ADDITIONAL NOTES ON THE COMPARATIVE STATICS OF CONSTRAINED OPTIMIZATION PROBLEMS

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Abstract: These are additional notes relating to the paper 'The Comparative Statics of Constrained Optimization Problems,' which is appearing in *Econometrica*. It gathers together material present in various earlier versions of the paper, as well as some new material, which are not found in the published article or in its Supplement (both available at the *Econometrica* website). In particular, these notes establish conditions for the existence of increasing selections. It also includes two simple applications of our techniques: to find conditions for gross substitutability of a demand function and to find conditions that guarantee increasing best response in a model of Bertrand competition. These notes are not intended to be a self-contained account of the theory, so are best consulted after reading the main paper.

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1. Results on $\mathcal{C}$-quasisupermodular functions and the $\mathcal{C}$-flexible set order

The central concepts in 'The Comparative Statics of Constrained Optimization Problems' (Quah, 2006) are $\mathcal{C}$-quasisupermodularity and the $\mathcal{C}$-flexible set order. In this section we develop more results relating to these concepts.

Turning first to $\mathcal{C}_i$-supermodularity and $\mathcal{C}$-supermodularity, it is clear that both these properties are preserved by positive scalar multiplication and by addition. In particular, let X be a convex sublattice of R^l and $f : X \to R$ a $\mathcal{C}_i$-supermodular function; then for any w in R^l, the map $g_w : X \to R$ given by $g_w(x) = f(x) - w \cdot x$ is also a $\mathcal{C}_i$-supermodular function. The next result gives the converse of this observation: if the functions g_w are $\mathcal{C}_i$-quasisupermodular for all w then f must be $\mathcal{C}_i$-supermodular.[1]

Proposition Ad(1): *Let $X \subseteq R^l$ be a convex sublattice of R^l.*
(i) The function $f : X \to R$ is $\mathcal{C}_i$-supermodular if, for all w_i in R, the function g_{w_i} mapping x in X to $f(x) - w_i x_i$ is $\mathcal{C}_i$-quasisupermodular.
(ii) The function $f : X \to R$ is $\mathcal{C}$-supermodular if, for all w in R^l, the function g_w mapping x in X to $f(x) - w \cdot x$ is $\mathcal{C}$-quasisupermodular.
(iii) Suppose that f is increasing. Then f is $\mathcal{C}_i$-supermodular ($\mathcal{C}$-supermodular) if for all w in R^l_+, the function g_w mapping x in X to $f(x) - w \cdot x$, is $\mathcal{C}_i$-quasisupermodular ($\mathcal{C}$-quasisupermodular).

Proof: (i) Suppose that f is not $\mathcal{C}_i$-supermodular. Then there is λ in $[0, 1]$ such that for some x and y with $x_i > y_i$, we have

$$f(x \bigtriangledown_i^\lambda y) - f(y) < f(x) - f(x \bigtriangleup_i^\lambda y). \tag{1}$$

Note that $(x \bigtriangleup_i^\lambda y)_i = y_i < x_i$, so there is a scalar $\bar{w}_i$ such that $\bar{w}_i[x_i - (x \bigtriangleup_i^\lambda y)_i] = f(x) - f(x \bigtriangleup_i^\lambda y)$. Furthermore, since $x - (x \bigtriangleup_i^\lambda y) = (x \bigtriangledown_i^\lambda y) - y$, we have

[1]Note that Milgrom and Shannon (1994) establishes an analogous result (Theorem 10) for quasisupermodular functions.

$w_i[x_i - (x \triangle_i^\lambda y)_i] = w_i[(x \bigtriangledown_i^\lambda y)_i - y_i]$. Deducting this term from both sides of (1) we obtain

$$g_{\bar{w}_i}(x \bigtriangledown_i^\lambda y) - g_{\bar{w}_i}(y) < g_{\bar{w}_i}(x) - g_{\bar{w}_i}(x \triangle_i^\lambda y) = 0.$$

So $g_{\bar{w}_i}$ violates $\mathcal{C}_i$-quasisupermodularity and we have a contradiction.

Clearly (ii) follows from (i), so we only have to prove (iii). Note that if (1) holds for $\lambda = 0$, then the right hand side of (1) is nonnegative (since f is increasing), while $(x - x \wedge y)_i = x_i - y_i > 0$. We could then use the proof given for (i), choosing $\bar{w}_i \geq 0$ and setting the other entries of the vector $\bar{w}$ at zero. So we consider the case when (1) is true for $\lambda > 0$. Note that (1) can hold only if x and y are unordered, and with $\lambda > 0$, x and $x \triangle_i^\lambda y$ must also be unordered. Thus, $x - x \triangle_i^\lambda y$ has both positive and negative entries, and there is $\bar{w}$ in R^l_+ such that $\bar{w} \cdot [x - x \triangle_i^\lambda y] = f(x) - f(x \triangle_i^\lambda y)$. Now repeating the steps in our proof of (i), we see that the function $g_{\bar{w}}$ must violate $\mathcal{C}_i$-quasisupermodularity. QED

The significance of Proposition Ad(1) is that in those situations where we require $\mathcal{C}$-quasisupermodularity for all functions in the class $\{g_w\}_{w \in R^l}$ or $\{g_w\}_{w \in R^l_+}$, then we must necessarily impose $\mathcal{C}$-supermodularity on f. Of course these classes of functions do indeed arise naturally in comparative statics problems, since it can be interpreted as a profit function, with $f(x)$ as the revenue of the firm when it produces the output vector x and with w_i as the unit cost of producing good i (so $w \cdot x$ is the total cost of producing x).

We turn now to a characterization of the $\mathcal{C}$-flexible set order. Suppose H and G are subsets of R^2; representing the second variable on the vertical axis, we have shown that H dominates G in the $\mathcal{C}_2$-flexible set order provided (in some specific sense) H has a steeper boundary than G (see Lemma A1 in the main paper, and also the discussion in the Supplement). The next result is a higher-dimensional version of Lemma A1.

Proposition Ad(2): *Let S' and S'' be subsets of a convex sublattice X of R^l which are both closed, obey free disposal and satisfy $S' \subseteq S''$.*[2] *Then $S'' \geq_i S'$ if and only if the following property ($\star$) holds:*[3]
whenever x and u are vectors with $u > 0$, $u_i = 0$, $x \in S'$, $x + u \in S''$, and $x + tu \notin S'$ for all $t > 0$, then for any scalar $\mu > 0$, and $\hat{u} > 0$ which is orthogonal to u with $\hat{u}_i > 0$,

$$x - \mu u + \hat{u} \in S' \Longrightarrow (x + u) - \mu u + \hat{u} \in S''.$$

Proof: We first prove that ($\star$) implies that $S_i'' \geq_i S'$. Let x be in S' and y be in S'' with $x_i > y_i$. If $x > y$, the condition for $S'' \geq_i S'$ requires x to be in S'' and y to be in S': the first is true since $S' \subset S''$, while the second follows from free disposal. So we assume that x and y are unordered. If y is in S', the condition for $S'' \geq_i S'$ holds with $\lambda = 1$. This leaves us with the case of x and y are unordered, with y not in S'. Since $x' \wedge y$ is in X and less than x', we know that it is in S'. Define $v = y - x \wedge y$. By the closedness of S' and free disposal, there is λ^* in $[0, 1)$ such that $x \wedge y + \lambda^* v$ is in S' and $x \wedge y + \lambda v$ is not in S' for λ in $(\lambda^*, 1]$. Define $u = (1 - \lambda^*)v$. Choose $\mu = \lambda^*/(1 - \lambda^*)$ and $\hat{u} = x - x \wedge y$. We have $u_i = 0$, $\hat{u}_i > 0$. We then have $x \wedge y + \lambda^* v$ in S', $(x \wedge y + \lambda^* v) - \mu u + \hat{u} = x$ in S', and $x + u = y$ in S''. So by ($\star$), $(x + u) - \mu u + \hat{u} = x \vee y - \lambda^*$ must be in S''. Thus $S'' \geq_i S'$.

For the other direction, let $x' = x - \mu u + \hat{u}$. By assumption, this is in S'; also by assumption, $x + u$ is in S'' and $x_i' > x_i + u_i$. Note that $x' \wedge (x + u) = x - \mu u$. Since $S'' \geq_i S'$, there must be a positive t smaller than μ such that $x - tu$ is in S' and $x + u + \hat{u} - tu$ is in S''. Note that t cannot be negative because it is assumed that x is at the 'edge' of S'. Since S'' obeys free disposal, the fact that $x + u + \hat{u} - tu$ is in S'' implies that $x + u + \hat{u} - \mu u$ is also in S'', which establishes ($\star$). QED

[2] A subset S of X has the free disposal property if whenever x is in S and y in X satisfies $y < x$ then y is in S.

[3] The notation $S'' \geq_i S'$ means that S'' dominates S' in the $\mathcal{C}_i$-flexible set order.

Proposition Ad(2) says, in a specific formal sense, that the set of substitution possibilities which favor variable x_i in the constraint set S'' is larger than the set of substitution possibilities which favor x_i in the constraint set S'. Property ($\star$) considers two points x in S' and $x+u$ in S'', where u is positive and orthogonal to the direction i; furthermore, the point x is on the 'edge' of S' in the sense that it is not possible to add anything in the direction of u and still stay within S'. Suppose that it is possible at x to substitute μu with u' and still stay within the constraint set S' - note that this is a substitution which 'favors i' because $u_i = 0$ and $\hat{u}_i > 0$ - then property ($\star$) requires that it is possible to make the *same substitution* at the point $x+u$ in S'' and stay within the S''.

2. Results on increasing selections

Let X be a lattice and $F : X \to R$ a supermodular function. It is well-known that if the constraint set S is a sublattice of X then $\text{argmax}_{x \in S} F(x)$ is also a sublattice. Indeed, in many contexts it is possible to show that this set is a subcomplete sublattice, so that one could sensibly speak of the smallest or largest element of $\text{argmax}_{x \in S} F(x)$ (see Topkis (1998, Corollary 2.7.1)). This feature is significant for the following reason. Let $(T, \succeq)$ be a partially ordered set, $\{F(\cdot, t)\}_{t \in T}$ a family of real-valued functions defined on X, and $\{S_t\}_{t \in T}$ a family of constraint sets in X such that the family $\{\text{argmax}_{x \in S_t} F(x,t)\}_{t \in T}$ is increasing in t (in the strong set order). Standard comparative statics results identify conditions under which this is true (see Milgrom and Shannon (1994)). Provided $\text{argmax}_{x \in S_t} F(x,t)$ is also a nonempty subcomplete sublattice, then it has a largest and a smallest element; the fact that $\{\text{argmax}_{x \in S_t} F(x,t)\}_{t \in T}$ is an increasing family guarantees that the map from t to the largest element in $\text{argmax}_{x \in S_t} F(x,t)$ is an increasing function (and so is the map from t to the smallest element). Such maps are called *increasing selections*. The ability to construct an increasing selection is useful in some contexts (see, for example, Milgrom and Roberts (1990)).

The constraint sets considered in our theory are not as a rule sublattices (with respect to the product order, or indeed any lattice order). For this reason, the optimal sets need not be sublattices and may not contain a largest or smallest element. Nonetheless, there is a quick and easy way to guarantee the existence of an increasing selection.[4]

PROPOSITION Ad(3): *(i) Let* $(T, \succeq)$ *be a partially ordered set and let* $\Phi : (T, \succeq) \to R^l$ *be a compact-valued correspondence with the following property:* $\Phi(t'') \geq_i \Phi(t')$ *if* $t'' \succeq t'$*. Then there is a function* $\phi : (T, \succeq) \to R^l$ *such that* $\phi(t) \in \Phi(t)$ *for all* t *in* T*, and* $\phi_i(t'') \geq \phi_i(t')$ *if* $t'' \succeq t'$*.*
(ii) Suppose that $\Phi : (T, \succeq) \to R^l$ *is a convex and compact-valued correspondence with the following property:* $\Phi(t'') \geq \Phi(t')$ *if* $t'' \succeq t'$*.*[5] *Then there is a function* $\phi : (T, \succeq) \to R^l$ *such that* $\phi(t) \in \Phi(t)$ *for all* t *in* T*, and* $\phi(t'') \geq \phi(t')$ *if* $t'' \succeq t'$*.*

Proof: (i) Since $\Phi(t)$ is a compact set, $\text{argmax}_{x \in \Phi(t)} x_i$ is a nonempty set. By the axiom of choice, there exists a function $\phi : (T, \succeq) \to R^l$ where, for all t, $\phi(t)$ is in $\text{argmax}_{x \in \Phi(t)} x_i$. By assumption, if $t'' \succeq t'$, we have $\Phi(t'') \geq_i \Phi(t')$; by Proposition 3 in the main paper, this also means that $\phi_i(t'') \geq \phi_i(t')$.

(ii) Define $g : R^l \to R$ by $g(x) = -\sum_i^l e^{-x_i}$. This function is supermodular and strictly concave, hence $\mathcal{C}$-supermodular; it is also continuous. The continuity of g and the compactness of $\Phi(t)$ guarantee that $\text{argmax}_{x \in \Phi(t)} g(x)$ is nonempty; since g is a *strictly* concave function and $\Phi(t)$ is a convex set, $\text{argmax}_{x \in \Phi(t)} g(x)$ is a singleton, which we shall denote by $\phi(t)$. By assumption, if $t'' \succeq t'$, we have $\Phi(t'') \geq \Phi(t')$; Theorem 2 in the main paper then guarantees that $\phi(t'') \geq \phi(t')$. QED

It should be quite clear that one can usefully combine Proposition Ad(3) with the comparative statics theorems in the main paper. As an illustration, we shall apply

[4] I suspect that the requirement in Proposition Ad(3-ii) that Φ be convex-valued can be dropped or considerably weakened, but the proof will be more involved.

[5] $\Phi(t'') \geq \Phi(t')$ means that $\Phi(t'')$ dominates $\Phi(t')$ in the $\mathcal{C}$-flexible set order.

this result to the standard consumer problem (Example 1 in the main paper), with the consumer maximizing the utility function $U : R^l_+ \to R$. It is fairly common in this context to assume that U is strictly quasiconcave, i.e., whenever $U(x') = U(x)$ for x and x' in R^l_+, then $U(\alpha x' + (1-\alpha)x) > U(x)$, where α is in $(0,1)$. (If the inequality is weak, we say that U is quasiconcave.) If U is strictly quasiconcave, demand is unique, so the issue of selection does not arise. But when it is not, it is natural to ask whether there exists a selection from the demand correspondence with normality properties. The next proposition identifies conditions sufficient for this property.

Proposition Ad(4): *Suppose that the utility function $U : R^l_+ \to R$ has closed preferred sets, i.e, $U^{-1}([k,\infty))$ is closed for all k. Then at all (p,w) with $p \gg 0$ and $w > 0$, the demand set $D(p,w)$ is nonempty. Furthermore, at any $p \gg 0$,*
(i) if U is $\mathcal{C}_i$-quasisupermodular, there is $\phi(p,w) \in D(p,w)$ such that $\phi_i(p,\cdot) : R_{++} \to R_+$ is an increasing function;
(ii) if U is $\mathcal{C}$-quasisupermodular and quasiconcave, there is $\phi(p,w) \in D(p,w)$ such that $\phi(p,\cdot) : R_{++} \to R^l_+$ is an increasing function.

Proof: For the non-emptiness and compactness of $D(p,w)$ consult Mas-Colell et al (1995). The assumptions in (i) guarantee that $D(p,w'') \geq_i D(p,w')$ whenever $w'' > w' > 0$ (see Example 1 in the main paper). By Proposition Ad(3-i), there is a function $\phi : R^l_{++} \times R_{++} \to R^l_{++}$ such that $\phi(p,w) \in D(p,w)$ for all (p,w) and with $\phi_i(p,\cdot) : R_{++} \to R_+$ being an increasing function.

For (ii), note that the assumptions guarantee that $D(p,w'') \geq D(p,w')$ whenever $w'' > w' > 0$ (see Example 1 in the main paper). Furthermore, $D(p,w)$ is convex-valued because U is quasiconcave (though $D(p,w)$ need not be singleton). The result follows immediately from the application of Proposition Ad(3-ii). QED

3. Applications to Comparative Statics Problems

We give two more applications of our techniques. For other applications, see the main paper and the Supplement.

Example Ad(1). A demand function is said to exhibit the gross substitutability property if a fall in the price of good i causes the demand for all other goods to decrease. This property is important because, amongst other things, it helps to guarantee the uniqueness and stability of the equilibrium price in general equilibrium models (see, for example, Mas-Colell et al (1995)). The best known condition guaranteeing gross substitutability is the following. Let $U : R^l_{++} \to R$ be of the form $U(x) = \sum_{i=1}^l u_i(x_i)$ where each $u_i : R_+ \to R$ is C^2, with $u_i'(x_i) > 0$ and $u_i'' \leq 0$. Then the demand function $f : R^l_{++} \times R_+ \to R^l_{++}$ generated by U will obey gross substitutability if $-x_i u_i''(x_i)/u_i'(x_i) < 1$ for all i and $x_i > 0$.[6]

One can easily obtain this result using the techniques developed here. First, it is useful to give a different formulation of the demand problem. Suppose that x^* is the demand at (p, w); formally, x^* solves the problem: (i) maximize $\sum_i^l u_i(x_i)$ subject to $p \cdot x \leq w$. Clearly, x^* solves (i) if and only if $(s_1^*, x_2^*, ..., x_l^*)$, where $s_1^* = p_1 x_1^*$, solves the problem: (ii) maximize $u_1(s_1/p_1) + \sum_{i=2}^l u_i(x_i)$ subject to $s_1 + \sum_{i=2}^l p_i x_i \leq w$.

Assume that income is held fixed at w and consider a price change from p' to p'', where $p_i'' = p_i'$ for $i \geq 2$ and $p_1'' < p_1'$. Suppose that demand exists at both prices, with x' being a demand at p'. We wish to show that there is a demand at p'' in which the demand for good i rises and that of all other goods fall. Since x' solves (i) at $p = p'$ we know that $(s_1', x_2', x_3', ..., x_l')$, with $s_1' = p_1' x_1'$ is a solution to (ii) at $p = p'$. Provided the map from $(s_1, 1/p_1)$ to $u_1(s_1/p_1)$ is supermodular, and since demand exists at p'' by assumption, we know from Lemma A2 (in the main paper) that there is a solution $(s_1'', x_2,'', ..., x_l'')$ to (ii) at $p = p''$ such that $s_1'' \geq s_1'$. In other words, there must be a demand at $p = p''$ in which the expenditure on good 1 is higher than that at $p = p'$. In particular, $x_1'' > x_1'$.

[6] For a proof of this result see, for example, Hens and Loffler (1995).

Since U is additive, we know that $(x_2', x_3', ...x_l')$ maximizes $\bar{U}(x_2, x_3, ...x_l) = \sum_{i=2}^{l} u_i(x_i)$ subject to $\sum_{i=2}^{l} p_i x_i \leq w - s_1'$. If u_is are concave, so is $\bar{U}$; furthermore, $\bar{U}$ is additive and therefore supermodular. From Example 1, we know that $\bar{U}$ generates normal demand. When more is spent on good 1, the expenditure available for other goods is reduced from $w - s_1'$ to $w - s_1''$, and so there must be $(x_2''', x_3''', ..., x_l''')$ which maximizes $\bar{U}(x_2, x_3, ...x_l)$ subject to $\sum_{i=2}^{l} p_i x_i \leq w - s_1''$ such that $x_i''' \leq x_i'$ for $i \geq 2$. Furthermore, $(s_1'', x_2''', x_3''', ..., x_l''')$ solves (ii) at $p = p''$, which establishes gross substitutability.

It remains for us to point out what it means for the map from (s_1, a) in R^2_{++} to $u_1(as_1)$ to be supermodular. It is not hard to check that this is equivalent to the convexity of the map $\tilde{u}_1 : R \to R$ given by $\tilde{u}_1(z_1) = u_1(e^{z_1})$. In short, we have shown that the additive utility function U will generate demand satisfying gross substitutability if for all $i \geq 1$, u_i is concave and $\tilde{u}_i$ is convex. When u_i is C^2 with $u_i' > 0$, then $\tilde{u}_i$ is convex if and only if $-x_i u_i''(x_i)/u_i'(x_i) \leq 1$ for all $x_i > 0$. So we have obtained the non-differentiable version of the well known result.

The Supplement to the main paper concentrated on developing and applying the general theory in a two-dimensional context. The main comparative statics result in the Supplement is Theorem S1, which is a corollary of Theorem 2 in the main paper. We now present a more general version of Theorem S1, after which we shall provide an application. The objective function is $f : X_1 \times X_2 \to R$ where $X_1 = (\underline{x}_1, \overline{x}_1)$ and $X_2 = (\underline{x}_2, \overline{x}_2)$ are nonempty open intervals in R. We wish to specify conditions on the function f and the constraint sets H and G which guarantee that $\text{argmax}_{x \in H} f(x) \geq_2 \text{argmax}_{x \in G} f(x)$. As in Theorem S1, we assume that the indifference curves of f obey the declining slope property, but we shall impose weaker conditions on the constraint sets H and G. Let I_H and I_G be two nonempty and open intervals in X_2 such that I_H dominates I_G in the strong set order (this is equivalent to saying that the infimum and supremum of I_H are greater than the infimum and supremum respectively of I_G).

The set H has a boundary given by the graph of $h : I_H \to X_1$; formally,

$$H = \{(x_1, x_2) \in X_1 \times X_2 : x_2 \in I_H \text{ and } x_1 \leq h(x_2)\}.$$

The set G with boundary given by the function $g : I_G \to X_1$ is defined similarly.[7]

PROPOSITION Ad(5): *Suppose that $f : X_1 \times X_2 \to R$ has indifference curves that obey the declining slope condition. Then* $\text{argmax}_{x \in H} f(x) \geq_2 \text{argmax}_{x \in G} f(x)$ *if* **either**
(i) for all x_2'' and x_2' in $I_g \cap I_h$ with $x_2'' > x_2'$, we have $g(x_2') \leq h(x_2')$ and $g(x_2'') - g(x_2') \leq h(x_2'') - h(x_2') \leq 0$; **or**
(ii) f is strictly increasing in x_2 and for all x_2'' and x_2' in $I_g \cap I_h$ with $x_2'' > x_2'$, we have $g(x_2') \leq h(x_2')$ and $g(x_2'') - g(x_2') \leq h(x_2'') - h(x_2')$.

This result differs from Theorem S1 (in the Supplement) in two respects. In Theorem S1, we assume that the boundary functions of H and G are differentiable and decreasing and require $h'(x_2) \geq g'(x_2)$ whenever both are defined. In this proposition the boundary functions need not be differentiable and the derivative condition in Theorem S1 is replaced by an analogous condition on differences: $g(x_2'') - g(x_2') \leq h(x_2'') - h(x_2')$. So part (i) of Proposition Ad(4) is just a non-differentiable version of Theorem S1. There is a second, and perhaps more significant, difference between Theorem S1 and Proposition Ad(4): in part (ii) of the latter result, we remove the condition that h and g are *decreasing* functions, at the expense of requiring f to be strictly increasing in x_2.

Proof of Proposition Ad(5): We claim that the conditions in (i) imply that $H \geq_2 G$; the result then follows from Theorem 2 in the main paper. Let (x_1', x_2') in G and (y_1, y_2) in H satisfy $x_2' > y_2$ and $y_1 > g(y_2)$. (All the other cases are trivial.) Consider the points (x_1', x_2'), $(g(y_2), y_2)$, (y_1, y_2), and $(x_1' + [y_1 - g(y_2)], x_2')$; they form a parallelogram, indeed a backward-bending parallelogram since $x_1' \leq g(x_2') \leq g(y_2)$.

[7]The casual way we are using the word 'boundary' here may bother some readers. The set H has a boundary in the precise topological sense; the graph of h is part of, but generally not *all* of, this boundary.

Note that the second point, $(g(y_2), y_2)$, is in G, so we need only show that the last point is in H. This is true since

$$\begin{aligned} x_1' + [y_1 - g(y_2)] &\leq h(x_2') + [x_1' - h(x_2')] + [y_1 - g(y_2)] \\ &\leq h(x_2') + [g(x_2') - h(x_2')] + [h(y_2) - g(y_2)] \\ &\leq h(x_2'). \end{aligned}$$

The proof for case (ii) is just a modification of that for (i). Let (x_1', x_2') maximize f in the constraint set G and let (y_1, y_2) maximize f in the set H. As in the proof for (i), we assume that $x_2' > y_2$ and $y_1 > g(y_2)$ (the other cases being trivial) and consider the points (x_1', x_2'), $(g(y_2), y_2)$, (y_1, y_2), and $(x_1' + [y_1 - g(y_2)], x_2')$. The same argument as before guarantees that the second point is in G and the fourth is in H. They also form a backward-bending parallelogram. If not, $g(y_2) \leq x_1'$, so $x_1' + [y_1 - g(y_2)] \geq y_1$. Since $(x_1' + [y_1 - g(y_2)], x_2')$ is in H so must (y_1, x_2'). Since f is strictly increasing in x_2, $f(y_1, x_2') > f(y_1, y_2)$, which contradicts the assumption that (y_1, y_2) maximizes f in H. Thus $\{(y_1, y_2), (x_1' + [y_1 - g(y_2)], x_2')\} >_2 \{(x_1', x_2'), (g(y_2), y_2)\}$. Applying Theorem 2 again, we see that $f(x_1' + [y_1 - g(y_2)], x_2') \geq f(y_1, y_2)$ and $f(g(y_2), y_2) \geq f(x_1', x_2')$. So $(x_1' + [y_1 - g(y_2)], x_2')$ maximizes f in H and $(g(y_2), y_2)$ maximizes f in G. QED

Example Ad(2): Consider a profit-maximizing firm producing a single product. If it charges a price $p > 0$, its demand is $D(p, \theta) > 0$ where θ is some parameter. (In a Bertrand game with differentiated products θ will represent the prices of other firms.) The cost of producing output q is $C(q)$, so that the firm's objective is to maximize $pD(p, \theta) - C(D(p, \theta))$. Suppose that, as θ increases, $\ln D(p, \theta)$ increases and the difference $\ln D(p, \theta) - \ln D(p', \theta)$, for any $p' > p$, also increases; respectively, this means that demand increases and becomes less elastic with respect to its own price as θ increases. Suppose also that the firm has increasing marginal costs. With these assumptions, we can show that the profit-maximizing price charged by the firm must increase with θ. Note that this is not a new result; it has already been established by Milgrom and Shannon (1994), albeit with the additional assumption that the demand

D is a differentiable and decreasing function of its own price. Nonetheless, this result gives a good illustration of how to apply Proposition Ad(4).

Firstly, we reformulate the firm's problem as a constrained optimization problem, with the demand function acting as the boundary of the firm's constraint set. Let $\tilde{q}$ be log output. The firm maximizes $\Pi : R_{++} \times R \to R$ given by $\Pi(p, \tilde{q}) = pe^{\tilde{q}} - C(e^{\tilde{q}})$, subject to $(\tilde{q}, p)$ in $S(\theta) = \{(\tilde{q}, p) \in R \times R_{++} : \tilde{q} \leq \ln D(p, \theta)\}$. Note that the boundary of $S(\theta)$ is the map from p to $\ln D(p, \theta)$; if $\theta'' > \theta'$, the properties we have imposed on D guarantee that the conditions on the constraint set boundaries in Proposition Ad(5-ii) are satisfied. As for the objective function Π, it is clearly strictly increasing in p. It also satisfies the declining slope property since the slope of the indifference curve at $(\tilde{q}, p)$ is $C'(e^{\tilde{q}}) - p$, which increases with $\tilde{q}$ if $C'' \geq 0$. Thus, all the conditions of Proposition Ad(4-ii) are satisfied and we conclude that the profit-maximizing price charged by the firm increases with θ.

It is also worth highlighting that this conclusion holds for *any* objective function f with indifference curves that obey the declining slope property - and not just the standard one we have considered. A particularly simple case is the following. Suppose that marginal cost is constant and that, at price p, the log-demand is stochastic, taking the value $\ln D(p, \theta) + s$, with the distribution of s governed by the density function μ. Assume that the firm chooses the price p and thereafter meets the realized demand, which will be $D(p, \theta)e^s$ for some value of s. The firm has the Bernoulli utility function u and maximizes the expected utility of profit, i.e., it maximizes $\int u((p - c)D(p, \theta)e^s)\mu(s)ds$. This can be reformulated as a constrained optimization problem: maximize $U(\tilde{q}, p) = \int_R u((p - c)e^{\tilde{q}+s})\mu(s)ds$ subject to $(\tilde{q}, p)$ in $S(\theta)$. U is strictly increasing in p if u is strictly increasing. The slope of the indifference curve at $(\tilde{q}, p)$ is simply $c - p$, which is independent of $\tilde{q}$. So the declining slope property is satisfied and we conclude that the optimal price rises with θ (under the maintained assumptions on $\ln D(p, \theta)$).

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