$u_Y(\theta, y)$. So $y$ does not alter incentives about the choice of $x$. The second simplification is that there are constant differences in $(\theta, y)$, leading to value functions $V_t(\hat{\theta}, \theta)$ that have constant differences, so that fixing any $\hat{\theta}_0$, $V_t(\hat{\theta}, \theta) - V_t(\hat{\theta}_0, \theta)$ is a function of $\hat{\theta}$ only.

If the worker signals $\theta'$ in period $t$, he is thought to be $\psi(\theta')$ in period $t+1$; so we only need to keep track of the value function on beliefs of the form $\theta'$. We shall work with the reduced value function $V_t(\psi(\theta'), \theta)$:

$$V_t(\psi(\theta'), \theta) - V_t(\psi(\theta_{\text{min}}), \theta).$$

This the value in period $t$ of having previously signaled $\theta'$ rather than $\theta_{\text{min}}$.

Suppose $v_{t+1}$ lies in the following set:

**Definition** Let $\Pi^*$ be the set of weakly increasing $v^\sim : \Theta \to \mathbb{R}$, with $v^\sim(\theta_{\text{min}}) = 0$ and $v^\sim(\theta_{\text{max}}) \leq \max_x u_X(\theta_{\text{max}}, x) - u_X(\theta_{\text{max}}, x_{\text{max}})$.

$v_{t+1}$ gives rise to signaling incentives in period $t$. First we describe conditions that make signaling games tractable, before showing that the signaling game induced by $v_{t+1}$ satisfies these conditions.

### 2.1.1. Signaling game theory: general specifications

The static monotonic signaling game is specified in reduced form by a function $\hat{U}(\theta, x, \theta')$ giving payoff for type $\theta$ choosing signal $x$ and consequently being believed to be type $\theta'$. Higher types are more willing to take higher signals in return for better beliefs, and this is expressed in a single crossing condition:

**Definition** (Roddie (2011a)) Suppose $B$ is a weakly ordered set. A function $u(\theta, x, b)$ satisfies single crossing if whenever $\theta_1 < \theta_2$, $x_1 < x_2$, and $b_1 \leq b_2$, $u(\theta_1, x_2, b_2) \geq u(\theta_1, x_1, b_1)$ implies $u(\theta_2, x_2, b_2) > u(\theta_2, x_1, b_1)$.

All we need here is that this holds if $u$ is weakly supermodular, with strictly increasing differences in $(\theta, x)$. A few other technical conditions help to give the signaling game a tractable structure; the space of payoff functions satisfying these conditions is $\Phi_B$:

**Definition** (Roddie (2011a)) Let $\Phi_B$ be the set of $\hat{U} : \Theta \times X \times \Theta \to \mathbb{R}$ with $\hat{U}(\theta, x, \theta')$:

1. uniformly continuous in $(\theta, x)$, 2. weakly increasing in $\theta'$, 3. strictly quasi-concave in $x$, 4. with $\hat{U}(\theta, x, \theta_1) = \hat{U}(\theta, x, \theta_2)$ independent of $x$, 5. satisfying single crossing, and 6. satisfying $\max_x \hat{U}(\theta_{\text{max}}, x, \theta_{\text{min}}) \geq \hat{U}(\theta_{\text{max}}, x_{\text{max}}, \theta_{\text{max}})$. 


The last condition will guarantee existence of separating equilibrium. If \( \hat{U}_1 \) and \( \hat{U}_2 \) are signaling payoffs, and \( \hat{U}_2 - \hat{U}_1 \) is a function of \( \theta \), then \( \hat{U}_1 \) and \( \hat{U}_2 \) will give the same signaling incentives, and we shall call them equivalent, even if, because of the continuity requirement, only one is in \( \Phi_B \).

### 2.1.2. The worker’s signaling incentives

Suppose initial beliefs at time \( t \) are \( \hat{\theta}_t \). If the worker is of type \( \theta \) and chooses productivity \( x \), and signals type \( \theta' \), he receives payoff \( u_X(\theta, x) + s_{p2}(t)(\hat{\theta}_t) \) in period \( t \). He is subsequently believed to be \( \psi(\theta') \), and his type changes to \( \theta^* \), distributed with probability measure \( \psi(\theta) \), giving payoff is \( \delta_{p1} \cdot \int V_{t+1}(\psi(\theta'), \theta^*) \, d\psi(\theta) \). Taking \( \equiv \) to mean differing by a function of \( \theta \), we have signaling payoff:

\[
\begin{align*}
\hat{U}(\theta, x, \theta') &:= u_X(\theta, x) + \delta_{p1} \cdot \int \psi(\theta') \, d\psi(\theta) \\
&= u_X(\theta, x) + \delta_{p1} \cdot \int v_{t+1}(\theta') \, d\psi(\theta)
\end{align*}
\]

Since \( v_{t+1} \) and \( \psi \) are weakly increasing, \( \hat{U} \) is weakly increasing in \( \theta' \). It is supermodular, since \( u_X \) is, and so satisfies single crossing. Because of the bound on \( v^* \in \Pi^* \), we have:

\[
\begin{align*}
\max_x \hat{U}(\theta_{\max}, x, \theta_{\min}) - \hat{U}(\theta_{\max}, x_{\max}, \theta_{\max}) &= \max_x u_X(\theta_{\max}, x) - u_X(\theta, x) - \delta_{p1} \cdot v_{t+1}(\theta_{\max}) \\
&\geq 0
\end{align*}
\]

So \( \hat{U} \in \Phi_B \). It follows that \( \hat{U} \) has a dominant separating equilibrium. Again, we summarize the basic signaling theory before applying it to the current setting.

### 2.1.3. Signaling theory: Separating and Riley equilibria

A function \( f : \Theta \rightarrow X \) satisfies individual rationality if for all \( \theta \), \( \hat{U}(\theta, f(\theta), \theta) \geq \max_x \hat{U}(\theta, x, \theta_{\min}) \). It satisfies incentive compatibility (IC) if for all \( \theta, \theta' \), \( \hat{U}(\theta, f(\theta), \theta) \geq \hat{U}(\theta, f(\theta'), \theta') \). If both are satisfied, \( f \) describes a weakly separating equilibrium. This is a separating equilibrium iff it is injective, which it must be if \( \hat{U} \) is strictly increasing in \( \theta' \).

We know (Roddie, 2011a) that for any payoff in \( \Phi_B \) there is a weakly increasing
weakly separating equilibrium, and one of them is payoff-dominant, the Riley equilibrium given by $R: \Phi_B \rightarrow \Theta \rightarrow X$.

2.1.4. Equilibrium strategies and the previous value function

Assume the equilibrium of this game is the Riley equilibrium: $s_{P1}(t) = R(\hat{U})$. So $s_{P1}(t)(\theta)$ has no dependence on $\hat{\theta}_t$. Responding to this strategy, the market takes a best response $s_{P2}(t)(\hat{\theta}_t) = \rho(\hat{\theta}_t)(s_{P1})$ in period $t$. This is weakly increasing in $\hat{\theta}_t$. When beliefs are $\psi(\theta')$, the response is $s_{P2}^\sim(t)(\theta') := s_{P2}(t)(\psi(\theta')) = \rho'(\psi(\theta'))(s_{P1})$, weakly increasing in $\theta'$.

Then $V_t(\hat{\theta}, \theta) - V_t(\hat{\theta}_0, \theta) = s_{P2}(t)(\hat{\theta}) - s_{P2}(t)(\hat{\theta}_0)$ as we concluded earlier, and in particular $v_{i-1}(\theta') = s_{P2}^\sim(t)(\theta') - s_{P2}(t)(\theta_{\min})$.

Clearly $v_{i-1}(\theta_{\min}) = 0$. $v_{i-1}$ is weakly increasing since $s_{P2}^\sim(t)$ is. $v_{i-1}(\theta_{\max}) \leq \bar{y} - y \leq \max_x u_X(\theta_{\max}, x) - u_X(\theta_{\max}, x_{\max})$ by assumption, so $v_{i-1}$ satisfies the bound condition. So $v_{i-1} \in \Pi$.

2.1.5. Conclusion

We have a function $F^*: \Pi^* \rightarrow \Pi^*$ that gives $v_{i}^*$ in terms of $v_{i+1}^*$. Figure 1 represents $F^*$ in a commutative diagram, making the individual maps in the description above explicit.

\[ \text{\footnotesize 11} \text{The reason for introducing } s_{P2}^\sim \text{ is to improve consistency with the general case, where the simpler } s_{P2} \text{ will prove useful.} \]
2.2. The iterated Riley equilibrium

The value function in period $k+1$ (after the game is over) is the zero function $0 : \theta' \mapsto 0$. Value function iteration gives:

$$v_{t+1}^- = F^{k+1-t}(0)$$

Using the notation of Figure 1, we have strategies:

$$s_{P1}(t) = (\mathcal{R} \circ \alpha_2)(v_{t+1}^-)$$
$$s_{P2}(t) = (\alpha_{4+5} \circ \mathcal{R} \circ \alpha_2)(v_{t+1}^-)$$

This only gives the signaler’s strategy when $\hat{\theta} = \psi(\theta')$ for some $\theta'$. This will hold on the equilibrium path for $t \geq 2$. In general, if $\hat{\alpha}_{4+5}(s_{P1}) : \hat{\theta} \mapsto \rho'(\hat{\theta})(s_{P1})$, we have $s_{P2}(t) = (\hat{\alpha}_{4+5} \circ \mathcal{R} \circ \alpha_2)(v_{t+1}^-)$.

2.3. Strict signaling incentives

It will often be useful to guarantee that signaling incentives are strict. Strict incentives give rise to fully separating equilibria, ensure a positive reputational effect, and allow limiting statements about reputation to be made.
In the case when signaling is purely reputational\textsuperscript{12}, and all types prefer the minimal action \(x_{\text{min}}\), there can be no signaling incentives. In the last period, there are no signaling incentives, and then all types take the action \(x_{\text{min}}\), and so \(s_{P2}\), the best response to \(x_{\text{min}}\) is constant in \(\hat{\theta}\). So there are no signaling incentives in the previous period, and so on. This is an edge case which a number of conditions can rule out, and here will give two.

For strict signaling incentives, we need \(\rho'(\psi(\theta'))(s_{P1})\) to be strictly increasing in \(\theta'\). This is guaranteed if the respondent responds favorably to higher expectations of type, irrespectively of actions. Secondly it is guaranteed if \(s_{P1}\) can be guaranteed to be non-constant, and \(\psi(\theta)\) has full support and satisfies a monotone-likelihood-ratio type of property: then increasing \(\theta'\) increases expectations over actions given type \(\psi(\theta')\), so must strictly increase \(\rho'(\psi(\theta'))(s_{P1})\). These conditions are given precisely in Section 5.1 for the general model.

2.3.1. Beliefs and continual separation

If there are always strict signaling incentives, then the Riley equilibrium of each signaling game, which is known to be weakly separating, must be separating. So there is continual separation in equilibrium.

Note that equilibrium analysis did not depend on separation: all it assumed was weak separation. If there are not strict signaling incentives in period \(t\), this corresponds to regions of constancy of \(s_{P2}(t+1)(\theta')\): signaling different \(\theta'\) leads to the same response in the next period. Suppose a set of types \(S \subseteq \Theta\) generates the same response in the next stage, so that there is no advantage to signaling different types within this set. This set must be connected. Then if the signaler takes the action of type \(\theta \in S\), and initial beliefs in period \(t\) are \(\mu_0\), then initial beliefs in period \(t+1\) will be the update of \(\mu_0\) conditioning on \(\theta \in S\). So without strict signaling incentives, beliefs become a little more complex, but this does not affect strategies.

2.4. Computed results and reputational limit properties

2.4.1. Reputational distortions

For a signaling payoff \(\hat{U}\), let \(x^*_U\) be the complete information equilibrium strategy:

\textsuperscript{12}I.e. the respondent cares about the distribution of actions only.
**Definition**  For $\hat{U} \in \Phi$, $x_{\hat{U}}^*(\theta) := \arg\max_x \hat{U}(\theta, x, \theta)$.

We know that $R(\hat{U}) \geq x_{\hat{U}}^*$. So in the iterated Riley equilibrium, we have:

$$s_{P1}(t)(\theta) = R(\hat{U})(\theta) \geq x_{\hat{U}}^*(\theta) = \arg\max_x u_X(\theta, x)$$

So actions in the Iterated Riley equilibrium are weakly above the complete information Nash Equilibrium action, moved in the direction of Stackelberg actions.  

2.4.2. *Computed example*

Suppose the worker has value $u_X(\theta, x) = \theta \cdot x - x^2$ for quantity of work $x$, and the market gives a wage equal to expected $x$. This is a pure reputation model, since the respondent does not care directly about type.

Using a discrete type space to approximate the continuum, Figure 2 shows various computations. Figure 2.a) shows that in the final period, the complete information Nash Equilibrium action is taken. Figure 2.b) shows that signaling incentives are significant in the penultimate period. (Higher types start to approximate the Stackelberg strategy, but this is an artifact of the linear-quadratic specification and is not a general property.) It is not true to say that signaling incentives are increasing with the length of the game, but as the game becomes long certain limit properties emerge. 50 periods from the end, we get 2.d), which shows that apart from low types, strategies approximate the Stackelberg action. This result is proved in Roddie (2011b) for the general case. Figure 2.c) shows how discounting reduces the incentive to signal and ability to establish a reputation. Roddie (2011b) shows that in such a limit the signaler maximizes a discounted form of Stackelberg payoffs. Note that the lowest type’s action is not altered by signaling incentives, and this implies that subsequent types must take discontinuously higher signals so that the lowest type does not have an incentive to mimic them.

---

13For $\theta_{\min}$ we have equality in the equation. Strict signaling incentives will normally move types above $\theta_{\min}$ strictly above Nash equilibrium. This is guaranteed if signaling incentives are first order, and $u_X$ is differentiable.

14Whether this is significant depends on how low $\theta_{\min}$ is and whether $\theta_{\min}$ gains from committing.
Now suppose that work produced has quality $(1 + \theta/4 - x/8)$: now workers who can more easily produce a high quantity of output, for a fixed output work at a higher quality. Wage is the expected value of $x(1 + \theta/4 - x/8)$, which now depends on $\theta$ in addition to $x$. So there is a direct incentive to signal, in addition to the reputational incentive. For a sufficiently long game, with a patient signaler, work is strictly above the efficient level (Figure 3.b). Again, Roddie (2011b) gives a characterization of the limit $(n \to \infty; \psi \to \psi_{id})$ in terms of the separating incentive compatible outcome of a signaling game where the signaler has Stackelberg leadership.

Speaking imprecisely, if there is a direct incentive to signal ability via quantity of work, this may compensate for reputational incentives that are insufficient (Figure 3.c) - from the point of view of the signaler, or equivalently here, if efficiency is a goal.
2.5. Equilibrium refinement

The continually separating equilibrium described above is not the only equilibrium of the game. The choice of equilibrium is made when the Riley equilibrium was chosen as the equilibrium of the induced signaling game faced in period $t$ (Section 2.1.4). The same questions of equilibrium selection arise in the repeated signaling game as in the static signaling game. No indubitable conclusions can be reached, as with any economic modeling. As in the static signaling game, informally, separating equilibria may be considered to be focal, and formally, they are selected by additional modeling of beliefs off the equilibrium path.

2.5.1. Equilibrium refinement in static signaling games

To study separating equilibria we need only to know payoffs $\bar{U}(\theta, x, \theta')$ for degenerate signaled types $\theta'$. To study other equilibria we need to know payoffs $\hat{u}(\theta, x, \hat{\theta})$ for any posterior belief $\hat{\theta} \in \Delta \Theta$. This is normally derived as $\hat{u}(\theta, x, \hat{\theta}) = v(\theta, x, \rho(\hat{\theta}, x))$,
where \( v \) is a fundamental utility function and \( \rho \) is the best-response function of the respondent.

**Definition** (Roddie (2011a)) Let \( \Phi^* \) be the set of \( \hat{u} : \Theta \times X \times \triangle \Theta \rightarrow \mathbb{R} \) with \( \hat{u}(\theta, x, \hat{\theta}) = v(\theta, x, \hat{\theta}(\theta, x)) \), where:

1. \( \hat{\theta}(\theta, x) \in \mathbb{R} \) is continuous, weakly increasing in \( x \) and strictly increasing in \( \hat{\theta} \)
2. \( v(\theta, x, y) \) is continuous, strictly increasing in \( y \) and satisfies single crossing
3. \( \hat{u}(\theta, x, \hat{\theta}) \) is strictly quasi-concave in \( x \)
4. \( \hat{u}(\theta_{\text{max}}, x_{\text{max}}, [\theta_{\text{max}}]) \leq \max_x \hat{u}(\theta_{\text{max}}, x, [\theta_{\text{min}}]) \)

If \( \hat{u} \in \Phi^* \), then the reduced payoff \( \hat{U}(\theta, x, \theta') = \hat{u}(\theta, x, [\theta']) \) must lie in \( \Phi_B \).

The equilibrium refinement D1 models and restricts beliefs off the equilibrium path. Let \( \psi^* \in \Delta \Theta \) be the initial distribution of types. Take a strategy for the signaler \( f : \Theta \rightarrow \Delta X \), and a belief function for the respondent \( \beta : X \rightarrow \Delta \Theta \) satisfying Bayes rule, with \( f \) rational given \( \beta \): actions \( a \) in the support of \( f(\theta) \) must give type \( \theta \) (almost surely) the same payoff \( \hat{u}_{f, \beta}(\theta) := \hat{u}(\theta, x, \beta(x)) \). I.e. these form a perfect Bayesian Nash equilibrium.

Given an action \( x \), if any belief \( \hat{\theta} \) associated with \( x \) that would give some type outside \( S \) a weak incentive to deviate and play \( a \) would give all types in \( S \) strict incentives to deviate, then beliefs after observing \( x \) must lie in \( S \).

**Definition** Suppose \( \hat{u} \in \Phi^* \). \( f, \beta \) satisfy D1 if for any \( x \in X \) not in the support of \( f \) and any \( S \subseteq \Theta \) measurable with \( \psi^*(S) > 0 \), \( \bigcup_{\theta \in S} \{ \hat{\theta} : \hat{u}(\theta, x, \hat{\theta}) > \hat{u}_{f, \beta}(\theta) \} \subseteq \bigcap_{\theta' \in S} \{ \hat{\theta} : \hat{u}(\theta', x, \hat{\theta}) \geq \hat{u}_{f, \beta}(\theta') \} \) implies \( \beta(x)(S) = 1 \).

Cho and Sobel (1997) showed that D1 uniquely selects the Riley equilibrium. This result is generalized in Roddie (2011a) to weaker single crossing and technical conditions expressed in \( \Phi^* \).

**Proposition** (Roddie (2011a)) \( f = \mathcal{R}(\hat{U}) \) satisfies D1 for some \( \beta \). If \( \Theta \) is finite\(^{15}\), \( \psi^* \) has full support and \( f, \beta \) satisfy D1, then \( f = \mathcal{R}(U) \).

\(^{15}\)Continuum \( \Theta \) is allowed with an additional condition, but in this section we shall assume finite types for simplicity.
2.5.2. Equilibrium selection in the repeated signaling game

To apply the refinement to the repeated signaling game, we can ask whether the derived signaling games are in $\Phi^*$. We shall only be able to apply the refinement D1 when the current belief $\hat{\theta}_t$ has full support. Suppose that from periods $t = n + 1$ on, equilibrium play is given by the iterated Riley equilibrium strategies, whenever $\hat{\theta}_t$ has full support.

Then $v_{t+1}(\hat{\theta}) = V_{t+1}(\hat{\theta}, \theta) - V_{t+1}(\psi(\theta_{\min}), \theta)$ is independent of $\theta$, weakly increasing, and generates a signaling game with fully specified payoffs $\hat{u}(\theta, x, \hat{\theta}) := u_X(\theta, x) + \delta_{p1}$. $\nu_{t+1}(\psi(\theta))$.  

To apply D1, we require that $\hat{u} \in \Phi^*$. For finite $\Theta$, this holds provided we have guaranteed strict incentives (Section 2.3), and provided $\rho'(\hat{\theta})(f)$ is continuous in $\hat{\theta}$ for any $f$, which must hold if $\rho$ is the unique maximum of a continuous utility function.  

Then repeated application of the D1 refinement selects the dynamic Riley equilibrium. See Section 5.4 for a formal argument and definition of the dynamic refinement.

3. THE GENERAL MODEL

For reputation to be a question, there must be some strategic interaction between $x$ and $y$ in the respondent’s payoff. Often there will also be strategic interactions in the signaler’s payoff. This is necessary in any setting where strategic interaction is symmetric, for example in Cournot duopoly. To allow for this we extend the assumption $u_{p1} = u_X(\theta, x) + y$ to arbitrary supermodular $u_{p1}(\theta, x, y)$.

There are two players and $k < \infty$ periods. In each period both players act simultaneously; actions are observable. The signaler takes actions from $X = [x_{\min}, x_{\max}] \subseteq \mathbb{R}$; the respondent from $Y = [y_{\min}, y_{\max}] \subseteq \mathbb{R}$. The signaler has a type in each period and knows in each period knows his current and previous types. Each type lies in a set $\Theta \subseteq \tilde{\Theta} = [\theta_{\min}, \theta_{\max}]$ with $\theta_{\min}, \theta_{\max} \in \Theta$. Either $\Theta$ is finite or the entire interval $\tilde{\Theta}$.

$\text{16}\psi(\hat{\theta})$ denotes the composed measure $S \rightarrow \int \psi(\theta)(S) \, d\hat{\theta}(\theta)$ for measurable $S$. (This exists since $\psi(\theta)(S)$ is measurable for measurable $S$, which is true for $\Theta = \tilde{\Theta}$ since it is measurable for intervals $S$, which is true since it is monotonic and therefore measurable for intervals containing $\theta_{\max}$.)

$\text{17}$We can write $\hat{u}(\theta, x, \hat{\theta}) = v(\theta, x, \hat{\rho}(\hat{\theta}, x))$, where $v = u_{p1}$ and $\hat{\rho}(\hat{\theta}, x) = s_{p2}(t + 1)(\psi(\hat{\theta}))$. Conditions 2 and 3 for $\hat{u} \in \Phi^*$ are immediate, and 4 is already known, so it remains to show 1, namely that $s_{p2}(t + 1)(\psi(\hat{\theta}))$ is continuous in $\hat{\theta}$ and strictly increasing in $\hat{\theta}$.

Condition 1 follows from strict incentives, giving strict monotonicity of $s_{p2}(t + 1)(\psi(\hat{\theta}))$, and continuity follows from the assumption of continuity on $\rho'$, since $s_{p2}(t + 1)(\psi(\hat{\theta})) = \rho'(\psi(\hat{\theta}))(s_{p2}(t + 1))$ and $\psi$ is necessarily continuous for finite $\Theta$. 

---

5.4. Dynamic refinement
Types change according to the Markov process $\psi : \Theta \rightarrow \Delta \Theta$, with $\psi$ continuous and strictly increasing.\(^{18}\)

3.1. Assumptions

3.1.1. Basic assumptions for the signaler

The signaler has the discounted utility function $\sum_i \delta^i P_1 (\theta_i, x_i, y_i)$ from outcomes $O = (\Theta \times X \times Y)^k$ to $\mathbb{R}$, with $0 < \delta_1 \leq 1$. This signaler’s payoff is supermodular, and he prefers higher responses:

**Assumption 1** $u_{P_1}(\theta, x, y)$:

1. is continuous
2. is strictly quasi-concave in $x$
3. is strictly increasing in $y$
4. has strictly increasing differences in $(\theta, x)$ and weakly increasing differences in $(\theta, y)$ and $(x, y)$

Increasing differences in $(\theta, x)$ and $(\theta, y)$ are used to generate single crossing. Many applications would be modelled with constant differences in $(\theta, y)$: i.e. $u_{P_1} = \alpha(\theta, x) + \beta(x, y)$. Weakly increasing differences in $(x, y)$ is a strategic complementarity condition, generalizing the constant differences assumption of Section 2. Parts 1 and 2 of the assumption allow definition of a best response function:

**Definition** $\text{BR}_{P_1} (\theta, y) = \arg\max_x u_{P_1}(\theta, x, y)$

3.1.2. Basic assumptions for the respondent

We shall place assumptions on the static best response function of the respondent. At any stage the respondent does not know either the current type or the current action of the signaler, so he responds to a distribution over both:

**Assumption 2** The respondent has a best response function $\rho : \Delta (\Theta \times X) \rightarrow Y$, with $\rho$ continuous\(^{19}\) and weakly increasing.\(^{20}\)

---

\(^{18}\)Again let $\psi(\hat{\theta}) : S \rightarrow \int \psi(\theta) (S) d\hat{\theta}(\theta)$.

\(^{19}\)If $Z$ is a metric measure space, give $\Delta Z$ the topology of weak convergence. Then $\Delta Z$ is compact if $Z$ is compact.

\(^{20}\)We can derive from primitive assumptions similar to those for the signaler. Suppose the respon-
In the case where $\rho(\mu)$ is a function of $\mu X$ only, signaling incentives are purely reputational.\(^{21}\)

The highest and lowest best responses are:

**Definition** Let $y := \rho(\{ \theta_{\text{min}}, x_{\text{min}} \})$ and $\bar{y} := \rho(\{ \theta_{\text{max}}, x_{\text{max}} \})$. Let $\bar{Y} = [y, \bar{y}]$.

If player 1’s strategy in a stage game is a map $\sigma : \Theta \rightarrow X$, and player 2’s belief about 1’s type is $\hat{\theta}$, player 2’s belief over $\Theta \times X$ is $\hat{\theta} \circ f^{-1}$, where $f(\theta) := (\theta, \sigma(\theta))$. So define:

**Definition** For $\sigma : \Theta \rightarrow X$ measurable, $\rho'( \hat{\theta})(\sigma) := \rho(\hat{\theta} \circ f^{-1})$, where $f(\theta) := (\theta, \sigma(\theta))$.

3.1.3. Joint assumptions

Both players have Lipschitz constants regulating how their best responses adjust to the actions of each other. These constants multiply to something less than 1, giving uniqueness of equilibrium.

**Assumption 3** There exist $\kappa_{P1} \in [0, \infty)$ and $\kappa_{P2} \in [0, \infty]$ with $\kappa_{P1} \cdot \kappa_{P2} < 1$, such that:

1. The difference $u_{P1}(\theta, x + dx, y) - u_{P1}(\theta, x, y)$ is decreasing on any line $\{(x, y) = (x_0, y_0) + \lambda (\kappa_{P1}, 1)\}$\(^{22}\), and
2. $\rho'( \hat{\theta})(\sigma)$ has Lipschitz constant $\kappa_{P2}$ in $\sigma$\(^{23}\).

The assumption is trivially satisfied if $u_{P1}$ has constant differences in $(x, y)$ (take $\kappa_{P1} = 0$ and $\kappa_{P2} = \infty$).

To guarantee that there are costly enough signals for separating equilibria to exist, we assume that no change in the respondent’s action (within $\bar{Y}$) will compensate for taking $x_{\text{max}}$ over some more moderate action $x$, even for the highest type.

**Assumption 4** $u_{P1}(\theta_{\text{max}}, x_{\text{max}}, \bar{y}) \leq \max_x u_{P1}(\theta_{\text{max}}, x, \bar{y})$

\(^{21}\)This corresponds to constant differences of $u_{P2}$ in $(\theta, y)$.

\(^{22}\)This is implied by $\left( \kappa_{P1} \frac{\partial}{\partial x} \right)^2 + \frac{\partial}{\partial x y} u_{P1} \leq 0$.

\(^{23}\)This would be given by a condition on $u_{P2}$ symmetric to that on $u_{P1}$. This is proved in Section B.4.
3.1.4. The strength of the supermodularity assumptions

The supermodular assumptions model various directional strategic interactions. Allowing the technical condition that payoff differences in all pairs of variables are weakly monotonic, how constraining are these directional assumptions? There are a number of constraints, but we can also choose how we order $\Theta$ and $X$. The ordering of $Y$ is set by the assumption that higher $y$ benefits the signaler.

Start with the special case $u_P^1(\theta, x, y) = u_X(\theta, x) + y$, assuming pure reputation ($\rho$ independent of $\theta$). Here there are no directional constraints. Higher $x$ leads the respondent to choose higher $y$, which sets the $X$ direction, and higher $\theta$ leads the signaler to value higher $x$, which sets the $\Theta$ direction. Beyond this, we allow for $\rho$ to depend on $\theta$. The direction is constrained: $\rho$ must be weakly increasing in $\theta$. This happened in the work incentives example we considered in Section 2.4.2, where we assumed that higher ability generated higher quality of work in addition to higher quantity.

We also allow for strategic interaction of $(x, y)$ in $u_1$; again the direction is constrained: higher $y$ leads to a higher choice of $x$. This happens in Cournot competition, where lower $q_P^2$ (higher $y = -q_P^2$) leads to higher $q_P^1$. Here there are constant differences in $(\theta, y)$, i.e. $(c_{P1}, q_P^2)$. In a monetary policy setting, lower unemployment will lead to a dryer policy.

We also allow for different $\theta$ to value $y$ differently. Again the direction is constrained: a higher $\theta$ gains more from a higher $y$. In the monetary policy setting, higher $\theta$ means a tougher central banker, who may care more about higher $y$ meaning lower inflation.

If these assumptions are not all satisfied, the procedure and results below may still hold, but they are not guaranteed.

3.2. Strategies and equilibrium

At any point in the game there is a commonly-known belief $\hat{\theta}$ by the respondent about the signaler’s current type $\theta$. Strategies in the continually separating equilibrium we will study will be Markov. The respondent’s strategy will be $s_{P2}(t): \Delta \Theta \to Y$; the signaler’s strategy $s_{P1}(t): \Delta \Theta \times \Theta \to X$; and the belief updating rule $\beta(t): \Delta \Theta \times X \to \Delta \Theta$.

We will have a value function $V_t(\hat{\theta}, \theta)$ describing utility for the signaler from period $t$ on. In a perfect Bayesian Nash equilibrium, $\beta$ is a Bayesian update, $s_{P2}$ is
a myopic best response, and $x = s_{P_1}(t) (\hat{\theta}, \theta)$ maximizes $u_{P_1} (\theta, x, s_{P_2}(t)(\hat{\theta})) + \delta_{P_1} \cdot \int V_{t+1} (\theta_{t+1}, \psi (\beta(\hat{\theta}, x))) \, d\psi(\theta)(\theta_{t+1}).$

4. VALUE FUNCTION ITERATION

General supermodular $u_{P_1}$ will result in supermodular value functions $V_t (\hat{\theta}, \theta)$. Again, we shall work with a reduced value function $v^\sim_t (\theta', \theta) := V_t (\psi(\theta'), \theta) - V_t (\psi(\theta_{\min}), \theta)$, the gain to type $\theta$ from having previously signaled $\theta'$ compared with having previously signaled $\theta_{\min}$. This simplifies the value function and allows easy computation.

Again we have value function iteration described by a map $F$ from a set $\Pi$ of value functions into itself. This again decomposes into a number of component maps, a larger number than in Section 2 since there is a new strategic interaction to deal with.

**Definition** $B_V := \max_x u_{P_1} (\theta_{\max}, x, \bar{y}) - u_{P_1} (\theta_{\max}, x_{\max}, \bar{y})$

**Definition** Let $\Pi$ be the set of functions $\nu^\sim : \Theta \times \Theta \rightarrow \mathbb{R}$

1. weakly increasing, with $\nu^\sim (\theta_{\min}, \theta) = 0$ and $\nu^\sim (\theta_{\max}, \theta_{\max}) \leq B_V$,
2. weakly supermodular,
3. with $\nu^\sim (\theta', \theta)$ continuous in $\theta$.

Again, the bound $B_V$ ensures the existence of high enough signals to allow separating equilibria.

Figure 4 collects definitions and gives a commutative diagram of all of the maps involved in the analysis.

4.1. The current value of signaling

Suppose $\nu^\sim_{t+1} \in \Pi$. If type $\theta$ signals $\theta'$ in period $t$, in period $t + 1$ the respondent’s belief will be $\psi(\theta')$, so if the player is type $\theta_{t+1}$ next period, he gets utility $V_{t+1} (\psi(\theta'), \theta_{t+1})$ from period $t + 1$ on. So his expected payoff from period $t + 1$ from signaling $\theta'$ is the integral of this over the distribution of his types next period, $\psi(\theta)$. Noting again that adding a function of $\theta$ preserves signaling incentives, we have value of signaling:

---

24 The value function exists and has a recursive definition provided this integral exists. In general it exists provided that strategies are measurable.

25 It also gives continuity properties that are shown and used in Roddie (2011b).
\[
\delta_{P_1} \cdot \int_{V_{t+1}} \psi(\theta, \theta_{t+1}) \, d\psi(\theta) (\theta_{t+1})
\]

\[
\equiv \delta_{P_1} \cdot \int [V_{t+1} \psi(\theta, \theta_{t+1}) - V_{t+1} \psi(\theta_{\text{min}}, \theta_{t+1})] \, d\psi(\theta) (\theta_{t+1})
\]

\[
= \delta_{P_1} \cdot \int v'_t \, d\psi(\theta) (\theta_{t+1})
\]

So if \( \alpha_1 : (\theta', \theta) \mapsto \delta_{P_1} \cdot \int v'_t \, d\psi(\theta) (\theta_{t+1}) \) for any integrable function \( v \), we have a value of signaling \( v'_t = \alpha_1 (v'_{t+1}) \). In the special case considered in Section 2, \( v'_t = \delta_{P_1} \cdot v'_{t+1} \), since \( v'_{t+1} \) was constant in the second argument. Here \( v'_t \) still remains in the same space as \( v'_{t+1} \):

**Lemma** \( \alpha_1 : \Pi \to \Pi \).

**Proof:** All statements of this form are proved in Section B.1. \( Q.E.D. \)

### 4.2. Generating a signaling game, given \( y = s_{P2} (t) (\hat{\theta}_t) \)

Suppose the action \( y \in \hat{Y} \) is expected in period \( t \). Signaling incentives now depend on the expected current \( y = s_{P2} (\hat{\theta}_t) \). The value of signaling \( v'_t \) induces a static signaling game with reduced form payoff:

\[
\bar{U} (y) = \alpha_2 (v'_t) : (\theta, x, \theta') \mapsto u_{P_1} (\theta, x, y) + v'_t (\theta', \theta)
\]

Since \( u_{P_1} \) and \( v'_t \) are supermodular, so is \( \bar{U} (y) \). This implies single crossing, and the other conditions for the payoff to be in \( \Phi_B \) are also satisfied:

**Definition** Let \( S_3 \) be the set of functions \( \bar{U} : \hat{Y} \to \Phi_B \) with \( \bar{U} (y) \) weakly increasing in \( y \) and weakly supermodular.

**Lemma** \( \alpha_2 : \Pi \to S_3 \)

### 4.3. Riley equilibrium, given \( y \)

Given \( y \), the signaler plays according to the Riley equilibrium of the signaling game \( \bar{U} (y) \):

\[
\bar{U} (y)
\]
\[ s_Y = \alpha_3(\tilde{U}): (y, \theta) \mapsto \mathcal{R}(\tilde{U}(y))(\theta) \]

So \( s_Y: \tilde{Y} \times \Theta \to X \). Using comparative statics from Roddie (2011a), we can show that \( s_Y(y, \theta) \) is weakly increasing in \( y \), since \( \tilde{U}(y)(\theta, x, \theta') \) is supermodular, and that it has a Lipschitz constant in \( y \), which comes from Assumption 3.

**Definition** Let \( S_4 \) be the set functions \( s_Y: \tilde{Y} \times \Theta \to X \) with \( s_Y(y, \theta) \) weakly increasing in \( y \) and \( \theta \) and \( \kappa_{P1} \)-Lipschitz in \( y \).

**Lemma** \( \alpha_3: S_3 \to S_4 \)

### 4.4. Finding \( s_{P2}(t) \)

Suppose initial beliefs in period \( t \) are \( \psi(\theta') \). Given \( y \), the signaler takes strategy \( s_Y(y) \). The signaler takes action \( s_{P2}(t)(\theta') = s_{P2}(t)(\psi(\theta')) \). The respondent takes a best response to current expectations about the signaler’s type and action. So \( s_{P2}(t)(\theta') = \rho'(\psi(\theta'))(s_Y(y)) \), when \( y = s_{P2}(t)(\theta') \). So \( s_{P2}(t)(\theta') \) is the fixed point of the map \( \rho'(\psi(\theta')) \circ s_Y \).

Let \( M = \alpha_4(s_Y) \) be the map \( \theta' \to (\rho'(\psi(\theta')) \circ s_Y) \). Then for each \( \theta' \), \( M(\theta'): \tilde{Y} \to \tilde{Y} \) has Lipschitz constant \( \kappa_{P1} \cdot \kappa_{P2} < 1 \):

**Definition** Let \( S_5 \) be the set of functions \( M: \Theta \to \tilde{Y} \to \tilde{Y} \) with \( M(\theta')(y) \) weakly increasing in \( \theta' \) and \( y \) and \( \kappa_{P1} \cdot \kappa_{P2} \)-Lipschitz in \( y \).

**Lemma** \( \alpha_4: S_4 \to S_5 \)

Then \( M(\theta'): \tilde{Y} \to \tilde{Y} \) is a contraction, so has the unique fixed point \( s_{P2}^*(\theta') \), which is weakly increasing in \( \theta' \) since \( M \) is weakly increasing.\(^{26}\) Let \( \alpha_5(M): \Theta \to \tilde{Y}, \theta' \mapsto \text{FixedPoint}(M(\theta')) \). Then \( s_{P2}^* = \alpha_5(M) \). Collecting these results:

**Definition** Let \( S_6 \) be the set of weakly increasing functions \( \Theta \to \tilde{Y} \).

**Lemma** \( \alpha_5: S_5 \to S_6 \)

\(^{26}\)Without the Lipschitz assumptions on \( u_1 \) and \( u_2 \), there exists a fixed point (assuming \( \rho \) is continuous) but it may not be unique. Taking the highest or lowest equilibrium, for example, will preserve the necessary comparative for subsequent results.
4.5. Preserving a supermodular value function

The signaling game $\tilde{U}(y)(\theta, x, \theta')$ gives has Riley equilibrium $s_Y(y)$, and this gives value function $W = \beta_3(\tilde{U})$ with $W(y, \theta) = \tilde{U}(y)(\theta, s_Y(y)(\theta), \theta)$. We know that $\tilde{U}(y)(\theta, x, \theta')$ is supermodular, and by Roddie (2011a) $W(y, \theta)$ is also supermodular. In particular that makes $w(y, \theta) := W(y, \theta) - W(y, \theta)$ supermodular.

It also satisfies the bound $w(\bar{y}, \theta_{\text{max}}) \leq B_V$; to show this we need to use the information that $\tilde{U} = \alpha_2(v'_t)$, not just $\tilde{U} \in S_3$. So to formalize this conclusion, we define a map $\beta_2$ from $S_2$, so that $w = \beta_2(v'_t)$:

DEFINITION Let $\beta_2 = \alpha_2 \circ \beta_3$, where $\beta_3(\tilde{U}) : (y, \theta) \mapsto W(y, \theta) - W(y, \theta)$, where $W(y, \theta) := \tilde{U}(y)(\theta, \mathcal{R}(\tilde{U}(y))(\theta), \theta)$. Let $S_7$ be the set of weakly increasing, weakly supermodular functions $w : \hat{Y} \times \Theta \to \mathbb{R}$, continuous in $\theta$, with $w(y, \theta) = 0$ and $w(\bar{y}, \theta_{\text{max}}) \leq B_V$.

LEMMA $\beta_2 : S_2 \to S_7$

Since we know $s_{P2}$, we have the value function in period $t$: $v^*_t(\theta', \theta) = w(s_{P2}(\theta'), \theta)$. Equivalently $v^*_t = \alpha_{(6,7)}(s_{P2}, w)$ where $\alpha_{6,7}(s_{P2}, w) : (\hat{\theta}, \theta) \mapsto w(s_{P2}(\theta'), \theta)$. It follows from the properties of $w$ and the fact that $s_2$ is weakly increasing in $\theta'$, that $v^*_t \in \Pi$:

LEMMA $\alpha_{(6,7)} : S_6 \times S_7 \to \Pi$

PROOF: It is immediate that $v^*_{t}$ is supermodular, weakly increasing in $\theta'$, and Lipschitz in $\theta$. The bound condition also follows immediately from that on $w$: $v^*_t(\theta_{\text{max}}, \theta_{\text{max}}) \leq w(\bar{y}, \theta_{\text{max}}) \leq B_V$ \hfill \(\text{Q.E.D.}\)

4.6. Functional decomposition

Figure 4 shows the logic of the text above in a commutative diagram\textsuperscript{27}, defining the value function iterator $F$ as the composition of individual operations.

5. THE DYNAMIC RILEY EQUILIBRIUM

5.1. Equilibrium

As in Section 2, the value function in period $k + 1$ (after the game is over) is the zero function $0 : \theta' \mapsto 0$, and $0 \in \Pi$. Iteration gives us:

\textsuperscript{27}That $\alpha_{(6,7)}$ takes two arguments is non-standard but unproblematic.
Sets and functions

$\Pi$: Weakly increasing, weakly supermodular functions $v(\theta', \theta)$, continuous in $\theta$, with $v^{-}(\theta_{\min}, \theta) = 0$ and $v^{-}(\theta_{\max}, \theta_{\max}) \leq B_{V}$.

$S_{3}$: Functions $\tilde{U} : \tilde{Y} \to \Phi_{B}$, with $\tilde{U}(y)(\theta, x, \theta')$ weakly increasing in $y$, weakly supermodular.

$S_{4}$: Weakly increasing functions $s_{Y}(y, \theta), \kappa_{p1}$-Lipschitz in $y$.

$S_{5}$: Weakly increasing functions $M(\theta')(y), \kappa_{p1} \cdot \kappa_{p2}$-Lipschitz in $y$.

$S_{6}$: Weakly increasing functions $\theta \to \tilde{Y}$.

$S_{7}$: Weakly increasing, weakly supermodular functions $w(y, \theta)$, continuous in $\theta$, with $w(\tilde{y}, \theta) = 0$ and $w(\tilde{y}, \theta_{\max}) \leq B_{V}$.

$\alpha_{1}(v_{t+1}^{-}) : (\theta', \theta) \to \delta_{p1} \cdot f v_{t+1}^{-}(\theta', \theta_{t+1}) \cdot \Delta(\psi(\theta))(\theta_{t+1})$

$\alpha_{2}(v_{t}') : y \to (\theta, x, \theta') \to u_{p1}(\theta, x, y) + v_{t}'(\theta', \theta)$

$\alpha_{3}(f) = \mathcal{R} \circ f$

$\beta_{2} = \alpha_{2} \circ \beta_{3}$, where $\beta_{3}(U) : (y, \theta) \to \bar{U}(y)(\theta, \mathcal{R}(\bar{U}(y))(\theta), \theta) - \bar{U}(y)(\theta, \mathcal{R}(\bar{U}(y))(\theta), \theta)$

$\alpha_{4}(s_{Y}) : \theta' \to \rho'(\psi(\theta')) \circ s_{Y}$; $\alpha_{5}(M) : \theta' \to \text{FixedPoint}(M(\theta'))$

$\alpha_{6,7}(w, s_{p2}) : (\theta', \theta) \to w(s_{p2}^{-}(\theta'), \theta) - w(s_{p2}^{-}(\theta_{\min}), \theta)$

Figure 4: Value function iteration: decomposition
\[ v_i^\sim = F^{k+1-t}(0) \]
\[ s_Y(t) = (\alpha_3 \circ \alpha_2 \circ \alpha_1)(v_{i+1}^\sim) \]

So we have strategies:

\[ s_{P_2}^\sim(t) = (\alpha_5 \circ \alpha_4)(s_Y(t)) \]
\[ s_{P_1}^\sim(t)(\theta', \theta) = s_Y(s_{P_2}^\sim(t)(\theta'), \theta) \]

As in Section 2, for finite types, \( v_i^\sim \) is naturally represented by an array of real numbers, and so iteration of \( F \) allows us to compute \( v_i^\sim \) for finite types and games of finite length. A functional computer program that does this can be almost identical in form to the definitions given above.

5.1.1. Complete strategies

Again, this only tells us strategies when (the respondent’s) beliefs are of the form \( \psi(\theta') \). To find them for general beliefs (which is necessary off the equilibrium path and in the first period), modify \( \alpha_4 \) and \( \alpha_5 \) to the following functions:

\[ \hat{\alpha}_4(s_Y) : \hat{\theta} \mapsto \rho'(\hat{\theta}) \circ s_Y \]
\[ \hat{\alpha}_5(M) : \hat{\theta} \mapsto \text{FixedPoint}(M(\hat{\theta})) \]

The full strategies are then:

\[ s_{P_2}^\sim(t) = (\hat{\alpha}_5 \circ \hat{\alpha}_4)(s_Y(t)) \]
\[ s_{P_1}^\sim(t)(\hat{\theta}, \theta) = s_Y(s_{P_2}^\sim(t)(\hat{\theta}), \theta) \]

5.2. Strict signaling incentives

Here two conditions are given that guarantee strict signaling incentives. What we want to guarantee is that \( s_{P_2}^\sim(t)(\psi(\theta')) \) is strictly increasing in \( \theta' \). To ensure this we need to know some strictness properties of \( \rho \).

**Definition** Given \( \mu^1, \mu^2 \in \Delta(\Theta \times X) \), let the marginals be \( \mu_\Theta^i := \mu^i(\cdot \times X) \) and \( \mu_X^i := \)
\( \mu^i (\Theta \times \cdot) \). \( \rho \) is “strict in \( \Theta \)” if for any \( \mu^1 \leq \mu^2 \) with \( \mu^2_{\Theta} > \mu^1_{\Theta}, \rho (\mu^1) < \rho (\mu^2) \). \( \rho \) is “strict in \( X \)” if for any \( \mu^1 \leq \mu^2 \) with \( \mu^2_X > \mu^1_X, \rho (\mu^1) < \rho (\mu^2) \).\(^{28}\)

The first condition that guarantees strict incentives is that \( \rho \) strict in \( \Theta \): this is the combined case where the respondent cares about types in addition to actions. Then \( \rho' (\hat{\theta}) (\sigma) \) is strictly increasing in \( \hat{\theta} \) for any \( \sigma \in \text{Inc}(\Theta, X) \). Since \( \psi \) is strictly increasing, signaling higher \( \theta' \) in period \( t \) results in a strictly increased best response to any fixed strategy, so results in a strictly increased \( s_{P2} (\psi (\theta')) \) in the next period.

The second condition that gives strict incentives guarantees that the signaler always takes a strategy that is non-constant in \( \theta \) (i.e. the lowest types and the highest types take different signals), and that signaling a higher type reduces the chance of being one of the lowest types and increases the chance of being one of the highest types. Then reputational considerations alone imply strict signaling incentives, provided that \( \rho \) is strict in \( X \) (the normal case). For this argument a condition is made on the Markov process, so that as the type \( \theta \) increases, weight in \( \psi (\theta) \) is increasing on any upper interval. This condition almost requires full support of \( \psi (\theta) \), and is weaker than a monotone likelihood ratio property.\(^{29}\)

**Definition** If \( \hat{\theta}_1, \hat{\theta}_2 \in \Delta \Theta \), \( \hat{\theta}_2 \) strongly dominates \( \hat{\theta}_1 \) (\( \hat{\theta}_1 \ll \hat{\theta}_2 \)) if \( \hat{\theta}_1 (S) > \hat{\theta}_2 (S) \) whenever \( S = [\theta_{\min}, \theta^*] \cap \Theta \), \( \theta^* < \theta_{\max} \).

We have the result:

**Definition** “Strict signaling assumptions” hold if:

1. \( \rho \) is strict in \( \Theta \), or
2. \( \rho \) is strict in \( X \), \( \psi (\theta_1) \ll \psi (\theta_2) \) for \( \theta_1 < \theta_2 \), and \( \text{BR}_{P1} (\theta_{\min}, y) < \text{BR}_{P1} (\theta_{\max}, y) \) for all \( y \).\(^{30}\)

**Proposition 1** Under strict signaling assumptions, for any \( t \leq k \), and \( s_{P2} (t) (\psi (\theta')) \) is strictly increasing in \( \theta' \).

\(^{28}\)These conditions are implied by \( [\bar{y}, \bar{y}] \subseteq (y_{\min}, y_{\max}), \) and respectively \( \frac{\partial^2 u_{P2}}{\partial \theta \partial y} > 0 \) and \( \frac{\partial^2 u_{P2}}{\partial x \partial y} > 0 \).

\(^{29}\)But it also requires that \( \hat{\theta}_2 \) places positive weight on \( \theta_{\min} \). The condition can weakened to remove this aspect if a continuity assumption is made on \( \psi \) at \( \theta_{\min} \).

\(^{30}\)This must hold if \( u_{P1} \) is \( C_2 \) with \( \frac{\partial^2}{\partial \theta \partial x} u_{P1} > 0 \) and \( \left( \frac{\partial}{\partial x} \right)^2 u_{P1} < 0 \), and \( \text{BR}_{P1} (\theta_{\max}, y) > x_{\min} \).
If there are strict signaling assumptions, the signaler’s strategy must be separating in any non-final period:

**Corollary** For \( t < k \), \( s_{P1}(t)(\hat{\theta}, \theta) \) is strictly increasing in \( \theta \).

**Proof:** It follows from Proposition 1 that the derived signaling payoff in period \( t \) is strictly increasing in signaled type. So there can be no pooling in a weakly separating equilibrium. \( Q.E.D. \)

5.3. *Computed example: Cournot competition*

Imagine a firm with unknown cost is engaging in dynamic Cournot competition with other firms with known costs. This situation falls within the phenomenon of “limit pricing” as that has come to be understood, even though it doesn’t involve entry deterrence. This is not a situation that fits the additive separable framework, since the static best response of each firm depends on the expected outputs of the other firms.

Take a linear inverse demand function \( P(Q) = \max(6 - 2Q, 0) \), and let profits for firm \( i \in \{P1, P2\} \) be \( \Pi_i = (P(q_i + q_{-i}) - c_i)q_i \). Let \( c_{P2} = 1 \) and \( c_{P1} \) vary from 4 to 0. The signaler, \( P1 \), increases production to signal lower costs (Figure 5.b). Again, with a long horizon and a patient signaler, the firm gains Stackelberg leadership.\(^{31}\)

\(^{31}\)Unlike Section 2.4.2, there is no gap caused by separation by the low (high cost) type: the reason is that for high cost types the Stackelberg quantities are at the lower bound, so that the highest-cost type would not want to mimic a type producing a slightly higher quantity.
5.4. Equilibrium Refinement

As in Section 2.5, selection of the dynamic Riley equilibrium is given by selection of the Riley equilibrium of each induced signaling game. Suppose Markov strategies $s^*_P$, $s^*_P$ and beliefs $\beta^*$ constitute a perfect Bayesian equilibrium. Then they give value function $V^*$. Then in period $t$ the induced the signaling game is $(\theta, x, \hat{\theta}) \rightarrow u_{P_1}(\theta, x, s^*_{P_2}(t)(\hat{\theta}_t)) + \hat{\alpha}_1(V^*_{t+1})(\hat{\theta}, \theta)$, where, provided $V$ is integrable in the second argument:

$$\hat{\alpha}_1(V)(\hat{\theta}, \theta) := \delta_{P_1} \cdot \int V(\psi(\hat{\theta}), \theta_{t+1}) d[\psi(\theta)](\theta_{t+1})$$

We shall assume that the equilibrium $s^*_{P_1}$ of this signaling game satisfies D1. Formally:

**Definition** Perfect Bayesian strategies $s^*_{P_1}$, $s^*_{P_2}$, beliefs $\beta^*$ and associated value function $V^*$ satisfy recursive D1 if at each $t$ and $\hat{\theta}_t$, provided $V^*_{t+1}(\hat{\theta}, \cdot)$ is integrable, the strategy $s^*_{P_1}(t)$ and updating rule $\beta'(t)(\hat{\theta}_t, \cdot)$ satisfy D1 for the signaling game, $(\theta, x, \hat{\theta}) \rightarrow u_{P_1}(\theta, x, s^*_{P_2}(t)(\hat{\theta}_t)) + \hat{\alpha}_1(V^*_{t+1})(\hat{\theta}, \theta)$, with initial beliefs $\hat{\theta}_t$.

One can see D1 as viewing the signaler as having a small (infinitesimal) probability of making errors about the belief that results from a given signal, with a larger error being infinitely less likely than a smaller one. In this interpretation, recursive D1 involves the possibility of making this type of error, while the future value function is known and unmistakable. In taking the action $x$ in period $t$, the potential mistake is about the interpretation of $x$, not about the interpretation of any future signals, or about the Markov process or the current action $s_{P_2}(t)(\hat{\theta}_t)$ of the respondent.

By Roddie (2011a), this refinement will imply that $s^*_{P_1}$ is the Riley equilibrium of this game, provided we can guarantee the technical conditions of full support of the initial distribution at time $t$, continuity of the signaling payoff, and strict monotonicity in $\hat{\theta}$.

This allows us to guarantee inductively that $s^*_{P_1}$ and $s^*_{P_2}$ are the dynamic Riley equilibrium strategies $s_P$ and $s_{P_2}$, as defined in 5.1.1.

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32The assumption of Markov strategies is given for notational simplicity, but if strategies and beliefs (and therefore value functions) were allowed to depend also on the observed history, the above argument would still apply, showing recursively that the strategies are Markov, not depending on these extra variables.
Suppose \( \Theta \) is finite, \( \rho \) is continuous, strict signaling assumptions hold, and \( \psi(\theta) \) has full support for each \( \theta \). If \( (s^*_p, s^*_p, \beta^*) \) is a perfect Bayesian equilibrium satisfying recursive D1, \( s^*_p(\hat{\theta}, \theta) = s^*_p(\hat{\theta}, \theta) \) and \( s^*_p(\hat{\theta}, \theta) = s^*_p(\hat{\theta}, \theta) \) whenever \( \hat{\theta} \) has full support.

6. MODELING ISSUES AND EXTENSIONS

Dynamics, history independence and the random process on types

In the dynamic Riley equilibrium, actions only have a one-period effect, since the subsequent period will reveal the true type. This property follows from continual separation and the assumption of type being a Markov process.

In reality reputations are not always built up and lost instantly. If a string of good signals is followed by a bad signal, the bad signal may reflect a permanent change, or a temporary shock. This suggests allowing for higher order Markov processes, dependent on the last \( k \) realizations, for example a moving average process.

Another approach to capture history dependence would be to relax separation. But even static signaling games become intractable outside of perfect observability and separation. So the previous approach would seem the more productive way to capture history dependence.

Constant stage game

In the model the only thing that varies over time is type and type beliefs. In reality the situation in which the players are acting is often not constant. A central bank’s policy is influenced by a large number of changing and uncertain variables and will be very different in a recession from policy in a boom.

It is easy to accommodate the model to allow for this. For example stage game payoffs may depend on a state variable \( \xi_t \), randomly evolving over time, observable at the beginning of period \( t \). Assumptions on payoffs should hold across all possible values of \( \xi_t \). The value function \( V_t(\hat{\theta}_t, \theta_t, \xi_t) \), or if \( \xi_t \) is i.i.d. \( V_t(\hat{\theta}_t, \theta_t) \), can be iterated in the same way, increasing differences in \( (\hat{\theta}_t, \theta_t) \) and monotonicity in \( \hat{\theta}_t \) being preserved as before. So the dynamic Riley equilibrium and associated methods carry over to this setting.

If \( \xi_t \) is i.i.d., then the amount of signaling done - measured by cost - is independent of \( \xi_t \), which is generally not be the type of strategy that the signaler would want to
commit to. So there may be over-signaling in states where reputation is less valuable (good economic states in the monetary policy example) and under-signaling in states where it is more valuable (bad states).

Order of moves

The analysis has dealt with the case of simultaneous moves. If the signaler moves first in the stage game, we have a repetition of the static outcome, so there is no question of reputation. The remaining case is where the signaler moves second, as in Vincent (1998). Under additive separability \( u_{P1} = u_X(\theta, x) + u_Y(\theta, y) \) this is identical to the simultaneous moves case. But in general, this case can be dealt with using the same techniques as for simultaneous games\(^3\), and there are continually separating equilibria with permanent signaling incentives. Unlike the simultaneous moves case, simple limit characterizations are not known and may not exist.

Dimensionality of A

Often there is more than one way to signal the same variable. For example a worker may signal ability through productivity but also through achieving qualifications. If the action set \( A \) of the signaler is subset of \( \mathbb{R}^n \) rather than \( \mathbb{R} \), assumptions would be need for Riley equilibria to be increasing in \( \theta \) the Riley equilibrium of \( \tilde{U}(y) \) to be increasing in \( y \). It is conjectured that while the amount of signaling may have the optimality properties we have seen, the (directional) choice of signal may be suboptimal: for example too much effort may go into qualifications.

Two-way uncertainty

In symmetric settings, both players may have similar incentives to signal. This includes dynamic oligopoly, studied in Mester (1992). Work incentives in teams is another example. Theoretical and computational work is in progress to understand this setting.

\(^3\)The main new issue is guaranteeing that \( s_{P2}(\hat{\theta}) \) is weakly increasing. This would follow from an increasing difference property in \( y \) and \( \theta \) subject to the signaler choosing \( s_Y(y, \theta) \).
APPENDIX A: KEY

Variables and sets

<table>
<thead>
<tr>
<th>θ ∈ Θ ⊆ [θ_min, θ_max]</th>
<th>stage-game type</th>
</tr>
</thead>
<tbody>
<tr>
<td>x ∈ X = [x_min, x_max]</td>
<td>signaler’s (P1’s) action</td>
</tr>
<tr>
<td>y ∈ Y = [y_min, y_max]</td>
<td>respondent’s (P2’s) action</td>
</tr>
<tr>
<td>t ∈ Z⁺, t &lt; k</td>
<td>time index</td>
</tr>
<tr>
<td>σ : Θ → X ∈ Inc(Θ, X)</td>
<td>Weakly increasing functions</td>
</tr>
<tr>
<td>U : Θ × X × Θ → R ∈ Φ_B</td>
<td>static signaling payoff function</td>
</tr>
<tr>
<td>ˆu : Θ × X × ΔΘ → R ∈ Φ⁺</td>
<td>fully specified signaling payoff</td>
</tr>
<tr>
<td>V_t : ΔΘ × Θ → R</td>
<td>value function</td>
</tr>
<tr>
<td>ˆu_t : Θ × Θ → R ∈ Π</td>
<td>reduced value function</td>
</tr>
</tbody>
</table>

Main functions

<table>
<thead>
<tr>
<th>u_Pi : Θ × A × R → R</th>
<th>stage-game payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>δ_Pi ∈ R⁺</td>
<td>discount factor</td>
</tr>
<tr>
<td>ψ : Θ → ΔΘ</td>
<td>Markov process on type</td>
</tr>
<tr>
<td>BR_Pi : Θ × R → X</td>
<td>(myopic) best response of 1</td>
</tr>
<tr>
<td>ρ : Δ(Θ × X) → Y</td>
<td>best response of 2</td>
</tr>
<tr>
<td>ρ' : ΔΘ → Inc(Θ, X) → Y</td>
<td>best response of 2 to a type-belief and strategy</td>
</tr>
<tr>
<td>˜R : Φ_B → Θ → X</td>
<td>Riley equilibrium</td>
</tr>
<tr>
<td>F : Π → Π</td>
<td>value function iteration</td>
</tr>
</tbody>
</table>

APPENDIX B: PROOFS

B.1. Value function iteration

B.1.1. The map α₁

**Lemma** α₁ : Π → Π.

**Proof:** The fact that ˆu_t(θ', θ) is weakly increasing θ', continuous in θ, weakly supermodular, and equal to 0 when θ' = θ_min, follow directly from the same properties of V. The bound condition also follows, using supermodularity of ˆu_{t+1}.
\[ v'_t(\theta_{\text{max}}, \theta_{\text{max}}) \]
\[ = \delta_1 \int (v'_{t+1}(\theta_{\text{max}}, \theta_{t+1}) - v'_{t+1}(\theta_{\text{min}}, \theta_{t+1})) \, d(\psi(\theta_{\text{max}}))(\theta_{t+1}) \]
\[ \leq \int (v'_{t+1}(\theta_{\text{max}}, \theta_{t+1}) - v'_{t+1}(\theta_{\text{min}}, \theta_{t+1})) \, d(\psi(\theta_{\text{max}}))(\theta_{t+1}) \]
\[ \leq \int (v'_{t+1}(\theta_{\text{max}}, \theta_{\text{max}}) - v'_{t+1}(\theta_{\text{min}}, \theta_{\text{max}})) \, d(\psi(\theta_{\text{max}}))(\theta_{t+1}) \]
\[ = v'_{t+1}(\theta_{\text{max}}, \theta_{\text{max}}) \]
\[ \leq B \]
\[ \text{Q.E.D.} \]

B.1.2. The map \( \alpha_2 \)

**Lemma**: \( \alpha_2 : \Pi \rightarrow S_3 \)

**Proof**: \( \bar{U} \) is weakly increasing in \( y \) since \( u_{p_1}(\theta, x, y) \) is increasing in \( y \).

Claim: \( \bar{U}(y) \in \Phi_B \)

**Proof**: Supermodularity implies that \( \bar{U}(y) \) satisfies single crossing: more precisely single crossing is implied by weakly increasing differences in \( (\theta, y) \) and strictly increasing differences in \( (\theta, x) \), which follows from strictly increasing differences of \( u_{p_1} \) in \( (\theta, x) \).

\( v_t'(\theta', \theta) \) is weakly supermodular and continuous in \( \theta \), and therefore uniformly continuous in \( \theta \).

Strict quasi-concavity in \( x \) is implied by strict concavity of \( u_{p_1} \) in \( x \), uniform continuity in \( x \) by uniform continuity of \( u_{p_1} \) in \( x \) and uniform continuity in \( \theta \) by uniform continuity of \( u_{p_1} \) and \( v'_t \) in \( \theta \). That \( \bar{U}(y)(\theta, x, \theta') \) is weakly increasing in \( \theta' \) is implied by \( v'_t(\theta', \theta) \) being weakly increasing in \( \theta' \). Given \( y \), \( \bar{U}(y)(\theta, x, \theta_1) = \bar{U}(y)(\theta, x, \theta_2) \) if and only if \( v'_t(\theta_1, \theta) = v'_t(\theta_2, \theta) \), which is not a function of \( x \).

Finally the bound condition holds. Since \( v'_t \in \Pi \), using the assumed inequality and supermodularity, for \( y \in \bar{Y} \):

\[ v'_t(\theta_{\text{max}}, \theta_{\text{max}}) \leq \max_x u_{p_1}(\theta_{\text{max}}, x, \bar{y}) - u_{p_1}(\theta_{\text{max}}, x_{\text{max}}, \bar{y}) \]
\[ \leq \max_x u_{p_1}(\theta_{\text{max}}, x, y) - u_{p_1}(\theta_{\text{max}}, x_{\text{max}}, y) \]

So:

\[ \bar{U}(y)(\theta_{\text{max}}, x_{\text{max}}, \theta_{\text{max}}) = u_{p_1}(\theta_{\text{max}}, x_{\text{max}}, y) + v'_t(\theta_{\text{max}}, \theta_{\text{max}}) \]
\[ \leq \max_x u_{p_1}(\theta_{\text{max}}, x, y) \]
\[ = \max_x \bar{U}(y)(\theta_{\text{max}}, x, \theta_{\text{min}}) \]

**End**

\[ \text{Q.E.D.} \]
B.1.3. The map $\alpha_3$

We use two results in Roddie (2011a).

**Proposition (Roddie (2011a), Corollary 19)** Let $Z \subseteq \mathbb{R}$ and $\tilde{U} : Z \to \Phi_R$. If $\tilde{U}(z)(\theta,x,\theta')$ is weakly supermodular, then $\mathcal{R}(\tilde{U}(z))$ is weakly increasing.

**Proposition (Roddie (2011a), Proposition 20)** Let $Z \subseteq \mathbb{R}$ and $\tilde{U} : Z \to \Phi_{\text{Sep}}$. Suppose that for $d > 0$, the difference $\tilde{U}(z)(\theta,x+d,\theta') - \tilde{U}(z)(\theta,x,\theta')$ is decreasing on any line $\{(x,z) = (a_0 + \alpha \cdot \lambda, z_0 + \lambda)\}$. Then for $z_0 < z_1, \mathcal{R}(\tilde{U}(z_1))(\theta) - \mathcal{R}(\tilde{U}(z_0))(\theta) \leq \alpha \cdot (z_1 - z_0)$.

**Lemma** $\alpha_3 : S_3 \to S_4$

**Proof:** Taking $Z = \tilde{Y}$ in the first proposition, we have $s_Y(y)(\theta)$ is increasing in $y$. Assumption 3 implies that $\tilde{U}(z)(\theta,x+d,\theta') - \tilde{U}(z)(\theta,x,\theta')$ is decreasing on $\{(x,z) = (a_0 + \kappa_{P_1} \cdot \lambda, z_0 + \lambda)\}$, and so the second proposition implies that $s_Y(y)(\theta)$ has Lipschitz constant $\kappa_{P_1}$ in $y$. Q.E.D.

B.1.4. The map $\alpha_4$

**Lemma** $\alpha_4 : S_4 \to S_5$

**Proof:** It remains to be shown that fixing $\theta', \Phi(\theta')(y) = \rho'(\psi(\theta'))(s_Y(y))$ is $\kappa_{P_1} \cdot \kappa_{P_2}$-Lipschitz in $y$. This follows from the facts that $s_Y$ has Lipschitz constant $\kappa_{P_1}$, and $\rho'(\tilde{\theta})$ has Lipschitz constant $\kappa_{P_2}$. Q.E.D.

**Proposition (Roddie (2011a))** Let $R$ be an interval and $\tilde{U} : Z \to \Phi_{\text{Sep}}$. Suppose that $\tilde{U}(z)(\theta,x,\theta')$ is weakly (strictly) increasing in $\theta$ and weakly supermodular. Let $W(z,\theta)$ be the utility of type $\theta$ in the Riley equilibrium: $W(z,\theta) := \tilde{U}(z)(\theta,\mathcal{R}(\tilde{U}(z))(\theta),\theta)$. Then $W(z,\theta)$ is weakly (strictly) increasing in $z$ and weakly supermodular.

B.1.5. The map $\beta_2$

**Lemma** $\beta_2 : S_2 \to S_7$

**Proof:** Let $\hat{U} = a_2(v'_1)$. Then $\beta_2(v'_1) = \beta_3(\hat{U})$. We know $\hat{U} \in S_3$. Weak supermodularity follows from the proposition above, and the fact that $w$ is weakly increasing in $y$. $w(y,\theta) = 0$ is trivial. That $w$ is weakly increasing in $\theta$ then follows from weak supermodularity.

Continuity in $\theta$ follows from the fact that any weakly separating equilibrium $f$ of any $\hat{U} \in \Phi_R$ must give rise to a continuous resulting payoff function $\theta \to \hat{U}(\theta,f(\theta),\theta)$. In particular $\hat{U}(y)(\theta,\mathcal{R}(\hat{U}(y))(\theta),\theta)$ is continuous for any $y$. So $w$ is continuous in $\theta$.

It remains to show the bound holds. Here we need more than $\hat{U} \in S_3$ and need to use the information that $\hat{U} \in \text{Im} \alpha_2$. Then $\hat{U}(y) = u_{P_1}(\theta,x,y) + v'_1(\theta',\theta)$. Let $s_Y = \alpha_3(\hat{U})$, so $s_Y(y) = \mathcal{R}(\hat{U}(y))$. Let $x_h := \ldots$

---

For this reason it may be that $\text{Im} \beta_3 \not\subseteq S_7$. 

---
Suppose \( \psi \) holds since \( \mathrm{BR}_1 \left( \theta_{\max}, \bar{y} \right) \leq \psi \left( \theta_{\max} \right) \leq x_h \). Inequalities ii. and iv. hold by increasing differences of \( u_1 \) in \( (x, y) \), and iii. holds by Assumption 4. \( \text{Q.E.D.} \)

**B.2. Proof of Proposition 1**

**Proposition** If condition 1 or or condition 2 holds, for any \( t \leq k \), and \( s_{P_2} (t) \left( \psi \left( \theta' \right) \right) \) is strictly increasing in \( \theta' \).

**Proof:** The implication from condition 1 has been covered already.

Suppose that condition 2.b) holds. By continuity, \( \mathrm{BR}_1 \left( \theta_{\min}, y \right) < \mathrm{BR}_1 \left( \theta^*, y \right) \) for some \( \theta^* < \theta_{\max} \). Suppose \( \theta_1 < \theta_2 \). Let \( y = s_{P_2} \left( t \right) \left( \psi \left( \theta_1 \right) \right) \) and let \( \sigma \left( \theta \right) := s_{P_1} \left( t \right) \left( \psi \left( \theta_1 \right), \theta \right) \).

\[
\sigma \left( \theta_{\min} \right) = \mathrm{BR}_1 \left( \theta_{\min}, y \right) < \mathrm{BR}_1 \left( \theta^*, y \right) \leq \sigma \left( \theta^* \right)
\]

Take \( x^* \in (\sigma \left( \theta_{\min} \right), \sigma \left( \theta^* \right)) \). By assumption \( \psi \left( \theta_1 \right) \ll \psi \left( \theta_2 \right) \). So if \( S := \sigma^{-1} \left( (x^*, \bar{x}_{\max}) \right) \), then \( \psi \left( \theta_1 \right) \left( S \right) < \psi \left( \theta_2 \right) \left( S \right) \). So \( \psi \left( \theta_2 \right) \) induces a different distribution over \( X \) from \( \psi \left( \theta_1 \right) \), given \( \sigma \). Since \( \rho \) is strict in \( X \):

\[
s_{P_2} \left( t \right) \left( \psi \left( \theta_1 \right) \right) = \rho' \left( \psi \left( \theta_1 \right) \right) \left( s_{P_1} \left( t \right) \left( \psi \left( \theta_1 \right), \cdot \right) \right) < \rho' \left( \psi \left( \theta_2 \right) \right) \left( s_{P_1} \left( t \right) \left( \psi \left( \theta_1 \right), \cdot \right) \right) \leq \rho' \left( \psi \left( \theta_2 \right) \right) \left( s_{P_1} \left( t \right) \left( \psi \left( \theta_2 \right), \cdot \right) \right) = s_{P_2} \left( t \right) \left( \psi \left( \theta_2 \right) \right)
\]

\( \text{Q.E.D.} \)

**B.3. Proof of Proposition 2**

Supplementing dynamic Riley equilibrium strategies \( s_{P_1, P_2} \), define the value function difference

\[
\bar{v}_t \left( \bar{h}, \theta \right) := V_t \left( \bar{h}, \theta \right) - V_t \left( \psi \left( \theta_{\min} \right), \theta \right) = w \left( s_{P_2} \left( \bar{h} \right), \theta \right), \text{ where } w := \rho_2 \circ \alpha_1 \left( v_{t+1}^\sim \right).
\]

Let \( \bar{\alpha}_1 \)

**Proposition** Suppose \( \Theta \) is finite, \( \rho \) is continuous, strict signaling assumptions hold, and \( \psi \left( \theta \right) \) has
full support for each $\theta$. If $(s^*_p, s^*_p, \beta^*)$ is a perfect Bayesian equilibrium satisfying recursive D1, $s^*_p(t)(\hat{\theta}, \theta) = s^*_p(t)(\hat{\theta}, \theta)$ and $s^*_p(t)(\hat{\theta}) = s^*_p(t)(\hat{\theta})$ whenever $\hat{\theta}$ has full support.

**Proof:** Suppose $(s^*_p, s^*_p, \beta^*)$ is a perfect Bayesian equilibrium satisfying recursive D1. Let the associated value function be $V^*$, and let $\hat{\nu}_t^*(\hat{\theta}, \theta) := V_t^*(\hat{\theta}, \theta) - V_t^*(\psi(\theta_{\min}), \theta)$.

Suppose $s^*_p(t)(\hat{\theta}, \theta) = s^*_p(t)(\hat{\theta}, \theta)$ and $s^*_p(t)(\hat{\theta}) = s^*_p(t)(\hat{\theta})$ for $\hat{\theta}$ with full support for $t \geq n + 1$. This is trivial for $n \geq k - 1$, since there are no signaling incentives in period $k$, so assume $n < k - 1$. Then $\hat{\nu}_t^*(\hat{\theta}, \theta) = \hat{\nu}_t^*(\hat{\theta}, \theta)$ for $\hat{\theta}$ with full support for $t \geq n + 1$.

In period $n$, the signaler faces the signaling payoff $\{\theta, x, \hat{\theta}\} \rightarrow u_{p1}(\theta, x, s^*_p(t)(\hat{\theta})) + \alpha_1(V_{n+1}^*)(\hat{\theta}, \theta)$. Any signaled $\hat{\theta}$ results in future beliefs $\psi(\hat{\theta})$ that have full support; and so $V_{n+1}^*(\psi(\theta), \cdot) = V_{n+1}^*(\psi(\theta), \cdot)$, so $\hat{\alpha}_1(V_{n+1}^*) = \hat{\alpha}_1(V_{n+1})$. So up to addition of a function of $\theta$, the signaler signaling payoff is $\hat{u} : (\theta, x, \hat{\theta}) \rightarrow u_{p1}(\theta, x, s^*_p(t)(\hat{\theta})) + \hat{\nu}_n^*(\hat{\theta}, \theta)$, where $\hat{\nu}_n^* = \hat{\alpha}_1(\theta_{\theta}^\theta)$.

Claim: $\hat{u} \in \Phi^*$

Proof: $\hat{u}$ can be rewritten in the form $v(\theta, x, \hat{\theta}(\theta))$, where $\hat{\rho}(\hat{\theta})$ is the resulting response $\hat{\rho}(\hat{\theta}) = s^*_p(t + 1)(\theta, \theta)$. In the next period. If $y = \hat{\rho}(\hat{\theta})$, the payoff difference in period $t + 1$ for type $\theta_{t+1}$ is $w(y, \theta_{t+1})$, where $w = (\beta_3 + \alpha_2 \circ \alpha_1)(V_{t+2})$. Taking $v(\theta, x, y) = u_{p1}(\theta, x, y) + w' (y, \theta), where $w'(y, \theta) = \delta_{p1} \cdot f w(y, \theta_{t+1}) d\psi(\theta)(\theta_{t+1}), we have $\hat{u}(\theta, x, \hat{\theta}) = \hat{v}(\theta, x, \hat{\theta})$.

Since $\Theta$ is finite, and $\rho$ is continuous, for any $\sigma \in \text{Inc}(\Theta, X), \rho'(\hat{\theta})(\sigma)$ is continuous in $\hat{\theta}$. So $s^*_p(t + 1)$, the unique fixed point of the contraction $y \rightarrow \rho'(\hat{\theta})(\sigma)(y_{t+1}(\hat{\theta})(y))$, is continuous in $\hat{\theta}$. Also $\psi$ is automatically continuous for finite $\Theta$. So $\hat{\rho}(\hat{\theta}) = s^*_p(t + 1)(\theta, \theta)$ is continuous.

Since strict signaling assumptions hold, $\hat{\rho}$ is strictly increasing.

$w$ is already known to be uniformly continuous and strictly increasing in $y$ (Section 4.5), and that implies the necessary properties for $v$. **End**

So by Roddie (2011a), D1 selects the Riley equilibrium of this game in period $t$, provided $\hat{\theta}$ has full support. The fact that $y'$ must be a best response to the signaler’s strategy implies that $y^* = s^*_p(t)(\hat{\theta})$ is the dynamic Riley response.

So the statement is true for $t = n$, and by induction, for all $t \in \{1, \ldots, k\}$. **Q.E.D.**

**B.4. Lipschitz constant for the respondent**

**Claim** Suppose the respondent’s best response $\rho$ is given uniquely by maximizing expected utility with utility function $u_{p2}$. Suppose that the difference $u_{p2}(\theta, x, y) - u_{p1}(\theta, x, y + d y)$ is decreasing on any line $\{(x, y) = (x_0, y_0) + \lambda (x, y)\}$. Then for any $\hat{\theta}, \rho'(\hat{\theta})$ has Lipschitz constant $\kappa_{p2}$.

**Proof:** Suppose not. Then for some $s^1 : \Theta \rightarrow X$ and $s^2, y_2 - y_1 > \kappa_{p2} \cdot \|s^2 - s^1\|$, where $y_1 = \rho'(\hat{\theta})(s^1)$. Let $y'_2 := y_1 + \kappa_{p2} \cdot \|s^2 - s^1\| < y_2$ and $y'_2 := y_2 - \kappa_{p2} \cdot \|s^2 - s^1\| > y_1$. For $y_2$ to be preferred to $y'_2$ at $s^2$, we must have: $\int W d\hat{\theta}(\theta) > 0$ where $W := u_{p2}(\theta, s^2(\theta), y_2) - u_{p2}(\theta, s^2(\theta), y'_2)$. Then:

\[
W = u_{p2}(\theta, s^1(\theta), y_2 - \kappa_{p2} \cdot \lambda) - u_{p2}(\theta, s^1(\theta), y'_2 - \kappa_{p2} \cdot \lambda), \text{ where } \lambda = s^2(\theta) - s^1(\theta)
\]

\[
\leq u_{p2}(\theta, s^1(\theta), y_2 - \kappa_{p2} \cdot \|s^2 - s^1\|) - u_{p2}(\theta, s^1(\theta), y'_2 - \kappa_{p2} \cdot \|s^2 - s^1\|)
\]

\[
= u_{p2}(\theta, s^1(\theta), y'_2) - u_{p2}(\theta, s^1(\theta), y_1) := W'
\]
The first inequality holds by Assumption 3, with $\lambda$ as given. The second holds by convexity of $u_P$ in $y$, since $\lambda \leq \|s^2 - s^1\|$. So $\int P^d\hat{\theta}(\theta) > 0$, contradicting optimality of $r_1$ at $s^1$. Q.E.D.

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