Irrelevant Alternatives*

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Abstract

This paper examines the notion of 'irrelevant' alternatives in the context of social choice problems. It is shown that seemingly irrelevant alternatives could be relevant because of their information content. This leads to a consideration of aggregation rules where an endogenous condition of independence of irrelevant alternatives is imposed. The analysis leads to the characterization of a unique procedure to aggregate preferences - Borda's rule. This new characterization provides insight into independence conditions and into the status of Borda's rule. An extension to a domain incorporating interpersonal comparisons is also pursued.

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1 Introduction

Conditions imposing an independence from "irrelevant alternatives" play a central role both in, what may be termed, individual choice theory and in social choice theory. The purpose of this essay is to examine different aspects of independence, to assess whether independence can be defended, and to examine the consequences of weaker notions of independence, particularly ones based upon the extent of independence being determined endogenously by the problem.

Consistency of choice is often associated with some notion that there is an independence of irrelevant alternatives (Sen (1970), ch 1*).[1] Nash (1950), in the presentation of his bargaining solution, uses a notion of independence relating to a consistency of choice. This has been referred to as "independence of irrelevant alternatives" by Luce and Raiffa (1957). The arguments in favour of such a condition are different from those relating to another notion of independence, invoked originally by Arrow (1963), and incorporated into a major part of social theory since that time.[2] This, of course, also goes under the name of "independence of irrelevant alternatives " (IIA). An attempt will be made in the next section to clarify the differences between these conditions and the arguments for and against each of them. It will be suggested that whilst Nash IIA relates to the nature of the choice problem, Arrow IIA relates the informational requirements underlying choice.[3] This requirement may imply that irrelevant alternatives become relevant, even when social choice is based upon some desire for independence. Examples of this will be developed in Section 3.

In social choice problems, an Arrow IIA condition is very powerful. Consider the original Arrow impossibility theorem. Arrow sought to aggregate a set of individual orderings into a social ordering, subject to the requirement that 1) aggregation is possible whatever the set of individual orderings - a condition of unrestricted domain U; 2) aggregation respects the weak Pareto criterion P; 3) Arrow IIA is satisfied so

[1] It is worthy to note that this text is now as old as was Amartya Sen when he wrote this classic of social choice theory.

[2] Sen (1977) surveys the results incoporating independence under different informational structures; for a more recent discussion of different structures, see Fleurbaey and Hammond (2004).

[3] When we speak of Arrow IIA, we mean a condition lifted from the informational constraints of aggregation based upon a set of individual orderings. See Sen (1977).

that the social ordering over a subset of alternatives is independent of the ranking of alternatives outside the subset. The only aggregation rule satisfy U, P and IIA is a dictatorship, social value reflecting one of the individual orderings. Whilst all three conditions play a role, the role of IIA is noteworthy. Condition U relates only to the domain of application of the aggregation rule, even though it is as demanding as it can be in this regard. Condition P serves only to rule out uninteresting rules, as is demonstrated by Wilson's (1972) characterization of the possibilities under U and IIA.[4] Given the power of IIA, it is not a condition to be invoked lightly.

If there are reasons to believe that IIA as a blanket condition is too strong, the question arises as how it could be weakened. Even with the informational structure of the Arrow problem, there are many aggregation rules satisfying U and P. With richer informational structures, there is a further embarrassment of riches.[5] Is there a half-way house which incorporates some desirable elements of independence without invoking something as strong as Arrow IIA? Recent work by Campbell and Kelly (2006) suggests that imposing an independence condition which states that there are some alternatives, specified exogenously, which are irrelevant in any ranking, can be a condition almost as strong as IIA. Another approach is needed.

Section 4 takes such an approach. First, a strong condition of neutrality is imposed, in essence forcing aggregation rules to be welfarist, independently of other conditions.[6] Within the Arrow informational structure, the nature of aggregation rules is investigated. An independence condition is introduced which determines irrelevant alternatives endogenously. A characterization theorem is then presented, showing that the aggregation rule must be the Borda rule. This serves as an alternative characterization to that provided by Young (1974) which was based upon consistency with respect to decisions of sub-populations.[7] Given that the Borda rule is often given as

[4]Specifically, one individual's ordering is dictatorial in that other individual orderings are essentially disregarded. This restriction admits rules which negate the single individual ordering, giving an anti-dictatorship, ones which are imposed, and ones based upon dictatorship. Some combination of these rules is also possible.

[5]See the discussion in Sen (1977).

[6]For the most part, we also restrict attention to anonymous rules.

[7]Campbell and Kelly (2006) lament the fact that all characterization theorems of the Borda rule utilize an axiom based upon sub-populations. The present result shows that another approach is possible.

an example of a rule that rides roughshod over notions of independence, it is interesting to see it emerge from an attempt to embody some element of independence. In other work, Young (1988, 1995) has used a condition, referred to as stability or local independence of irrelevant alternatives, which is an endogenous condition. Whilst the formal setting is similar in his work to that studied here, the interpretation of the social ranking is different and his condition does not seem appropriate in our setting (see Section 4 below).

Section 5 extends some of our analysis to a richer information structure where interpersonal comparisons are permitted. Concluding remarks follow in Section 6.

2 Independence of Irrelevant Alternatives

We start by investigating the nature of choice through the use of choice functions. Let S be domain of social states, taken to be the set of all outcomes that could be envisaged. A decision maker chooses from some subset X of S, and $C(X, S)$, is a non-empty subset of X - the choice set of the decision maker. The function C is the choice function which captures the process of choice.

As we have written it, C has two arguments, X and S. S describes the possible universe in the sense that information about states in S is potentially available and could inform a decision maker concerning the 'best' choice from X. In particular, in a social choice problem, information could include the well-being of individuals in states. We could think of S as being a subset of some more universal set T. Then, as S varies, so the information available to a decision maker could change. The set X describes the currently feasible alternatives. With fixed S, varying X varies the set of alternatives available but the information available to the decision maker is unchanged.

One notion of IIA relates to the idea that the choice set for some set of available alternatives should change with the set of alternatives only if chosen elements become excluded or alternatives are added which 'dominate' the chosen alternatives. One way of expressing this condition is

(IIA1): Let $X \subset Y$. Either $C(X,S) = C(Y,S) \cap X$ or $C(Y,S) \cap X$ is empty.

The either part of this statement corresponds to the idea that elements in Y/X do not 'dominate' $C(X,S)$ and the or part corresponds to the idea that elements in Y/X do 'dominate'. A justification for (IIA1) could come from the idea that the value of an outcome is to be judged only by the consideration of features related to that outcome rather than a value in part related to what else could have been chosen. An example of the latter would be a choice function which tries to avoid outcomes where individuals fare poorly compared to how they would have fared if other available alternatives had been chosen. Thus, (IIA1) is one way of capturing a consequentialist or end-state approach to decision making.

More conventional properties of a choice functions than (IIA1) are (see Sen (1970), ch1*):

Property α: Let $x \in X \subset Y$. If $x \in C(Y,S)$ then $x \in C(X,S)$.

Property β: Let $x, y \in X \subset Y$. If $x, y \in C(X,S)$ then $x \in C(Y,S) \leftrightarrow y \in C(Y,S)$.

These properties, often referred to as contraction and expansionist consistency conditions, are well-known and posses well-known properties.[8] Define a base relation R as a ranking of states such that xRy if $x \in C(\{x,y\},S)$. Then α implies that if $x \in C(X,S)$ then $x \in C(\{x,y\},S)$ for all $y \in X$ so xRy for all $y \in X|x$. Thus $C(X,S)$ is drawn from the best states under the ranking R. Moreover, under β, the ranking R must be transitive. For assume that xRy, yRz and zPx. This rules out $x \in C(\{x,y,z\},S)$, applying α. If $y \in C(\{x,y,z\},S)$ then yRx and this together with xRy implies $x \in C(\{x,y,z\},S)$, applying β, which is a contradiction. If $z \in C(\{x,y,z\},S)$ then zRy and this together with yRz implies $x \in C(\{x,y,z\},S)$, applying β again, a contradiction. Thus $C(\{x,y,z\}S)$ is empty, another contradiction. The implication is that xRy & $yRz \Rightarrow xRz$ so that the base relation is transitive.

What is the connection between (IIA1) and the properties α and β? We have

[8] Property α is the independence of irrelevant alternative condition used in Nash (1950).

Proposition 1 *(IIA1) is equivalent to α and β taken together.*

Proof. Consider $x \in X \subset Y$. If $x \in C(Y,S)$ then $C(Y,S) \cap Y/X$ is non-empty. Thus (IIA1) implies $C(X,S) = C(Y,S) \cap X$ so $x \in C(X,S)$. Thus (IIA1) implies α. To show the converse, if $x \in X \subset Y$ and $x \in C(Y,S)$ then $x \in C(Y,S) \cap X$ so the set is non-empty, and (IIA1) implies $x \in C(X,S)$ so α is satisfied.

Now consider $x, y \in X \subset Y$. If $x, y \in C(X,S)$, and $x \in C(Y,S)$ then $C(Y,S) \cap Y/X$ is non-empty and (IIA1) implies $C(X,S) \subseteq C(Y,S)$ so $y \in C(Y,S)$. Thus (IIA1) implies β. For the converse, if $x, y \in C(X,S) \subseteq X \subset Y$, then $x \in C(Y,S)$ implies that $C(Y,S) \cap X$ is non-empty and (IIA1) implies that $y \in C(Y,S)$, so giving β. ■

The implication of this is that standard consistency conditions of choice are equivalent to an attempt to treat unchosen available alternatives as irrelevant. That the underlying base relation is transitive is a manifestation of the fact that choice is based, for instance, upon a consequentalist approach to the evaluation of states. If the base relation underlying a choice function fails to be transitive then this does not necessarily imply that choice is 'irrational', it could be *prima facie* evidence that a non-consequentalist approach is being adopted.

So far, we have considered how $C(X,S)$ varies with changes in X. What about changes in S? Consider changes which permit the set of available alternatives to stay fixed at X. This can be interpreted as changes in the information which could be used to inform the choice to be made from the set X. If these (unavailable) alternatives are to be irrelevant then we have:

(IIA2): For some X, consider any $S_1, S_2 \subset T$ such that $X \subseteq S_1 \cap S_2$. Then $C(X,S_1) = C(X,S_2)$.

(IIA1) and (IIA2) are independent conditions relating to the different arguments of the function C. One implication of (IIA2) is that if $X \subseteq S$ then $C(X,S) = C(X,X)$ so that the choice function is determined once choices where all alternatives are available has been pinned down. (IIA2) may be thought of as the appropriate independence condition behind Arrow's IIA condition.[9] In terms of the base relation R underlying C, the binary ranking of x and y is determined independently of unavailable alternatives in the domain, irrespective of the information that could be imparted from the knowledge of such alternatives.

If unavailable states may provide information about what would be a good choice, so an argument is provided for not invoking (IIA2). In the next section we look at how information from other states could be introduced naturally into a choice problem. Then we will look at possibilities when (IIA2) is not invoked. However, if (IIA2) is not invoked then a decision maker may need to be aware of information from the whole domain of alternatives. This is problematic, not least because the fact that the domain is unconstrained by the set X of alternatives that can be chosen. Related to this, the set S may contain states which could never arise as an available alternative in a choice problem. In any particular problem where the set of available alternatives is X, these hypothetical alternatives are not dissimilar to any alternative outside the set X. Thus, if (IIA2) is not invoked then an awareness of hypothetical alternatives could be used as information to inform choices that have to be made.

3 Unavailable Alternatives and Information Extraction

Consider the well-known Borda rule which determines a ranking of states based upon a set of individual rankings of the states - from each individual ranking, no points are awarded to the lowest ranked state, one to the second lowest, etc.;[10] the aggregate ranking is then based upon the aggregate points received by a state. This procedure may be viewed as being applied across the domain of states S.[11] The aggregate ranking R can then be interpreted as creating a choice function over any subset of

[9] The structure of the problem studied by Arrow assumes a fixed domain but permits the (welfare) information associated with states to vary. But with unavailable alternatives being irrelevant, the separate conditions will place the same restrictions on a choice function.

[10] If a number of states are ranked as indifferent then they can share equally the points that would have been awarded if they had been strictly ranked.

[11] This is sometimes referred to as the Broad Borda Rule.

states X:

$$C^B(X,S) = \{x : x \in X \text{ and } xRy \ \forall y \in X\}.$$

The ranking R, by its construction is a transitive ordering and C^B satisfies (IIA1). The Borda rule is thus consistent with a consequentalist decision making process.

But C^B clearly fails (IIA2): R over X depends upon the ranking of states outside X, as is obvious from its construction. However, the rule may be interpreted as a utilitarian rule, which is consequentalist, with information being extracted from knowledge of states outside X. The decision maker would like to base decisions on utility information that incorporates interpersonally comparable utility differences. But with only a set of rankings of utility for each individual and no richer welfare information, the assignment of points as in the Borda rule can be justified as a reasonable approach in the context of, what is to the utilitarian decision maker, extreme parsimony of information. In essence, ignorance leads to the equal treatment of the utility difference between adjacently ranked states.

The Borda rule places equal information weight or all unavailable states - the set S/X. This has the unfortunate implication that if hypothetical states are introduced then they would be given equal weight. An augmentation of the rule would be to use a points system which reduced the power of information in some states in determining the social ordering. One example of this is that, if S is divided into S_1 and S_2 then, taking one individual's ranking, if x and y are adjacently ranked then their points difference is unity if $x, y \in S_1$ but one-half otherwise. Hypothetical alternatives could be assigned to S_2 and be given a low weight, particularly if individual rankings over such states have to be created by the decision maker, perhaps through some process of introspection.

Consider now a problem where the decision maker is utilitarian and the information available relates to a ranking of intrapersonal and interpersonal utility differences. For instance, if $u(x,i)$ is utility in state x of individual i then information may take the form:

$$u(x,1) - u(y,1) > u(y,2) - u(x,2) > u(y,3) - u(x,3) > 0.$$

It is a common belief that a ranking of utility differences is sufficient to determine a utilitarian ranking. However, if (IIA2) is invoked then, in a three person society and with the above ranking of differences, it is impossible to determine which state gives the higher sum of utilities. In fact, the only non-dictatorial rule that satisfies (IIA1), (IIA2) and the Pareto criterion for all rankings is a rule that gives equal utility weight to two individuals and zero-weight to all other individuals.[12] But if (IIA2) is not invoked then information can be gleaned from non-available states to provide further information concerning available states. For the above utility difference ranking, assume that there is a state z where

$$u(z,2) - u(y,2) = u(y,3) - u(x,3)$$

Using this we have

$$\sum_{i=1,3} [u(x,i) - u(y,i)] = (u(x,1) - u(y,1)) - (u(z,2) - u(x,2)).$$

Thus, the sum of utilities is greater in state x than in state y if 1's utility differences between y and x is greater than 2's utility difference between x and z. We therefore see that information involving a non-available state can be used to make consequentalist choices between available states. In principle, state z could be a hypothetical state.

How does this example differ from our information acquisition interpretation of the Borda rule? In the Borda rule case, available information is ordinal and the construction of a interpersonally comparable cardinal index of utility is, in major part, conjectural. However, in the utility difference example, there is nothing conjectural and exact restrictions upon utility differences between two states are provided by invoking information from other states.

4 Social Choice without Independence

The purpose of this section is to examine aggregation rules that allow unavailable alternatives to influence the social ranking. It will be assumed that (IIA1) is satisfied and, instead of investigating possible choice functions, we concentrate on an examination

[12] See Roberts (2006).

of transitive base relations underlying choice functions. We will also restrict attention to an informational structure based upon a set of individual orderings defined over a set of domains S being all subsets of some set T. The cardinality of T is assumed to be sufficiently large.[13] If (IIA2) is invoked then one is quickly led to the Arrow impossibility theorem. Here, we dispense with such a condition and invoke strictly weaker conditions.

If R_i is individual i's ordering then we seek a rule $f(< R_i >_{i=1,n})$ which is a single ordering. f is a function of S and orderings defined over this S. Most of the conditions imposed upon f relate to a fixed S, the exception is EIIA defined below. For convenience, it will be assumed that there is no indifference in the individual orderings - each R_i is a strict order, P_i. As (IIA2) is not imposed, we impose some other strong conditions. In particular, we impose a strong condition of neutrality which makes the rule insensitive to any features of the problem other than the individual rankings. Let $\pi : S \to S$ be a permutation of states. Define the ordering $R(\pi, R)$ as

$$xR(\pi, R)y \text{ iff } \pi(x)R\pi(y).$$

A standard neutrality condition is

Neutrality (N). If π is a permutation of states then for all $< R_i >$:

$$R(\pi, f(< R_i >)) = f(< R(\pi, R_i) >).$$

We will also exploit a stronger notion of neutrality:

Strong Neutrality (SN). If π is a permutation of states, $\pi(x) = x$ and $\pi(y) = y$, and j is any individual, then

$$f(< R_i >)|_{\{x,y\}} = f(< R_i' >)|_{\{x,y\}}.$$

where $R_i = R_i' \;\; \forall i \neq j$ and $R_j' = R(\pi, R_j)$.

[13] This is for convenience of proofs. When this is not the case, proofs are more involved.

Strong neutrality says that a permutation of *one* individual's ranking which preserves the position of states x and y gives rise to an invariance in the social ranking over $\{x, y\}$.[14] This condition rules out the possibility that the social ranking over $\{x, y\}$ is influenced by a correlation across individuals of the hierarchical ranking of states other than x and y. Under an independence condition like IIA2, SN is always satisfied - it relates to neutrality only with respect to alternatives outside the choice set. A consequentalist decision maker, seeking to use information from irrelevant states to inform his choices, is unlikely to find SN objectionable and the examples of the last section lend support to SN.

We will also impose standard conditions on f:

Unrestricted Domain (U). f is defined for all individual strict orders over all subsets S of a set T.

Pareto (P). $xP_iy\ \forall i \Rightarrow x\ P\ y$ where P is the strict preference derived from $f(< R_i >)$.

Anonymity (A). Let σ be a permutation of the set of individuals. Then

$$f(< R_i >) = f(< R_{\sigma(i)} >).$$

An endogenous independence condition will be added later.

We start by considering the determinants of the social ordering over some pair of states x and y within some fixed set S where $\mid S \mid= m$.[15] Condition N will then ensure that the same determinants apply to all pairs. Assume that all states are labelled $z_{1,} \dots z_m$ and that individual preferences satisfy

$$z_k P_i z_\ell \quad \forall i, \ \ \forall k > \ell$$

[14] SN is different from a condition sometimes called strong neutrality which embodies independence. The condition SN bears a similar relation to N as a condition sometimes called strong anonymity bears to anonymity. See Section 5 below.

[15] We have yet to impose a condition which relates social rankings with different domains S.

where states x and y are excluded from this requirement. Under this restriction, individual preferences are totally determined by where x and y lie in each individual's ordering. Let

$$v_i(x) = |k : xP_i z_k|$$

$v_i(y)$ is defined symmetrically. In this restricted problem, the social ranking is determined by the vector $v(x)$ and $v(y)$ where v_i is an integer between zero and $m-1$. By N the social ranking is independent of the pair $\{x, y\}$ or the labelling of other states. Thus, the pairwise social ranking can be viewed as a function of the vectors v, the domain of this function being $[0, m-1]^N$. Let this ranking be $\widetilde{R}$. As only strict individual preferences are permitted, $\widetilde{R}$ cannot rank $v(x)$ and $v(y)$ when $v_i(x) = v_i(y)$ for some i. Thus $\widetilde{R}$ is an incomplete ranking.

Now consider an unrestricted problem where each individual has any strict order P_i over S and $R = f(< P_i >)$ is the social ranking. Recall that the social ranking is transitive. Consider the ranking over some pair $\{x, y\}$. Take each individual's ranking in turn and permute the states, other than the states x and y, so that with the labelling of states $z_1, \ldots, z_m$, we have $z_k P z_\ell$ for all $k > \ell$. Notice that $v_i(x)$ and $v_i(y)$ remain unchanged. After each permutation, the social ranking exists, by U, and is unchanged, by SN, so, after all individuals rankings have been permuted, the social ranking will be determinable by the ranking $\widetilde{R}$ applied to $v(x)$ and $v(y)$. The same process can be applied to all pairs of states. We thus have $f\left(< P_i >\right)|_{\{x,y\}} = \widetilde{R}\Big|_{v(x),v(y)}$. If three vectors are pairwise ranked under $\widetilde{R}$, transitivity of R implies transitivity of $\widetilde{R}$. In particular, if $v\widetilde{R}v'$ and $v'\widetilde{R}v''$ then $v''Pv$ is ruled out in all cases. Without loss of generality, $\widetilde{R}$ can always be extended to ensure that it is reflexive: vRv.

We have shown that F can be represented by a ranking $\widetilde{R}$:

Proposition 2 *If $f(< P_i >)$ satisfies U, N and SN then the social ranking is representable by a real-valued function W_S such that*

$$xf(< P_i >)y \quad \textit{iff} \quad v(x)\widetilde{R}_S v(y)$$

This implies that the social choice rule is equivalent to a points based voting rule where the social ranking is based upon the number of states ranked below that state for each individual. We have indexed $\widetilde{R}$ by S to denote the fact that, so far, we have kept the state domain of f fixed at S. We note, first, that xP_iy for all i implies $v(x) >> v(y)$ so if condition P is invoked, $\widetilde{R}$ must display strict preference with an increase in all arguments. Second, under A, a permutation of preferences between individuals does not change the social ordering. Thus, if $\sigma(v)$ is some permutation of the vector v then, under A, $v\widetilde{R}_Sv' \Leftrightarrow \sigma(v)\widetilde{R}_S\sigma(v')$ and the ranking will be symmetric.

The representation theorem in Proposition 2 is different from theorems which present an equivalence between social choice rules and rankings defined over utilities achieved in a state (Roberts (1980b)). Such theorems critically depend upon an independence condition like (IIA2). Here, $v(x)$ and $v(y)$ depend upon the ranking of x and y with all other states in S.

How can aggregation rules be further pinned down? First, one could demand that the rule possessed aggregation consistencies with respect to subsets of the population. This is the approach adopted first by Young (1974) in an environment when (IIA2) is not imposed. In the context of the present analysis, a separability condition could be imposed which demands that if an individual is indifferent between x and y then his overall ranking of all states does not influence the social ranking.[16] This separability condition requires the domain of f to be extended to include indifference.[17] Proposition 2 can then be used to impose a restriction of additive separability on $\widetilde{R}$.

Taking another route, is it possible to impose any condition which treats some alternatives as irrelevant without being led to dictatorship? If there is an exogenously determined subset of states that are irrelevant in the decision concerning the pair $\{x, y\}$ then this imposes extreme restrictions on $\widetilde{R}$, quickly leading to a dictatorship result.[18] Another approach is to consider some states to the irrelevant based upon how they are ranked by individuals - this can be thought of as endogenous independence. In this

[16] See Deschamps and Gevers (1978) for a use of this condition under IIA2.

[17] This is most easily accomplished by imposing a continuity assumption in the rule f. See, for instance, Maskin (1978).

[18] See Campbell and Kelly (2006).

setting, a state z could be considered irrelevant if everybody ranks $\{x, z\}$ in the same way as they rank $\{y, z\}$. Thus, information from state z brings no information which could permit further discrimination between x and y. Let f be the rule over some domain of states S and let f^+ be the rule when the domain is extended to consider a new state z. We have:

Endogenous Independence of Irrelevant Alternatives (EIIA): Let f be defined over some S, f^+ defined over $S \cup \{z\}$ $z \notin S$. If, for all i, $P_i = P_i^+\big|_S$ and

$$\begin{array}{ll} \text{either} & zP_i^+x \ \&\ zP_i^+y \\ \text{or} & xP_i^+z \ \&\ yP_i^+z, \end{array}$$

then

$$f(< P_i >) = f^+(< P_i^+ >)\big|_S .$$

In terms of the analysis of Section 2, EIIA is a condition that relates to how the social ranking, and so the implied choice function, changes with changes in S. Thus, it is in the spirit of IIA2 though independence is implied only under strict conditions.

Another endogenous independence condition which has appeared in the literature is the condition of stability (Young (1988)) or limited independence of irrelevant alternatives (Young (1995)). In essence, this condition states that if x and y are adjacent to each other in the social ranking then the ranking of x *vis-a-vis* y should be independent of other alternatives. Under anonymity, neutrality and Pareto, this directly implies that adjacently ranked states are ranked as under majority rule—see the discussion of majority rule below. In the context of our analysis, it could be thought that information from other states is most valuable in the ranking of x and y when there is little difference between them as judged by the social ranking so this condition may be inappropriate. One similarity between the two endogenous conditions comes from the fact that states that are overwhelmingly superior or inferior to states x and y are viewed as 'irrelevant' under both endogenous independence conditions.

Given that Proposition 2 can be applied, first we investigate the dependence of $\widetilde{R}$ on the domain of social states S. If a new state z is added and new preference rankings P_i^+ are such that, for each i, $P_i = P_i^+\big|_S$ and $z \; P_i^+ \; w$ for all $w \in S$, then $v(w)$ is the same for all w in S. Furthermore, EIIA implies that $f(< P_i >) = f^+(< P_i^+ >)\big|_S$ so f and f^+ are both representable by the same ranking $\widetilde{R}$. A similar analysis applies with regard to the removal of a state from the domain S. We thus have

Proposition 3 *If $f(< P_i >)$ satisfies U, N, SN and EIIA then it is representable by a ranking $\tilde{R}$ that is independent of the domain of social states over which rankings are defined.*[19]

Standard IIA2 conditions ensure independence of irrelevant alternatives both with respect to the aggregation rule used and with respect to how the rule operates when faced with a set of available alternatives in the domain S. EIIA ensures that the aggregation rule itself is independent even though the operation of the rule admits the influence of unavailable alternatives through the way that the function v is constructed.

We have thus far considered the implications of changes to the domain S involving states that dominate available alternatives. Now let consider the addition of a state which, for some individual i, is dominated by the pair $\{x, y\}$ and, for everybody else, dominates $\{x, y\}$. This change has the effect of increasing $v_i(x)$ and $v_i(y)$ by unity. Similarly, deletion of a state with this property will lead to unity being subtracted from v_i and v_i'. But by combining a series of additions and subtractions, with different individuals affected, we have:

Lemma 1. *Let t be an n-dimensional vector of (positive and negative) integers. Under U, N, SN and EIIA, the representation ranking $\widetilde{R}$ satisfies*

$$v\widetilde{R}v' \Longleftrightarrow (v+t)\widetilde{R}(v'+t)$$

(whenever the arguments are in the domain of $\widetilde{R}$).

[19] The individual arguments of $\tilde{R}$, the arguments of the vector v, are limited in magnitude by the size of set S minus unity. Thus, for any S, one could set $\tilde{R}_S = \tilde{R}_T$.

Lemma 1 implies a translation invariance property of the ranking by $\widetilde{R}$. It imposes strong restrictions on this ranking and this we now investigate. It is convenient also to invoke A which permits us to restrict attention to symmetric rankings.

What is implied by Lemma 1? Consider two rankable vectors v and v' such that $\sum v_i = \sum v'_i$. Assume that $v\widetilde{P}v'$. Let k be a constant vector such that $k_i = k_j$ for all i, j. This vector is chosen to ensure that vectors to be created are admissible, i.e. have non-negative arguments.[20] Applying Lemma 1:

$$v + (k - v')\ \widetilde{P}\ v' + (k - v')$$

$$\Rightarrow\quad v - v' + k\ \widetilde{P}\ k.$$

Let σ be a circular permutation of vectors such that $\sigma_{i+1}(v) = v_i$ and $\sigma_1(v) = v_n$. Under A, $\widetilde{R}$ is symmetric:

$$\sigma(v - v') + k\ \widetilde{P}\ k$$

(recall that k is a constant vector). Applying Lemma 1 again gives

$$(v - v') + \sigma(v - v') + k\ \widetilde{P}\ (v - v') + k\ \widetilde{P}\ k.$$

If the three vectors are pairwise rankable, transitivity and symmetry give

$$\sigma(v - v') + \sigma^2(v - v') + k\ \widetilde{P}\ k.$$

Repeating the initial permutation $n - 1$ times, we have

$$v - v' + \sigma(v - v') + \ldots + \sigma^{n-1}(v - v') + k\ \widetilde{P}\ k.$$

But from the construction of σ:

$$v - v' + \sigma(v - v') + \ldots + \sigma^{n-1}(v - v') = \sum v_i - \sum v'_i = 0$$

Thus, $k\ \widetilde{P}\ k$ which is a contradiction. Thus we have shown that $v\widetilde{I}v'$.

The above argument depends upon each of the $n-1$ created vectors being rankable with the vector k, i.e. not equal in any argument. This depends upon the form of

[20] The magnitude of arguments can be constrained by a more indirect proof. In particular, we could work with v, v' such that $|v_i - v_j| \leq 2$.

the vector $v - v'$ and cannot be guaranteed. To overcome this problem, a more cumbersome approach is required. Consider any rankable v, v' with $\sum v_i - \sum v'_i$ and let $w = v - v'$. By A, we can assume that $w_i > 0$, $i \leq m$, and $w_i < 0$, $i > m$. Apply a circular permutation among the first m arguments of the vector so that, after $m - 1$ permutations, we attain a vector with constant positive arguments in its first m places. At each stage, full rankability is retained. A similar circular permutation can be applied amongst the last $n - m$ arguments. The net result is that the ranking of v and v' will be the same as the ranking of two vectors whose difference $\widetilde{w}$ takes the form $\widetilde{w}_i = \overline{w}$, $i \leq m$, $\widetilde{w}_i = -\underline{w}$, $i > m$. It will be the case that $m\overline{w} = (n - m)\underline{w}$.

Take the case where n is even and assume $m < \frac{n}{2}$ (the case $m > \frac{n}{2}$ is symmetric). Consider two vectors such that their difference $\widetilde{w}'$ takes the form $\widetilde{w}'_i = w'$ $i \leq \frac{n}{2}$, $\widetilde{w}'_i = -w'$, $i > \frac{n}{2}$. By A, these vectors must be ranked as indifferent, or $\widetilde{w}' + k \ \widetilde{I} \ k$ for some constant vector k. By Lemma 1, $\widetilde{w} + \widetilde{w}' + k \ \widetilde{I} \ \widetilde{w} + k$. Let $w' > \underline{w}$ so the vector $\widetilde{w} + \widetilde{w}'$ will be strictly positive in its first $\frac{n}{2}$ arguments, strictly negative otherwise. How will $\widetilde{w} + \widetilde{w}' + k$ be ranked with k? Taking circular permutations among the first $n/2$ arguments, it will be ranked the same as a vector $\widetilde{w}'' + k$ where $\widetilde{w}''$ has constant positive arguments for $i \leq \frac{n}{2}$, constant negative thereafter. By $A, \widetilde{w}'' + k \ \widetilde{I} \ k$, so $\widetilde{w} + \widetilde{w}' + k \ \widetilde{I} \ k$ and, as $\widetilde{w} + k$ and k are rankable, $\widetilde{w} + k \ \widetilde{I} \ k$. Thus $v \ \widetilde{I} \ v'$.

When n is odd, we can consider vectors $\widetilde{w}$ where $\widetilde{w}'_i = w'$, $i \leq \frac{n+1}{2}$, $\widetilde{w}'_i = -\frac{(n+1)}{n-1}w'$ otherwise. Both w' and $\frac{(n+1)w'}{n-1}$ must be integers. It can be checked that a circular permutation amongst all arguments gives rankability at all stages so our initial argument gives us $\widetilde{w} + k \ \widetilde{I} \ k$. The argument then follows the lines of the n even case. The net effect is that if $\sum v_i = \sum v'_i$ then $v \ \widetilde{I} \ v'$.

Finally, when $\sum v_i > \sum v'_i$ then we can use a similar analysis to show that, under P, $v\widetilde{P}v'$. Thus, the ranking $\widetilde{R}$, although it is incomplete, is representable itself by a real-valued function $W = \sum v_i$.

Starting with preferences $< P_i >$, if each P_i is used to award zero points to the lowest ranked state, one to the next, etc., then the rule represented by W will rank states according to the total number of points awarded to each state - we will have the Borda rule. Collecting together the conditions which underlie the result, we have:

Proposition 4 *If $f(< P_i >)$ satisfies U, N, SN, A, P and EIIA then f is the Borda rule.*

This result is of some interest. First, it provides a characterization of the Borda rule that is different in flavour to previous approaches and is based upon conditions that are easily comparable to conditions much utilized in social choice theory. The only conditions which are not entirely straightforward are SN and EIIA. SN is only a mild strengthening of N and is in the same spirit. EIIA is a weakening of IIA2. Weakening IIA2 but retaining some exogenous independence of 'irrelevant' alternatives is, in essence, as strong as IIA and, as Campbell and Kelly (2006) have shown, there are no rules that satisfy such a condition together with (N), (A) and (P).[21] If any independence condition is to be imposed then it needs to incorporate some notion of endogenous independence and EIIA is a weak condition in this regard.[22]

It is also useful to compare Proposition 4 with May's (1952) characterization of simple majority rule. May showed that the unique aggregation rule applied to individual orderings which satisfied N, A, P and a condition capturing IIA2 was majority rule. IIA2 implies that both SN and EIIA are satisfied. However, majority rule is not a permissible rule in Proposition 4 because it can induce cycles in the social ranking - condition U is not satisfied. With a desire for a rule giving rise to transitive orderings, and so satisfaction of IIA1, a natural question to ask is by how much IIA2 must be relaxed before a rule can be found satisfying transitivity. Proposition 4 provides an answer and also tells us that the 'nearest' transitive rule to majority rule in the direction of relaxing independence is the Borda rule.[23]

We now make some further observations relating to Proposition 4. First, it should be stressed that the result applies only to strict individual orderings. With the possibility of indifference, the characterization result still holds over the sub-domain of strict preferences but, when there is indifference, a variety of rules can be utilized.[24]

[21] Of course, this result is closely related to Arrow's (1963) impossibility theorem.

[22] If no independence condition is to be incorporated then we are left with the representation result of Proposition 2.

[23] If we relax anonymity then Arrow's analysis shows that the 'nearest' transitive rule is dictatorship.

[24] With indifference, there are difference ways of assigning points to different states which are then aggregated to create the social ordering.

Second, we can consider whether it is possible to relax the conditions in the Proposition without a major impact upon the characterization. Condition P is important only in ensuring that the social choice is positively responsive to increases in $\sum v_i$. Condition A is used in a fundamental way in the proof but this relates mostly to the method of proof. If A is not invoked then it is possible to show that $W = \sum \gamma_i v_i$ where the γ_i are positive weights. Proving this seems to be rather more involved than in the anonymity case. We give a brief sketch: consider extending the domain of possible v vectors to $\mathbb{R}^n$. Taking the set of v such that $v\widetilde{R}v^*$, one can form the convex hull of this set, $\overline{V}$. Using translation invariance, one can show that no v such that $v^*\widetilde{P}v$ is contained in the interior of this set. Similar properties apply to $\underline{V}$, the convex hull of v such that $v^*\widetilde{R}v$. A separating hyperplane theorem can be applied to show that a hyperplane separates these sets, given by $\sum \gamma_i v_i =$ constant. This forms an indifference curve in $\mathbb{R}^n$. Translation invariance then implies that the same sloped hyperplane can be used for all v^* and the result follows. The class of rules characterized extends now to weighted Borda rules but no further.

5 Interpersonal Comparisons

The results of the last section have been developed under the (welfare) information restriction of ordinality and non-interpersonal comparability. If the information structure is richer then the set of possibilities expands and the potential application of information drawn from 'irrelevant' alternatives is enhanced—recall the discussion in Section 3. On the other hand, the richer the information structure, so the need to extract information from 'irrelevant' alternatives is diminished. The appeal of any independence condition will thus depend upon the information structure of the problem being investigated.

In this section, we will consider the information structure of ordinality combined with interpersonal comparability: welfare information is captured by an ordinal function $u(x, i)$ defined over state/individual pairs so $u(x, i) > u(y, j)$ is taken to mean that i in state x has higher welfare than j in state y. Conditions U, P, A, N and

SN can be straightforwardly applied in this set-up. It is useful to strengthen the anonymity condition to a condition of strong anonymity which has similarities to the connection between N and SN:

Strong Anonymity (SA). Let σ be a permutation of the set of individuals. If u and u' are such that for some $x \in S$:

$$u(x,i) = u'(x, \sigma(i)) \quad \forall i$$

and

$$u(y,j) = u'(y,j) \quad \forall j, \quad \forall y \in S/x$$

then $f(u) = f(u')$.

This condition is due to Sen (1977). Importantly, it says that a permutation of individuals in any state does not change the social ranking. A welfarist should be happy with such a condition. Recall that SN related to a permutation of states for some individual. If SA is applied with respect to each state in turn using the same σ then we see that SA implies A but not *vice versa*.

Conditions SA and SN taken together allow us to impose conditions upon the social ranking f and we adopt an approach similar to that adopted in the last section. We consider the determinants of the social ordering over some pair of states x and y with different welfare information: $f\left|_{\{x,y\}}\right.$. Assume that all states are labelled $z_1, \ldots, z_m$ and all individuals are labelled $1, \ldots, n$. Fixing x and y, consider u such that

i. $\forall z_k \forall i, j, i < j : \qquad u(z_k, i) < u(z_k, j)$

ii. $\forall z_k, z_\ell \neq x, y, k < \ell, \quad \forall i, j: \quad u(z_k, i) < u(z_\ell, j)$

Condition (i) says that, in every state, the ordering of individuals by their welfare level is the same. Condition (ii) says that states other than x and y can be ordered by their welfare level, everybody in one state is better off then everybody in another state.

We will consider the function $f\left|_{\{x,y\}}\right.$ over the sub-domain of individual welfares satisfying (i) and (ii). How large is this sub-domain? All rankings not including x and y are given, for x and y the ranking of individuals in each state are given. Thus u is totally determined given vectors $v(x)$ and $v(x)$ where

$$v_i(x) = |\{k, j\} : u(x, i) > u(z_k, j)|$$

and with $v(y)$ defined symmetrically. Notice that the vector $v(x)$ is restricted to be increasing across its arguments. We note that if u and u' are such that for some w, x, y, z : $u'(w, \cdot) = u(x, \cdot)$, $u'(x, \cdot) = u(w, \cdot)$, $u'(z, \cdot) = u(y, \cdot)$, $u'(y, \cdot) = u(z, \cdot)$ and $u = u'$ over other states then N gives $f(u)\left|_{\{x,y\}}\right. = f(u)'\big|_{(w,z\}}$ so $f(u)\left|_{\{x,y\}}\right.$ is determined by $v(x)$ and $v(y)$ not by the identity of x and y. We can therefore define a ranking $\widetilde{R}$ over v vectors such that $\widetilde{R}\left|_{v(x),v(y)}\right. = f(u)\big|_{\{x,y\}}$ where u satisfies (i) and (ii) and $v(x), v(y)$ are distilled from the function u.

We now proceed to show that $\widetilde{R}$ determines the ranking of states even when (i) and (ii) are not satisfied. Consider any u. Assume that the lowest state individual pair not relating x and y is (z, j). Consider a permutation of states which swaps (z, j) for (z_1, j). If u' is the transformed welfare information, $f(u)\left|_{\{x,y\}}\right. = f(u')\big|_{\{x,y\}}$ by SN. Now consider a permutation of individuals which swaps (z_1, j) with $(z_1, 1)$. If u'' is the transformed information, (SA) implies that $f(u'') = f(u')$ so $f(u'')\left|_{\{x,y\}}\right. = f(u)\big|_{\{x,y\}}$. Now moving to the second lowest state/individual pair, a state and then a person permutation can transform the welfare information so that this state/individual pair is $(z_1, 2)$ and if u''' is the transformed welfare information, $f(u''')\left|_{\{x,y\}}\right. = f(u)\big|_{\{x,y\}}$. This can be repeated for all state/individual pairs excluding x and y. Finally, a permutation of individuals in state x to give $\widetilde{u}(x, i) < \widetilde{u}(x, j)$ if $i < j$ and a similar permutation for y gives us, by SA, transformed welfare information $\widehat{u}$ such that $f(\widehat{u})$ $\left|_{\{x,y\}}\right. = f(u)\big|_{\{x,y\}}$, where $\widehat{u}$ satisfies the conditions (i) and (ii). The vectors $\widehat{v}(x)$, $\widehat{v}(y)$ that apply to $\widehat{u}$ relate directly to u: $\widehat{v}_i(x)$ is the number of state/individual pairs giving welfare below the i'th lowest welfare levels in state x under u—it relates to a position in the welfare hierarchy in state x rather than a particular individual in state x.

Taking any u, $f(u)\left|_{\{x,y\}}\right. = \widetilde{R}\Big|_{\widehat{v}(x),\widehat{v}(y)}$ and for three pairwise rankable vectors, $\widetilde{R}$

will be transitive. We have

Proposition 5 *If $f(u)$ satisfies U, N, SN and SA then the social ranking is representable by a ranking $\tilde{R}$ such that*

$$xf(u)y \qquad \textit{iff} \qquad \widetilde{v}(x)\widetilde{R}\widetilde{v}(y).$$

If P is added as a condition then there will be a strict preference with an increase in all arguments. If the ranking can be represented by a real-valued function W then, if $W(v) = v_d$, the social ranking follows the ranking of the k'th lowest welfare position in each state. For example, if $d = 1$ then W is the Rawlsian rule. If W is an affine function requiring separability, then $W = \sum \gamma_i v_i$ which is a generalised Borda rule based upon position in the welfare hierarchy rather than on particular individuals (Sen (1977)).

The characterization result in Proposition 5 has not invoked an independence condition. If IIA2 is added then W can be further restricted and it is known from Gevers (1979) and Roberts (1980a) that W must then take on the form $W = v_d$—a positional dictatorship. This can be derived directly or, more simply, by invoking Proposition 5. But if IIA2 is not invoked, is there a reasonable endogenous independence condition that could be imposed? Closest to EIIA is a condition which states that the ranking over $\{x, y\}$ is independent of the welfare information relating to states that are unambiguously dominated or unambiguously dominate states x and y. Consider a change in welfare information which shifts the welfare achieved in some state z from dominating both x and y to being dominated by x and y. Then $\widehat{v}(x)$ and $\widehat{v}(y)$ will be augmented by unity in all arguments. If this is done τ times then it will be required of W that

$$\widehat{v}\widetilde{R}\widehat{v}' \Longleftrightarrow \widehat{v} + (\tau, \tau, \ldots, \tau)\widetilde{R}\widehat{v}' + (\tau, \tau, \ldots, \tau).$$

This condition is much weaker than the translation invariance property of the last section—instead of linear indifference surfaces, one indifference surface is almost unrestricted but other indifference surfaces have the property that they are a translation up the 45$^{\rm o}$ line of the initial indifference surface. Further restriction would have to be based upon the imposition of a more restrictive independence condition.

6 Concluding Remarks

The purpose of this paper has been to explore aspects of the role of 'irrelevant' alternatives. One condition deems as irrelevant whether rejected states were or were not available to be chosen. The arguments for and against such a condition appear relatively straightfoward. However changes to the domain for potentially available states can lead to a change in the information that is available and deeming such alternatives as irrelevant is to deny the value of information that they may contain. We have seen, by example, how non-available alternatives can provide information about available alternatives.

If 'irrelevant' alternatives are not independent then the issue arises as to whether it is reasonable to impose any sort of independence condition. In the context of a welfare information structure of ordinality and interpersonal non-comparability, we have investigated weakening independence, motivated by the idea that alternatives could be deemed independent if welfare information relating to them is uninformative - the idea of endogenous independence. Adopting such a condition leads to the characterization of a unique aggregation procedure - Borda's rule. This analysis gives insights into the role of independence conditions and into the nature of Borda's rule.

It would be useful to examine further the possibilities with richer information structures. We have seen that some progress is possible under ordinality together with interpersonal comparability but the general issue that must be faced is that the appropriateness of any independence condition relates to the information structure of the problem being examined.

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