Understanding Preferences: "Demand Types", and the Existence of Equilibrium with Indivisibilities

Elizabeth Baldwin* and Paul Klemperer†

COMMENTS WELCOME!‡

9 October 2016 version–the latest version is at:

elizabeth-baldwin.me.uk and www.paulklemperer.org

Abstract

We propose new techniques for understanding agents' valuations. Our classification into "demand types", incorporates existing definitions (substitutes, complements, "strong substitutes", etc.) and permits new ones. Our Unimodularity Theorem generalises previous results about when competitive equilibrium exists for *any* set of agents whose valuations are all of a "demand type" for indivisible goods. Contrary to popular belief, equilibrium is guaranteed for more classes of purely-complements, than of purely-substitutes, preferences. Our Intersection Count Theorem checks equilibrium existence for combinations of agents with *specific* valuations by counting the intersection points of geometric objects. Applications include matching and coalition-formation; and the Product-Mix Auction, introduced by the Bank of England in response to the financial crisis.

Keywords: consumer theory; equilibrium existence; general equilibrium; competitive equilibrium; duality; indivisible goods; geometry; tropical geometry; convex geometry; auction; product mix auction; product-mix auction; substitute; complement; demand type; matching

JEL nos: C62 Existence and Stability Conditions of Equilibrium; D50 General Equilibrium and Disequilibrium; D51 Exchange and Production Economies; D44 Auctions

*Nuffield College, Oxford University, UK, and Grantham Research Institute, London School of Economics, UK; elizabeth.baldwin@nuffield.ox.ac.uk

†Nuffield College, Oxford University, UK; paul.klemperer@economics.ox.ac.uk

‡This paper extends and supersedes much of the material in Baldwin and Klemperer (2014). However, large parts of Sections 2.5, 6.2, and 6.3.1 of that paper will be incorporated and developed in Baldwin and Klemperer (in preparation-a), and Sections 4.2, 4.3, and 5, form the main part of Baldwin and Klemperer (in preparation-b). This work was supported by ESRC grant ES/L003058/1. Acknowledgements to be completed later.

1 Introduction

This paper introduces a new way to think about preferences for indivisible goods, and obtains new results about the existence of competitive equilibrium.

"Demand types." Our first key idea is to classify economic agents' individual and aggregate valuations into "demand types". A "demand type" is defined by a list of vectors that give the possible ways in which the individual or aggregate demand can change in response to a small generic price change. So the vectors defining a "demand type" are analogous to the rows of a Slutsky matrix; they specify the possible comparative statics of any demand of that "type".

For example, a purchaser of spectacles who values spare pairs might always buy lenses and frames in the ratio 2:1, so increase or reduce her demand in this ratio in response to any price change; her valuation is therefore of "demand type" $\pm\{(2,1)\}$. As another example, you might want to book a large hotel room, or a small room, or neither, but have no interest in both. So your response, if any, to a small change in prices would either be to substitute one room for the other, or to increase or decrease your demand for one of the rooms by 1 without altering your demand for the other. That is, your valuation is of "demand type" $\pm\{(1,-1),(0,1),(1,0)\}$.[1]

Our classification is parsimonious. For example, the "demand type" that comprises all possible substitutes preferences is defined by the set of all vectors with at most one positive integer entry, at most one negative integer entry, and all other entries zero; the "demand type" that is all complements preferences is defined by the set of all vectors in which all the non-zero entries (of which there may be any number) are integers of the same sign; the class of all "strong substitutes" preferences for n goods is a "demand type" with just $n(n+1)$ vectors.

Our classification clarifies the relationships between different classes of preferences. For example, the "demand types" descriptions above show clearly why the conditions for indivisible goods to all be (ordinary) substitutes are in general far more restrictive than the conditions for them to all be complements–although they are, of course, symmetric in the two-good case.

Our classification is very general. It permits multiple units of each good; the agents can include sellers, buyers, and traders who can both buy and sell; and it can also be applied to matching models.

Importantly, we will see the classification is also easy to work with.

Equilibrium existence. Our focus on how agents' demands change in response to small price changes yields two new theorems about the existence of competitive equilibrium with indivisibilities:

Our "Unimodularity Theorem" characterizes equilibrium existence for "demand types", that is, for any valuations in *classes* of preferences. It states that competitive equilibrium always exists, whatever is the market supply, if and only if all agents' valuations are

[1] In an auction in which goods' characteristics suggest natural rates of substitution, bidders might be asked to express valuations of the corresponding "demand type"; e.g., the Bank of England's Product-Mix Auction built one-for-one substitution into its design (see Klemperer, 2008, 2010).

concave and drawn from a demand type that is defined by a unimodular set of vectors.[2]

Our characterization immediately yields several earlier existence results, and extensions of them. It also identifies previously-unknown environments in which existence is assured. Moreover, it disproves the popular perception that existence requires substitutes valuations (or a "basis change" thereof). Indeed every demand type for which equilibrium is guaranteed can be obtained as a basis change of a demand type involving only complements preferences (and for which equilibrium is guaranteed)–and the corresponding result is *not* true for substitute preferences.

Our "Intersection Count Theorem", by contrast, concerns whether competitive equilibrium exists for combinations of agents with *specific* valuations. It relates whether equilibrium always exists whatever is the market supply, for the specific agents, to the number of price vectors at which more than one of these agents is indifferent between more than one bundle.

To illustrate our two Theorems, recall the hotel-room example above. Like you, Elizabeth is interested in either room (the hotel only has two rooms), but not both. Both your and her valuations are therefore of demand type $\pm\{(1,-1),(0,1),(1,0)\}$, which is unimodular (because any matrix formed by two of these vectors has determinant 0 or ± 1). So the Unimodularity Theorem tells us that whatever are your and Elizabeth's valuations (they will generally be different), there always exist competitive equilibrium prices, that is, prices such that demand exactly equals supply, if you and she are the only potential buyers.

Paul, however, requires two hotel rooms for his family; if they cannot have both, they will go elsewhere. So Paul's valuation is of demand type $\pm\{(1,1)\}$. The (smallest) demand type from which Elizabeth's and Paul's valuations are both drawn is therefore $\pm\{(1,1),(1,-1),(0,1),(1,0)\}$, which is not unimodular (because the determinant of $(1,1)$ and $(1,-1)$ is -2). So the Unimodularity Theorem tells us that, if Elizabeth and Paul are the potential buyers, then there are *some* valuation(s) of Elizabeth and Paul for which competitive equilibrium does not exist–but this Theorem does not tell us which those valuation(s) are.

However, our Intersection Count Theorem does tell us whether equilibrium exists for any specific valuations: equilibrium exists for Elizabeth and Paul if and only if *either* (i) there are exactly two price vectors at which both agents are indifferent between more than one bundle, *or* (ii) there exists a price at which one agent is indifferent between at least two bundles, and the other is indifferent between at least three (case (ii) is non-generic).[3]

For example, imagine Elizabeth would pay up to *£*40 for the large room, or £30 for the small. Paul is indifferent between paying *£*50 for both, and going elsewhere. Then there is only one pair of prices, (*£*30, *£*20) for the large and small rooms respectively, such that both agents are indifferent between more than one option. (Paul is indifferent between taking both rooms, and taking neither, while Elizabeth is indifferent between the two rooms so, also, each agent is indifferent between only two bundles.) So the Intersection Count Theorem predicts–and it is not hard to check (and Section 5.1.1 will

[2]A unimodular set of vectors in n dimensions is one for which every subset of n of them has determinant 0 or ± 1 (with an additional condition if they are not a spanning set).

[3]The Intersection Count Theorem actually tells us a little more: under these conditions equilibrium exists, for the specific valuations, whatever number of hotel rooms is available–see Section 5.

confirm)–that competitive equilibrium will fail: at any prices at which Paul is prepared to take both rooms, Elizabeth will also demand a room.

If, however, Paul is willing to pay up to £100 for the rooms, then there are exactly two price vectors, (£40, £60) and (£70, £30), such that both Elizabeth and Paul are indifferent between more than one option (Paul between taking both rooms and neither, and Elizabeth between taking no room and the one room she considers good value). So the Intersection Count Theorem now predicts that competitive equilibrium does exist. (In fact, any prices that exceed £40 for the large and £30 for the small, and add to less than £100, clear the market.) Section 5.1.1 gives full details.

Outline of the paper. Our basic tools are convex geometry and, in Section 5 in particular, the "tropical geometry" recently developed by, among others, Mikhalkin (2004).[4] So we begin, in Section 2, by using the existing mathematics literature to develop an economic understanding of two dual geometric objects:

The first, the "Locus of Indifference Prices" (LIP) comprises the price vectors at which the agent is indifferent between two or more bundles, that is, the prices at which the agent's demand changes. Since any LIP corresponds to a valuation function, we can develop our understanding of demand by working directly with these geometric objects.

Our dual geometric object, the "Demand Complex", comprises the convex hulls of the sets of bundles (i.e., quantity vectors) among which the agent is indifferent at some price.

Section 3 then defines a "demand type" by using the set of vectors that describes the ways in which the bundles demanded by the agent can change with prices. These vectors are associated in simple ways with both the LIP and the Demand Complex. So we can easily check whether a demand type is, for example, substitutes, or complements, or "strong substitutes", or "gross substitutes and complements", etc.

Section 4 proves our Unimodularity Theorem. Danilov et al. (2001) provide a very similar sufficient condition for equilibrium, but our use of tropical ideas allows a simpler proof. We also show the necessity of the same condition, so that our theorem is a full characterization of when equilibrium exists. Perhaps more important, our concept of "demand types" also shows how this condition can be applied. For example, equilibrium existence results such as those in Sun and Yang (2006), Milgrom and Strulovici (2009), and Hatfield et al. (2013), are obvious special cases of the Unimodularity Theorem, but none of these papers present their results as specialisations of Danilov et al.'s earlier work, since the latter's relevance was unclear.[5]

Section 5 develops our Intersection Count Theorem by applying a version of Bézout's classic theorem that the number of intersection points of two curves, taking into account "multiplicities" such as tangencies, is equal to the product of the degrees of their defin-

[4]We believe the first version of this paper, Baldwin and Klemperer (2012), was the first to apply tropical geometry to economics. Matveenko (2014), Shiozawa (2015), Crowell and Tran (2016) and Weymark (2016) are other applications.

[5]We especially thank Gleb Koshevoy for very helpful discussions. Analysing "demand types" in price space (as well as, like Danilov et al., in quantity space) also allows us to develop additional implications. We discuss the relationships to Danilov and Koshevoy and their co-authors' work in detail in Section 4.3. Our Intersection Count Theorem addresses similar issues as Bikhchandani and Mamer (1997) and Ma (1998), but our methodology is completely different and, we believe, gives more insight.

ing polynomials.[6] (LIPs can be obtained as "tropical" transformations of "ordinary" geometric objects, and their intersection properties are preserved under these transformations.)

Section 6 discusses applications: Sections 6.1 shows that our model encompasses classic models, and clarifies the relationships between them. Sections 6.2-6.3 show how to find new "demand types" for which equilibrium always exists. For example, we exhibit a previously-unstudied "demand type" that might model demand for, e.g., different kinds of workers and managers who are complements. Equilibrium always exists for this "demand type", although it is unrelated to any substitutes preferences (including via any basis change). Section 6.4 shows that there are many other purely-complements "demand types" for which equilibrium is guaranteed, and Section 6.5 uses our Unimodularity and Intersection Count Theorems to provide an algorithm for determining when equilibrium exists.

Sections 6.6-6.8 explain that our geometric techniques yield new results in other contexts. These include matching models, and developing extensions of the Product-Mix Auctions introduced by the Bank of England during the financial crisis.[7]

Section 7 concludes. The Appendix contains additional examples, and proofs of all results not proved in the text.

This paper has been written for economists. Some of our ideas have been translated for a mathematical audience by Tran and Yu (2015).

2 Representing Indivisible Demand Geometrically

2.1 Assumptions

An agent has a valuation $u : A \to \mathbb{R}$ on bundles $\mathbf{x} \in A \subsetneq \mathbb{Z}^n$. That is, the bundles are formed of n distinct goods, which come in indivisible units. Note that a bundle may be negative or mixed-sign. So our model allows for sellers with non-trivial supply functions, and more general traders, as well as buyers.

The *domain* A of bundles that the agent considers possible, can be *any* finite set in $\mathbb{Z}^n$. Note that A need not contain every integer bundle in its convex hull. Nor need A include every bundle that is available in the economy. In particular, if a bundle is completely unacceptable to the agent, it is simply not in A. This is equivalent to (and mathematically more convenient than) allowing the agent to value some bundles at "$-\infty$".

The agent has quasilinear utility, so maximises $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$, where $\mathbf{p} \in \mathbb{R}^n$ is the price vector. Thus different units of the same good all have the same price. (If they did not we could treat them as different goods.) We do not specify that valuations are weakly

[6]For example, two lines intersect once (possibly at infinity). A quadratic and a line intersect at two points (possibly including points with complex coordinates and points at infinity, and double-counting tangencies). Two quadratics intersect four times (correctly counted), etc.

[7]Bidders in these auctions make sets of "either/or" bids for alternative objects. These bids can be represented geometrically as sets of points in multi-dimensional price space. The then-Governor of the Bank of England (Mervyn King) told *the Economist* that the Product-Mix Auction "is a marvellous application of theoretical economics to a practical problem of vital importance"; current-Governor Mark Carney announced plans for its greater use; and an updated version has been introduced–see Bank of England (2010, 2011), Milnes (2010), Fisher (2011), Frost et al (2015) and *the Economist* (2012).

increasing, or that valuations or prices are non-negative, so our model covers "bads" as well as goods.

We will later (from Section 3.3) extend our model to a finite set of agents: agent j will have valuation u^j on integer bundles in a finite domain A^j. We will consider competitive equilibrium among these agents, given an exogenous supply. Thus our framework will encompass the case in which all traders (including all sellers) are explicitly modelled as agents, that is, exchange economies (for which the exogenous supply is $\mathbf{0}$).

The remainder of Section 2 interprets existing mathematics literature in the context of our basic (single-agent) model.

2.2 The Locus of Indifference Prices (LIP)

We will be particularly interested in the prices at which the agent's *demand set*, $D_u(\mathbf{p}) = \arg\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}\}$, contains more than one bundle, that is, those prices at which the agent is *indifferent* among more than one bundle.

Definition 2.1. The *Locus of Indifference Prices (LIP)* is $\mathcal{L}_u := \{\mathbf{p} \in \mathbb{R}^n : |D_u(\mathbf{p})| > 1\}$

This set is known as a "tropical hypersurface" in the mathematics literature (see Mikhalkin, 2004, and others), but we are introducing new terminology to facilitate understanding among economists. We analyse the structure of this set in more detail because (by continuity of quasilinear utility) it comprises the *only* prices at which demand can change in response to a price change.

Definition 2.2.

(1) A *cell* of $\mathcal{L}_u$ is a non-empty set of the form $\left\{\mathbf{p} \in \mathcal{L}_u : \mathbf{x}^1, \ldots, \mathbf{x}^k \in D_u(\mathbf{p})\right\}$ where $|\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}| > 1$ and $\mathbf{x}^1, \ldots, \mathbf{x}^k \in A$.

(2) A *facet* is an $(n-1)$-dimensional cell of $\mathcal{L}_u$.[8]

Thus a cell of $\mathcal{L}_u$ is the subset of its prices at which the bundles that the agent demands include a particular collection of at least two bundles. The cell therefore specifies the prices at which the agent's demand can change between the bundles in this particular collection. By continuity, cells are closed.

At prices *not* in $\mathcal{L}_u$, the agent demands a unique bundle:

Definition 2.3. A *unique demand region (UDR)* of u is a connected component of the complement of $\mathcal{L}_u$ in $\mathbb{R}^n$.

Demand is generically unique, so UDRs are n-dimensional and open in $\mathbb{R}^n$. The bundle defining the UDR is, again by continuity, also demanded in its closure. It is straightforward that a UDR is convex and each UDR corresponds to a different bundle. Thus the closure of a UDR gives the only points at which this particular bundle is demanded (that is, it has the form $\{\mathbf{p} \in \mathcal{L}_u : \mathbf{x} \in D_u(\mathbf{p})\}$), while a cell of $\mathcal{L}_u$ gives the only points at which some collection of multiple bundles is demanded.

Fig. 1 shows a simple example of a LIP. The agent uniquely demands one of the

[8] We always use the natural dimensions. Thus the dimension of a cell is the dimension of its affine span, i.e. the dimension of the smallest linear subspace $U \subseteq \mathbb{R}^n$ such that the cell is contained in $\{\mathbf{c}\}+U$ for some fixed vector $\mathbf{c}$. Here, and throughout the text, we use Minkowski (set-wise) addition on sets.

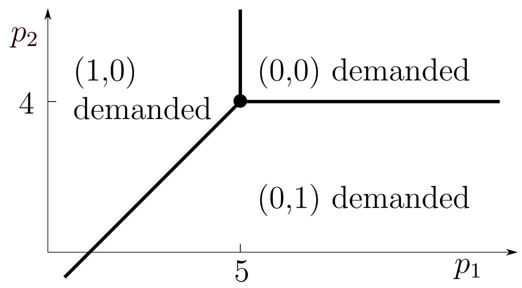


Figure 1: The LIP, $\mathcal{L}_u$, of the valuation u for which $u(0,0) = 0$, $u(1,0) = 5$ and $u(0,1) = 4$. The bundle demanded in each UDR is labelled.

bundles $(0,0)$, $(0,1)$, and $(1,0)$, in the correspondingly-labelled 2-dimensional region, so these regions are the UDRs. The agent demands both bundle $(0,0)$ and bundle $(0,1)$ on the line segment $\{(p_1, 4) \in \mathbb{R}^2 : p_1 \geq 5\}$ so this is a facet (1-dimensional cell). There are two other facets. Meanwhile, the price $(5,4)$ is a 0-dimensional cell (or "0-cell")–it is the only price at which the agent is indifferent between all three bundles.

If, instead, bundles were formed from three distinct goods, i.e. $n = 3$, the facets would be the plane-segments, separating 3-dimensional UDRs; the facets would then meet in line segments, i.e. 1-cells, which would themselves meet in 0-cells.

We give the facets a specific name because of the economic information they contain. At any price $\mathbf{p}$ in a given facet, F, the agent is indifferent between the bundles $\mathbf{x}$ and $\mathbf{x}'$ demanded in the UDRs on either side of F. That is, $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = u(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}', \forall \mathbf{p} \in F$. So $\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})$ is constant across all $\mathbf{p} \in F$. Therefore F is normal to the vector that gives the change in demand, $\mathbf{x}' - \mathbf{x}$, between the UDRs on either side of F. For example, in Fig. 1, the facet $\{(p_1, 4) \in \mathbb{R}^2 : p_1 \geq 5\}$ contains prices at which demand can change by $(0,0) - (0,1) = (0,-1)$, which vector is normal to this facet.

So the geometry of the LIP tells us the *directions* of demand changes between pairs of prices. To know *how much* demand changes in any direction that the LIP specifies, we need one more piece of information:

Definition 2.4. Let $\mathbf{x}, \mathbf{x}'$ be the bundles demanded in the UDRs on either side of facet F. The *weight* of F, $w_u(F)$, is the greatest common divisor of the entries of $\mathbf{x}' - \mathbf{x}$.

Now $\frac{1}{w_u(F)}(\mathbf{x}' - \mathbf{x})$ is a primitive integer vector (the greatest common divisor is 1). It points from the UDR where $\mathbf{x}'$ is demanded, to the UDR where $\mathbf{x}$ is demanded, and is in the opposite direction to the change denoted by $\mathbf{x}' - \mathbf{x}$. But since F is $(n-1)$ dimensional, there is a unique primitive integer vector normal to F and pointing in this direction. So we have shown:

Proposition 2.5.

(1) If $\mathbf{x}$, $\mathbf{x}'$ are uniquely demanded on either side of facet F, then $\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})$ is constant for all $\mathbf{p} \in F$.

(2) The change in demand as price changes between the UDRs on either side of F, is $w_u(F)$ times the primitive integer vector that is normal to F, and points in the opposite direction to the change in price.

That is, the LIP and its vector, $\mathbf{w}_u$, of weights, taken together, provide full information about how demand changes between UDRs.

2.2.1 The correspondence between LIPs and valuations

The previous subsection showed that, starting with a valuation, we could derive a geometric object, encoding economic information. We now show, conversely, that we can start with a purely geometric object and associate economic meaning.

We will use some standard terminology from convex geometry:

Definition 2.6.

(1) A *rational polyhedron* is the intersection of a finite collection of half-spaces $\{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{v}^j \leq \alpha^j\}$ for some $\mathbf{v}^j \in \mathbb{Z}^n$ and $\alpha^j \in \mathbb{R}$.[9]
(2) A *face* of a polyhedron C maximises $\mathbf{p} \cdot \mathbf{v}$ over $\mathbf{p} \in C$, for some fixed $\mathbf{v} \in \mathbb{R}^n$.
(3) The *interior* of polyhedron C is $C^\circ := \{\mathbf{p} \in C : \mathbf{p} \notin C' \text{ for any face } C' \subsetneq C\}$.
(4) A *rational polyhedral complex* Π is a finite collection of sets $C \subseteq \mathbb{R}^n$ such that:
   (i) if $C \in \Pi$ then C is a rational polyhedron and any face of C is also in Π;
   (ii) if $C, C' \in \Pi$ then either $C \cap C' = \emptyset$ or $C \cap C'$ is a face of both C and C'.
(5) A *k-cell* is a cell of dimension k.
(6) A polyhedral complex is *k-dimensional* if all its cells are contained in its k-cells.
(7) A *weighted* polyhedral complex is a pair $(\Pi, \mathbf{w})$ where Π is a polyhedral complex and $\mathbf{w}$ is a vector assigning a weight $w(F) \in \mathbb{Z}_{>0}$ to each facet $F \in \Pi$.

Since the cells of $\mathcal{L}_u$, and the closures of its UDRs, are defined by collections of linear equalities and weak inequalities, it is straightforward that they are all polyhedra and fit together as a "complex" (details in Appendix A.1). In particular:

Proposition 2.7. *The set of all cells of $\mathcal{L}_u$ is an $(n-1)$-dimensional rational polyhedral complex.*

So if C is a cell of $\mathcal{L}_u$, then every face C' of C satisfying $C' \subsetneq C$ is also a cell of $\mathcal{L}_u$. It follows that at prices in such C', the agent demands additional bundles to those that she demands in C. But the agent's demand set is constant in the interior of the cell. That is:

Lemma 2.8. *$D_u(\mathbf{p}^\circ)$ is constant across all $\mathbf{p}^\circ$ in the interior C° of a cell C. Moreover $D_u(\mathbf{p}^\circ)$ defines the cell: $C = \{\mathbf{p} \in \mathbb{R}^n : D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})\}$.*

Fig. 1 illustrates all these points.

Prop. 2.5 tells us that once we know the demand in one particular UDR, and we know the weights of the LIP, we can infer the demand in every UDR, by stepping across a series of facets. But if we follow an agent along a price path that ends where it started, the demand at the end must be the same as that at the beginning. So the weights on the facets must satisfy the *balancing condition*:

Definition 2.9 (Mikhalkin, 2004, Defn. 3)**.** An $(n-1)$-dimensional weighted polyhedral complex Π is *balanced* if for every $(n-2)$-cell $G \in \Pi$, the weights $w(F_j)$ on the facets

[9] We follow Mikhalkin (2004) in *not* restricting α^j to be rational for rationality of the complex.

$F_1, \ldots, F_l$ that contain G, and primitive integer normal vectors $\mathbf{v}_{F_j}$ for these facets that are defined by a fixed rotational direction about G, satisfy $\sum_{j=1}^{l} w(F_j)\mathbf{v}_{F_j} = 0$.[10]

This balancing condition is, in fact, the *only* condition that a weighted rational polyhedral complex has to satisfy to be the LIP of some valuation function.[11] That is:

Theorem 2.10 (Mikhalkin, 2004, Prop. 2.4)**.** *Suppose that* $(\Pi, \mathbf{w})$ *is an* $(n-1)$*-dimensional balanced weighted rational polyhedral complex in* $\mathbb{R}^n$*, and let* $\mathcal{L}$ *be the union of its cells. Then there exists a finite set* $A \subsetneq \mathbb{Z}^n$ *and a function* $u : A \to \mathbb{R}$ *such that* $\mathcal{L}_u = \mathcal{L}$ *and* $\mathbf{w}_u = \mathbf{w}$*.*

By contrast with Afriat's theorem (see e.g. Vohra, 2011, Thm. 7.2.1), which starts with a (finite) set of prices paired with demands, this theorem uses only information about the geometrical divisions in price space.

Thm. 2.10 is not mathematically novel, but its economic implications are important (and, we believe, novel). It shows that a set in $\mathbb{R}^n$ is the LIP of a quasilinear valuation *if and only if* it has some easily-checked geometric properties. It is also easy to identify the cells of this LIP, in particular its facets, and so understand the economics of the valuation, since:

Lemma 2.11. $C \subseteq \mathcal{L}_u$ *is a cell iff it is the intersection of the closures of a set of UDRs of* u*.*

In practice it is often much easier to develop ideas and intuitions by working with these geometric objects, than by working out explicit examples of valuations. Subsequent sections will show in more detail how simply describing the geometry of the LIP, and of related objects, gives insight into the economics.

The next section explores the uniqueness of valuations identified in this way.

2.2.2 The correspondence between LIPs and *concave* valuations

We define concavity of the valuation u in the standard "concave-extensible" sense, but with an extra property since we allow the domain to be *any* finite subset of $\mathbb{Z}^n$:

Definition 2.12.

(1) A set $A \subseteq \mathbb{Z}^n$ is *discrete convex* if it contains every integer point within its convex hull, that is, $\mathrm{conv}(A) \cap \mathbb{Z}^n = A$.

[10] This is just the n-dimensional generalisation of the requirement in 2 dimensions that, when moving in a sufficiently small circle around any point, the vectors $\mathbf{v}_F$ all point in the direction of travel. To choose a rotational direction around G, pick a 2-dimensional affine subspace H of $\mathbb{R}^n$ orthogonal to G, such that the intersection of each F_j with H is 1-dimensional. The intersection of H with the LIP is then a collection of 1-cells meeting at the 0-cell which is $G \cap H$. An ordinary choice of rotational direction in this two-dimensional picture gives a rotational direction around G in $\mathbb{R}^n$.

[11] There do not necessarily exist weights to balance a general rational polyhedral complex. For example, in two dimensions, consider three points (0-cells), each contained in three facets, such that each pair of points are both contained in a common facet. There are six weights, which must satisfy six equations (three balancing conditions in each of the two dimensions). But since the conditions are trivially satisfied by setting all weights equal to zero, the conditions can only be satisfied by positive integer weights if the conditions are not linearly independent–which is non-generic.

(2) We write $\operatorname{conv}(u) : \operatorname{conv}(A) \to \mathbb{R}$ for the minimal weakly-concave function everywhere weakly greater than u (sometimes called the "concave majorant" of u).
(3) $u : A \to \mathbb{R}$ is *concave* if A is discrete-convex and $u(\mathbf{x}) = \operatorname{conv}(u)(\mathbf{x})$ for all $\mathbf{x} \in A$.

It is a standard result that concave valuations are precisely those for which every possible bundle is demanded at some price, and for which the demand set at any price is discrete-convex, just as for divisible, weakly-concave valuations, and for essentially the same reasons:[12]

Lemma 2.13. *$u : A \to \mathbb{R}$ is concave*
iff *for all $\mathbf{x} \in \operatorname{conv}(A) \cap \mathbb{Z}^n$ there exists $\mathbf{p}$ such that $\mathbf{x} \in D_u(\mathbf{p})$*
iff *$D_u(\mathbf{p})$ is discrete-convex for all $\mathbf{p}$.*

For a simple example of failure of concavity, consider the 1-dimensional valuation $u(0) = u(1) = 0$; $u(2) = 10$. Then $D_u(5) = \{0, 2\}$ is not discrete-convex, and there exists no price p such that $1 \in D_u(p)$.

If we weakly increase a valuation until it becomes concave, it is easy to see that the only values we need change are those for bundles which were previously never demanded. And increasing any never-demanded bundle's value has no effect on the agent's behaviour until the bundle is just marginally demanded, when the value function becomes locally affine. The marginally defined bundle is then added to the demand at some prices, but is never demanded uniquely, and all other bundles are demanded exactly as they were previously, so the LIP is unchanged. That is:

Lemma 2.14. *Let $u : A \to \mathbb{R}$. Then:*
(1) for each $\mathbf{x} \in A$, $u(\mathbf{x}) = \operatorname{conv}(u)(\mathbf{x})$ iff there exists $\mathbf{p}$ such that $\mathbf{x} \in D_u(\mathbf{p})$;
(2) $\mathcal{L}_u = \mathcal{L}_{u'}$, where u' is the restriction of $\operatorname{conv}(u)$ to $\operatorname{conv}(A) \cap \mathbb{Z}^n$.

However, restricting to concave valuations does not resolve the only ambiguity in associating valuations to weighted rational polyhedral complexes. Adding a constant to $u(\mathbf{x})$ leaves the LIP unchanged, as does increasing every available bundle by a fixed bundle and making a corresponding shift in the valuation.[13] So to give a full equivalence between weighted LIPs and concave valuation functions we must specify the exact demand set at some price, and the value of one bundle.

Theorem 2.15 (Mikhalkin, 2004, Remark 2.3)**.** *Let $(\Pi, \mathbf{w})$ be an $(n-1)$-dimensional balanced weighted rational polyhedral complex in $\mathbb{R}^n$, let $\mathcal{L}$ be the union of the cells of Π, and let $\mathbf{p}$ be any price not contained in $\mathcal{L}$. Then there exists a unique concave valuation u such that $D_u(\mathbf{p}) = \{\mathbf{0}\}$, $u(\mathbf{0}) = 0$, $\mathcal{L} = \mathcal{L}_u$ and $\mathbf{w} = \mathbf{w}_u$.*

In sum, Thms. 2.10 and 2.15 tell us that we can develop our understanding of valuations by working directly with geometric pictures of unions of cells which form balanced weighted rational polyhedral complexes. Any such geometric picture corresponds to a concave valuation, which is unique up to the ambiguity described. However, we will *not* restrict attention to concave valuations.

[12] See Appendix A.1; these results are illustrated by the example in the next subsection (2.3). For the divisible case see, e.g., Mas-Colell et al. (1995) pp. 135-8, especially Prop. 5.C.1(v), since a quasilinear valuation is equivalent to a standard profit function with a single-output technology.

[13] Of course, the bundle demanded at any price is then increased by the fixed bundle.

2.3 The Demand Complex

We constructed the LIP in price space. It is now useful to construct a dual geometric object–the demand complex–in quantity space.

We saw that the LIP consists of cells, each of which is a set of price vectors at which a given set of bundles is demanded. Conversely, the demand complex is a collection of cells, each cell being the convex hull of a set of bundles (quantity vectors) which are demanded at a given set of price vectors. (We will see–Prop. 2.21–that working with the convex hulls of demand sets, rather than with the sets themselves, yields a valuable duality with weighted LIPs.)

Definition 2.16.
(1) The *demand complex* Σ_u is the set of all *cells* $\sigma := \text{conv}\,(D_u(\mathbf{p}))$ where $\mathbf{p} \in \mathbb{R}^n$.
(2) The *vertices* of the demand complex are its 0-cells.
(3) The *edges* of the demand complex are its 1-cells.
(4) The *length* of an edge is the number of primitive integer vectors, in its direction, of which it is formed (i.e., its Euclidean length divided by the Euclidean length of the primitive integer vector in its direction).

It is easy to see that every cell in Σ_u is a rational polyhedron. Furthermore,

Proposition 2.17. *The demand complex is a rational polyhedral complex, with dimension equal to that of* $\text{conv}(A)$*.*

We will understand this proposition via an alternative description of the demand complex, which aids intuition and also makes it easy to quickly develop examples.

First, note it is clear that:

Lemma 2.18. $D_{\text{conv}(u)}(\mathbf{p}) = \text{conv}\,(D_u(\mathbf{p}))$ *for all* $\mathbf{p} \in \mathbb{R}^n$*.*

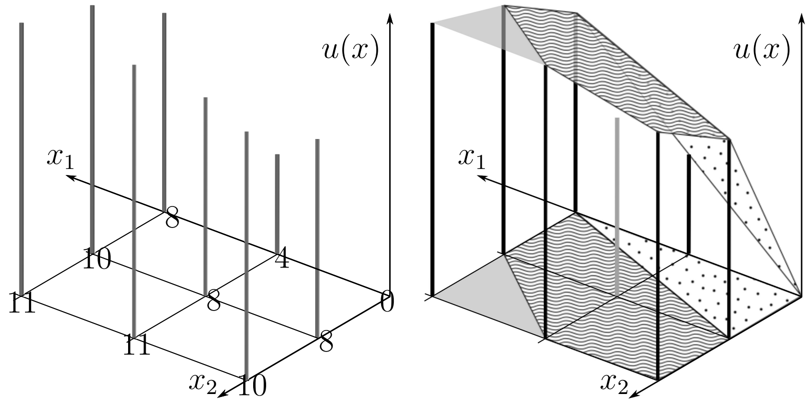
Now, $\text{conv}(u)$ can be understood as a valuation function on divisible goods. So we can use the standard construction for a concave valuation: any price vector defines a hyperplane, tangent to the graph of the agent's valuation, which meets this graph at the agent's demand set for that price. But because $\text{conv}(u)$ is only *weakly*-concave, some tangent hyperplanes meet the graph at more than one point, and some demand sets are multi-valued.

For example, Fig. 2a shows a valuation function, u, and Fig. 2b illustrates, using bars to represent the valuations, $u(\mathbf{x})$, of bundles $\mathbf{x}$. We will always present the feasible bundles increasing to the *left*, and *down.* This will reveal the duality between the demand complex and the weighted LIP most clearly.

Fig. 2c shows the graph of $\text{conv}(u)$. We call this the "roof" of the valuation. At any price $\mathbf{p}$, the bundles, $\mathbf{x}$, demanded under the valuation $\text{conv}(u)$, are those that maximise $\text{conv}(u)(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = (-\mathbf{p}, 1) \cdot (\mathbf{x}, \text{conv}(u)(\mathbf{x}))$. That is, $\mathbf{x}$ is demanded at $\mathbf{p}$ if the point $(\mathbf{x}, \text{conv}(u)(\mathbf{x}))$ is "farthest out" from the origin in the "direction of that price" (i.e., in the direction $(-\mathbf{p}, 1)$). So an intersection between the roof and a supporting hyperplane is a set of the form $\hat{\sigma} = \{(\mathbf{x}, \text{conv}(u)(\mathbf{x})) \in \mathbb{R}^{n+1} : \mathbf{x} \in D_{\text{conv}(u)}(\mathbf{p})\}$, where $\mathbf{p}$ is such that $(-\mathbf{p}, 1)$ is normal to the hyperplane. We call these sets the *faces of the roof* (cf. Defn. 2.6(2)). And projecting such a face from $\mathbb{R}^{n+1}$ to its first n coordinates (in $\mathbb{R}^n$) just yields the set $D_{\text{conv}(u)}(\mathbf{p}) = \text{conv}\,(D_u(\mathbf{p}))$ for that $\mathbf{p}$. So:

x_1: *2*	*1*	*0*	$u(\mathbf{x})$	
8	4	0	*0*	x_2
10	8	8	*1*	x_2
11	11	10	*2*	x_2

(a) Tabular representation of a valuation, $u(\mathbf{x})$.



(b) A representation of the valuation using bars.

(c) The "roof" of the valuation (the graph of $\mathrm{conv}(u)$).

Figure 2: A valuation and its roof.

Lemma 2.19. *$\hat{\sigma} \in \mathbb{R}^{n+1}$ is a face of the roof iff the projection of $\hat{\sigma}$ to its first n coordinates is a cell $\sigma \in \mathbb{R}^n$ of the demand complex.*

So projecting the faces of the roof into $\mathbb{R}^n$ yields the collection of all demand complex cells $\mathrm{conv}\,(D_u(\mathbf{p}))$. This is illustrated by the projection beneath the roof in Fig. 2c, and the demand complex in Fig. 3a.[14] Moreover, it is clear that the faces of the roof are faces of a polyhedron, namely, the convex hull of the points $(\mathbf{x}, u(\mathbf{x}))$. So these faces form a polyhedral complex. Prop. 2.17 follows from the fact that the projection of this complex to its first n coordinates is one-to-one. (Details are given in Appendix A.2).

Fig. 3a shows the three 2-cells (areas), shaded to match the corresponding pieces of planes of the roof in Fig. 2c. The 2-cells are separated by nine *edges* (line-segments that are 1-cells), that themselves meet in the seven *vertices* (0-cells) of the demand complex.

Note that only the "white" circles represent vertices. The grey and black circles represent bundles that are *not* at vertices of the demand complex, since they are not uniquely demanded at any price. Indeed the demand complex cannot tell us whether non-vertex bundles such as these are ever demanded. However, it *does* tell us that if a non-vertex bundle is demanded at any price, then it is demanded at the price(s) corresponding to those cells in which it lies. This follows straightforwardly from Lemma 2.14(1)'s result that if a bundle, $\mathbf{x}$, is demanded at *any* price, then $u(\mathbf{x}) = \mathrm{conv}(u)(\mathbf{x})$, together with the observation that $D_u(\mathbf{p})$=$\{\mathbf{x} : u(\mathbf{x}) = \mathrm{conv}(u)(\mathbf{x})\} \cap D_{\mathrm{conv}(u)}(\mathbf{p})$, and Lemma 2.18. So we have proved:

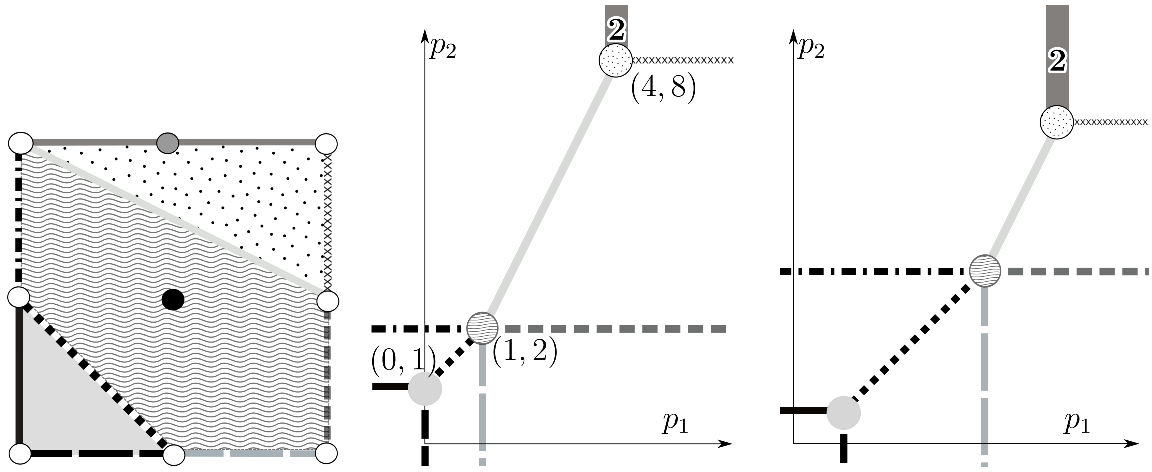
Lemma 2.20. *If there is* any *price $\mathbf{p}$ at which $\mathbf{x}$ is demanded, and if $\mathbf{x} \in \mathrm{conv}\,(D_u(\mathbf{p}))$, then $\mathbf{x} \in D_u(\mathbf{p})$.*

We discuss the example in more detail in the next section.

[14]We depict the demand complex by drawing its top-dimensional cells, on a grid of integer bundles. The remaining cells are easily identified as faces of the top-dimensional cells, while the grid allows us to identify the "lengths" of edges and the bundles in any cell. We omit axes, since replacing A with $A + \mathbf{x}$ for some $\mathbf{x} \in \mathbb{Z}^n$, and re-defining u correspondingly, yields a demand complex dual to the same weighted LIP.

2.4 Duality

We can now see an instructive (and beautiful) duality between the demand complex and the weighted LIP.[15]



(a) Σ_u, with the grid of integer bundles in $\mathrm{conv}(A)$.

(b) The weighted LIP $(\mathcal{L}_u, \mathbf{w}_u)$, which is dual to Σ_u.

(c) A different weighted LIP dual to Σ_u.

Figure 3: (a)-(b) The demand complex and weighted LIP of the valuation u given in Fig. 2a; dual geometric objects have the same style and shading. The weighted LIP of a *different* valuation from u, *also* dual to the demand complex of (a), is shown in (c).

Since the vertices of the demand complex are at bundles which are uniquely demanded for some price, they correspond to UDRs. And an edge of the demand complex between vertices $\mathbf{x}$ and $\mathbf{x}'$ indicates the existence of prices, $\mathbf{p}$, for which the demand set contains both these bundles. Moreover, such $\mathbf{p}$ form an ($(n-1)$-dimensional) facet of the LIP, as they are defined by only one equality constraint $u(\mathbf{x})-\mathbf{p}\cdot\mathbf{x} = u(\mathbf{x}')-\mathbf{p}\cdot\mathbf{x}'$.[16] And as we saw in Prop. 2.5, $\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x}) =$ *constant*, for all these price vectors, $\mathbf{p}$. So each edge of the demand complex is normal to the facet that corresponds to it in the LIP. And more generally:

Proposition 2.21 (**Duality**). *There is a bijective correspondence between: vertices of the demand complex and closures of UDRs; between edges of the demand complex and weighted facets of the LIP; and, for $k \geq 1$, between k-cells σ of the demand complex and $(n-k)$-cells C_σ of the LIP; such that:*

(1) $\sigma = \mathrm{conv}\,(D_u(\mathbf{p}))$ iff $\mathbf{p} \in C_\sigma^\circ$;
(2) $C_\sigma = \{\mathbf{p} \in \mathbb{R}^n : \sigma \subseteq \mathrm{conv}\,(D_u(\mathbf{p}))\}$;
(3) inclusion relationships reverse: $\sigma \subsetneq \sigma' \Leftrightarrow C_{\sigma'} \subsetneq C_\sigma$;
(4) dual cells are orthogonal: $(\mathbf{p}'-\mathbf{p})\cdot(\mathbf{x}'-\mathbf{x}) = 0$ for all $\mathbf{p},\mathbf{p}' \in C_\sigma$, $\mathbf{x},\mathbf{x}' \in \sigma$;
(5) facets F_σ correspond to edges σ of length $w_u(F_\sigma)$.

[15]The construction uses Legendre-Fenchel duality; e.g. see Murota (2003). For more on these 'regular subdivisions' and on polytopes in general see Thomas (2006) and De Loera et al. 2010.

[16]If there are additional points in A lying on the edge, they do not impose additional linearly independent constraints on such $\mathbf{p}$; see the discussion following relating to the "dark-grey edge".

The demand complex and weighted LIP of the valuation of Fig. 2a are pictured in Figs. 3a and 3b respectively; cells which are dual are depicted in the same style.

Thus the 0-cells of the LIP at the prices $(4, 8)$, $(1, 2)$, and $(0, 1)$ are dual to the dotted-, wavy-, and light-grey-, shaded 2-cells of the demand complex, respectively; the nine facets of the LIP are dual to the nine correspondingly-styled edges of the demand complex; and each of the seven UDRs, around the LIP, is dual to one of the seven bundles at the white circles that are the seven vertices of the demand complex.

Notice that the dark-grey horizontal edge at the top of the demand complex passes through a bundle, and has *length* 2 (in the sense of Defn. 2.16(4)). It is dual to the dark-grey vertical facet of the LIP, which correspondingly has *weight* 2, and is so labelled. Recall from Prop. 2.5 that a facet's "weight" times its primitive integer normal vector is the change in demand between the UDRs it separates. All other edges of this demand complex have length 1; all other facets of the LIP correspondingly have weight 1.

As we noted in the previous subsection, neither the grey bundle, nor the black bundle, is at a vertex of the demand complex, since neither is ever *uniquely* demanded for any price, so nor do they correspond to any UDRs.

Furthermore, *neither* the LIP *nor* the demand complex can tell us whether a non-vertex bundle such as one of these is ever demanded. However we do know, from Lemma 2.20, that because the central wavy-shaded (five-sided) demand complex cell is the only demand complex cell that the black bundle lies in, the corresponding wavy-shaded 0-cell of the LIP in which that bundle is "hidden" indicates the only price, (1,2), at which that bundle *might* be demanded. Similarly, because the dark-grey horizontal edge at the top of the demand complex is the lowest-dimensional demand complex cell that the grey bundle lies in, the corresponding dark-grey vertical facet of the LIP in which that bundle is "hidden" indicates the only prices ($(4, p_2)$ for $p_2 \geq 8$–see Fig. 3b) at which that bundle might be demanded.

In fact, $(\mathbf{x}, u(\mathbf{x}))$ is in the roof for a non-vertex bundle, $\mathbf{x}$–and so the bundle is demanded–if and only if the valuation, u, is affine in the relevant range. The grey bundle is an example of this. It is at $(1, 0)$, and its valuation, 4, is the average of the valuations, 0 and 8, of the bundles $(0, 0)$ and $(2, 0)$, so it is demanded at the prices $\{(4, p_2) : p_2 \geq 8\}$.

However, if u is non-concave at a non-vertex bundle, the bundle's value lies strictly below the roof, so it is never demanded–it is "jumped over" as we cross between UDRs. The black bundle in the centre of demand complex illustrates this. Its value under u is strictly below its value under $\operatorname{conv}(u)$, so it lies strictly under the "roof" (see Fig. 2c) and is never demanded at any price. (See Appendix A.2 for more discussion.)

Prop. 2.21, and the remark above Prop. 2.7, allow us to characterise the set of prices at which a bundle $\mathbf{x}$ is demanded, if it is demanded at any prices:

Corollary 2.22. *If σ is the* minimal *cell of the demand complex such that $\mathbf{x} \in \sigma$, and if $\mathbf{x}$ is demanded for any price, then $\mathbf{x} \in D_u(\mathbf{p})$ iff $\mathbf{p} \in C_\sigma$. In particular, the set of prices at which $\mathbf{x}$ is demanded forms a polyhedron.*

Finally, note that, for any single demand complex, there are multiple weighted LIPs which satisfy the correspondences and orthogonality relationships of Prop. 2.21. For example, Figs. 3b and 3c give two different weighted LIPs–and therefore two different valuations–that are both dual to the demand complex of Fig. 3a.[17] So it is natural to group together all valuations whose demand complexes are either the same, or differ only by a constant shift by some bundle, $\mathbf{x}$:

Definition 2.23. Two valuations u, u' have the same *combinatorial type* if they have the same demand complex, or if there exists $\mathbf{x} \in \mathbb{Z}^n$ such that $\sigma \in \Sigma_u$ iff $\{\mathbf{x}\} + \sigma \in \Sigma_{u'}$.

It is easy to list all the possible demand complexes, and examples of dual weighted LIPs which exhibit the combinatorial type (thus giving all "essentially-different" structures of demand) if the domain is not too large–see Figs. 11-12 in Appendix A.2.

2.5 Representation in Price Space *vs.* Quantity Space

Although the weighted LIP and demand complex are dual, there is an important distinction. In price space, any rational polyhedral complex satisfying the simple "balancing condition" of Defn. 2.9 corresponds to some valuation (see Thm. 2.10). But, in quantity space, it is *not* true that every way of subdividing $\text{conv}(A)$ into a rational polyhedral complex yields a demand complex. (See Maclagan and Sturmfels, 2015, Fig. 2.3.9 for an example of a subdivision which corresponds to no LIP, and therefore to no valuation.) Nor does there seem to be any simple check of which polyhedral complexes in quantity space correspond to any valuation function.

So while we can develop examples to, e.g., test conjectures, by working with geometric objects in price space, and be certain that the corresponding valuations will exist, it is hard to do this in quantity space. Furthermore, a demand complex shows only collections of bundles among which the agent is indifferent for *some* prices, while LIPs show the *actual* prices at which bundles are demanded. So we have found in practice that it is usually easier to develop ideas by working with our geometric objects in price space, than by working either in quantity space, or directly with valuation functions.

It is also much easier to aggregate agents' valuations in price space (see Section 3.3).

So we mostly work in price space.

However, some information that is only implicit in the weighted LIP becomes obvious in the demand complex. For example, we will see in Sections 4.1 and 5 that a low-dimensional cell of the LIP sometimes "hides" important detail that is much more easily seen in the higher-dimensional dual object in the demand complex, in quantity space.

Moreover, the easiest way to compute the LIP of a specific valuation is often by first finding the demand complex–it is easy to go from Fig. 2a to Fig. 3a in our example, and then also easy to use the duality to find a weighted LIP of the correct combinatorial type, and from that to find the exact LIP (that is, Fig. 3b is easily found from Fig. 3a, see Appendix A.2). It is generally much harder to construct the LIP directly from the valuation, especially for more complicated examples than ours.

The fact that the different representations are useful in different contexts makes the ability to move easily between them, using duality, especially valuable.

[17] We are here using category-theoretic "duality", thus allowing an object to have multiple, equivalent, "duals".

3 "Demand Types"

3.1 Definition of Demand Types, and Comparative Statics

We saw in the previous section that the LIP's facet normals describe how demand changes between UDRs (Prop. 2.5). They therefore give all the possible directions of change in demand that can generically result from a small change in prices. So it is natural to classify valuations into "demand types" according to these facet normals. A valuation's demand type then gives us comparative statics information that is analogous to the information that the Slutsky matrix provides for a valuation on divisible goods. (The dimensionality is low enough with indivisibilities that we can characterise a class of valuations globally in this way, by contrast with the divisible case for which the Slutsky matrix provides information only at a point.)

Definition 3.1. Let $\mathcal{D} \subsetneq \mathbb{Z}^n$ be a set of non-zero primitive integer vectors such that if $\mathbf{v} \in \mathcal{D}$ then $-\mathbf{v} \in \mathcal{D}$. The *demand type* defined by $\mathcal{D}$ comprises valuations u such that every facet of $\mathcal{L}_u$ has normal vector in $\mathcal{D}$.

(We will slightly abuse notation by also writing "u is of demand type $\mathcal{D}$.")

For example, the valuation of Fig. 1 is of demand type $\pm\{(1,0),(0,1),(-1,1)\}$, as are many other valuations, for example, all those shown in Figs. 9a-c. Note that a valuation is of *any* demand type which contains the facet normals of its LIP; we do *not* restrict to the minimal such set.[18]

By duality (Prop. 2.21), we could equivalently classify valuations according to the directions of their demand complexes' edges.[19] But our description makes clear that the demand type provides the generic comparative statics.

Proposition 3.2. *The following are equivalent for a valuation u:*

(1) u is of demand type $\mathcal{D}$.

(2) For valuation u and generic $\mathbf{p}, \mathbf{t}$, if $\exists\ \epsilon > 0$ such that $\mathbf{p}$ and $\mathbf{p} + \epsilon\mathbf{t}$ are in distinct UDRs, and such that $\nexists\ \epsilon' \in (0, \epsilon)$ such that $\mathbf{p} + \epsilon'\mathbf{t}$ is in a third distinct UDR, then the difference between bundles demanded at $\mathbf{p}$ and $\mathbf{p} + \epsilon\mathbf{t}$ is an integer multiple of some vector in $\mathcal{D}$.

That is, the change in demand between a generic starting price, $\mathbf{p}$, and the *next* UDR in any given generic direction of price change, $\mathbf{t}$, is described by one of the demand type's vectors. (The conditions of the proposition ensure there is no third UDR between those containing $\mathbf{p}$ and $\mathbf{p} + \epsilon\mathbf{t}$.) Furthermore, since the domain A is finite, the response to any specific price change can, generically, be broken down into a series of steps of this form.

---

[18] Thus, the valuations of Figs. 1 and 9a-c are also of demand type $\pm\{(1,0),(0,1),(-1,1),(-2,1)\}$ which is the minimal demand type of the valuations of Figs. 2–3.

Note our definition does not consider the weights on facets; see Baldwin and Klemperer (2012, note 25, and 2014, note 42).

[19] Danilov, Koshevoy and their co-authors' work (see Section 4.3) examine these vectors in quantity space. However, they do not use them to create a taxonomy of demand or, e.g., interpret them as giving comparative statics information. We, by contrast, develop a general framework to understand them in economic terms (see also Baldwin and Klemperer, 2012, 2014 and in preparation-b).

That (1)⇒(2) is immediate from Prop. 2.5. For (2)⇒(1), assume (1) fails, so $\mathcal{L}_u$ has a facet F with primitive integer normal $\mathbf{v} \notin \mathcal{D}$. Then we can violate (2) by choosing $\mathbf{p}, \mathbf{p}'$ in the UDRs adjacent to F and positioned close to F, and letting $\mathbf{t} = \mathbf{p}' - \mathbf{p}$.

Baldwin and Klemperer (2014 and in preparation-b) give a full discussion of *non*-generic price changes (which, e.g., pass through a lower-dimensional cell than a facet) and also give other equivalent characterisations of demand types, but Prop. 3.2 will suffice for our purposes.

3.2 Substitutes, Complements, and other "Demand Types"

It follows straightforwardly that demand types provide simple characterisations of familiar concepts such as ordinary substitutes, ordinary complements, and "strong substitutes". These characterisations are easier to generalise than standard ones based on imposing restrictions on u directly. Moreover, they more clearly reveal and explain features such as the lack of symmetry between substitutes and complements. We begin by recalling standard definitions:

Definition 3.3 (Standard)**.**

(1) A valuation u is *ordinary substitutes* if, for any UDR prices $\mathbf{p}' \geq \mathbf{p}$ with $D_u(\mathbf{p}) = \{\mathbf{x}\}$ and $D_u(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \geq x_k$ for all k such that $p_k = p'_k$.[20]
(2) A valuation u is *ordinary complements* if, for any UDR prices $\mathbf{p}' \geq \mathbf{p}$ with $D_u(\mathbf{p}) = \{\mathbf{x}\}$ and $D_u(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \leq x_k$ for all k such that $p_k = p'_k$.
(3) A valuation u is *strong substitutes* if, when we consider every unit of every good to be a separate good, it is a valuation for ordinary substitutes.[21]

It is easy to use Prop. 3.2 to provide alternative, equivalent, definitions of these concepts, as demand types. For substitutes:

Definition 3.4. The (n-dimensional) *ordinary substitutes vectors* are the set of non-zero primitive integer vectors $\mathbf{v} \in \mathbb{Z}^n$ with at most one positive coordinate entry, and at most one negative coordinate entry. They define the *ordinary substitutes demand type* (for n goods).

Proposition 3.5. *A valuation is an ordinary substitutes valuation* iff *it is of the ordinary substitutes demand type.*

[20]We write, as is standard, $\mathbf{p}' \geq \mathbf{p}$ when the inequality holds component-wise.

We call "ordinary substitutes" what most others (e.g., Ausubel and Milgrom, 2002, Hatfield and Milgrom, 2005) simply call "substitutes". We do this for clarity, since some have defined "substitutes" in other ways. In particular, although Kelso and Crawford's (1982) definition is equivalent in their model, it is not generally equivalent if it is extended to multiple units of three or more goods (which yields Milgrom and Strulovici's, 2009, definition of "weak substitutes"); see Danilov et al., 2003, Ex. 6 and Thm. 1. Our definition (3.3(1)) seems the most natural one in the general case. It is also equivalent to several properties that seem to naturally characterise "substitutes", and to the indirect utility function ($\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}\}$) being submodular–see Baldwin, Klemperer and Milgrom (in preparation). See also Baldwin and Klemperer (2014). Hatfield et al. (2013)–see our Section 6.1–and Danilov et al. (2003) use definitions equivalent to 3.3(1), and the latter authors make a similar observation to our Prop. 3.5 when they say "each cell of a valuation's parquet is a polymatroid".

[21]This is equivalent to Milgrom and Strulovici's (2009) definition–see Danilov et al (2003, Cor. 5). There are many other equivalent definitions (see Shioura and Tamura, 2015), the most important being $M^\natural$-concavity" of the valuation (Murota and Shioura, 1999).

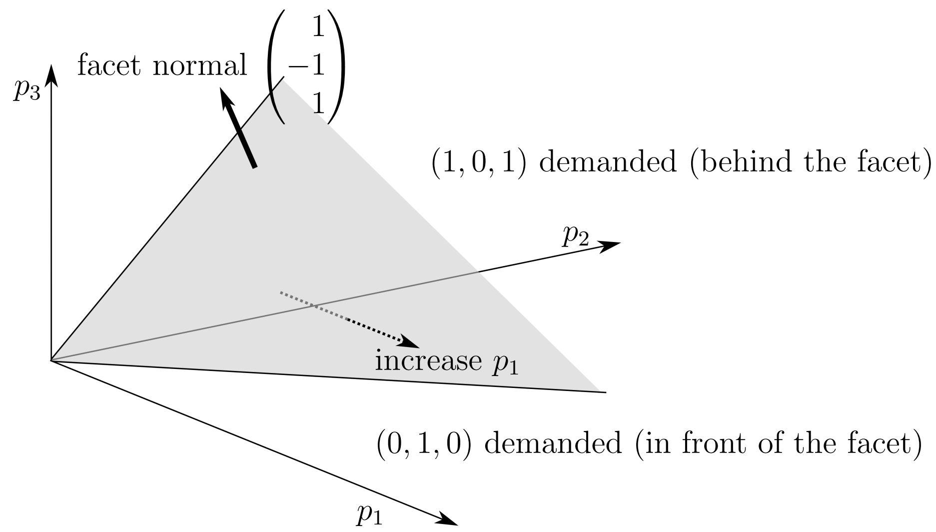


Figure 4: A facet (shaded) defined by $\{\mathbf{p} \in \mathbb{R}^3 : p_1 + p_3 = p_2;\ p_1, p_2, p_3 \geq 0\}$, with its normal (1,-1,1) (the arrow shown in bold). Increasing either p_1 (as shown with a dotted arrow), or p_3, demonstrates complementarities between goods 1 and 3, as the bundle demanded switches from (1,0,1) to (0,1,0).

(Figs. 1 and 3b-3c illustrate the substitutes property holding, and Fig. 4 shows it failing.) So a vector that is normal to a facet of a substitutes LIP cannot have two non-zero entries of the same sign. To understand the necessity of this, see Fig. 4, which depicts a facet whose primitive integer normal vector's first and third coordinates have the same sign. Increasing the price on either good 1 (as pictured) or good 3, can therefore take us across the facet–decreasing demand for *both* goods 1 and 3. So any such facet generates complementarities at some prices, and so cannot be part of a substitutes LIP.

To prove that being of the ordinary substitutes demand type is sufficient for a valuation to be ordinary substitutes, we apply Prop. 3.2 as we cross the finite number of facets between prices $\mathbf{p}$ and $\mathbf{p}' \geq \mathbf{p}$. (The fact that Prop. 3.2 refers only to generic $\mathbf{p}, \mathbf{t}$ does not matter, because Defn. 3.3(1) only requires us to examine UDR prices, and UDRs are open and dense in $\mathbb{R}^n$. So we can always pick "close-by" prices $\widetilde{\mathbf{p}}$ and $\widetilde{\mathbf{p}}'$ so that Prop. 3.2 does apply for this starting price and direction of price change, and so that demand is the same as at $\mathbf{p}$ and $\mathbf{p}'$, respectively.) Now recall the standard result that $(\mathbf{x}'' - \mathbf{x}) \cdot (\mathbf{p}'' - \mathbf{p}) < 0$, where $\mathbf{x}'', \mathbf{x}$ are the bundles demanded at any prices $\mathbf{p}'', \mathbf{p}$, respectively (see e.g. Mas-Colell et al., 1995, Prop. 2.F.1). So, as we cross each facet, demand is strictly reduced for some good whose price has strictly increased (since the remaining prices are constant). Since this good corresponds to a negative entry in the facet's normal vector, and since there is at most one such entry in an ordinary substitutes vector, demand weakly increases at each facet crossing for all goods whose price does not change, so Defn. 3.3(1) is satisfied.

For complements, a price change that reduces demand for a good can of course reduce (but not increase) demand for other goods. So, applying Prop. 3.2 in the same way as for Prop. 3.5, we define and then prove:

Definition 3.6. The (n-dimensional) *ordinary complements vectors* are the set of non-zero primitive integer vectors $\mathbf{v} \in \mathbb{Z}^n$ whose non-zero coordinate entries are all of the same sign. They define the *ordinary complements demand type* (for n goods).

Proposition 3.7. *A valuation is an ordinary complements valuation* iff *it is of the ordinary complements demand type.*

The lack of symmetry between substitutes and complements, and the reason for it are now clear: ordinary complements vectors may have *any number* of non-zero entries (of the same sign), but any pair of non-zero entries in an ordinary substitutes vector must be of opposite signs (recall Fig. 4), so ordinary substitutes vectors can have *at most two* non-zero entries (see Ex. A.3 for more discussion).

The characterisation of strong substitutes as a demand type also gives an intuitive description of them:

Definition 3.8. The *strong substitute vectors* are those non-zero $\mathbf{v} \in \mathbb{Z}^n$ which have at most one $+1$ entry, at most one -1 entry, and no other non-zero entries. They define the *strong substitutes demand type.*

Proposition 3.9 (See Baldwin and Klemperer, 2014, Cor. 5.20; and Shioura and Tamura, 2015, Thm. 4.1(i))**.** *A valuation is strong substitutes iff it is concave and is of the strong substitutes demand type.*

So Figs. 1, 5a, and 9a-c show examples of LIPs of strong substitutes valuations. Note that this characterisation is also parsimonious; with n goods, the strong substitutes demand type is defined by just $n(n+1)/2$ vectors (and their negations).[22]

We will see in Sections 6.2 and 6.3 that demand types also allow us to characterise significant new classes of valuations.

In Section 4 we show that analysing the properties of demand types helps us understand when competitive equilibrium exists.

3.3 Aggregate Demand, and the "Demand Type" of the Aggregate of Multiple Agents

An important feature of our "demand types" classification–that, in particular, greatly facilitates the study of equilibrium–is that the demand type when we aggregate valuations from multiple agents is just the union of the sets of vectors that form the individual agents' demand types.

So we now consider a finite set J of agents: agent $j \in J$ has valuation u^j for integer bundles in a finite set, A^j. Their *aggregate demand* is, of course, the (Minkowski) sum of the individual demands, but to apply our techniques to this, we want to treat it as the demand of a single "aggregate" agent.

Definition 3.10. An *aggregate valuation* of $\{u^j : j \in J\}$ is a valuation u^J with domain $A := \sum_{j\in J} A^j$ such that $D_{u^J}(\mathbf{p}) = \sum_{j\in J} D_{u^j}(\mathbf{p})\ \forall \mathbf{p} \in \mathbb{R}^n$.

Note that aggregate valuations are not uniquely defined. However, this does not matter: since the aggregate demand sets are unambiguous, properties such as concavity of aggregate valuations are also unambiguous, and the aggregate weighted LIP is unique.

[22]Our description is closely related to the "step-wise gross substitutes" of Danilov et al. (2003), which they link to the edge vectors of (what we call) the demand complex; these edge vectors are also linked to $M^\natural$-concavity by Murota and Tamura (2003).

(The fact that we can construct the aggregate LIP from the individual LIPs *without* knowing the form of u^J–so without using any cumbersome formula for u^J–is an important advantage of aggregation in price space.[23])

The rest of this subsection proves and discusses:

Lemma 3.11. *Given a finite set of valuations $\{u^j : j \in J\}$:*
(1) an aggregate valuation u^J exists;
(2) $\mathcal{L}_{u^J} = \bigcup_{j \in J} \mathcal{L}_{u^j}$;
(3) If F is a facet of $\mathcal{L}_{u^J}$, then $w_{u^J}(F) = \sum_{F^j \in \mathcal{F}} w_{u^j}(F^j)$, in which $\mathcal{F}$ is the set of all facets of the individual $\mathcal{L}_{u^j}$ which contain F.

Corollary 3.12. *A collection of individual valuations are all of demand type $\mathcal{D}$ iff every aggregate valuation of every finite subset of them is of demand type $\mathcal{D}$.*

For example, Figs. 5a-b show the LIPs of Elizabeth's and Paul's valuations for the hotel rooms of our introductory example. Both valuations have domain $\{0, 1\}^2$. Elizabeth regards the rooms as substitutes; her valuation is $u^s(x_1, x_2) = \max\{40x_1, 30x_2\}$ (Fig. 5a). Paul regards them as complements; his valuation is $u^c(x_1, x_2) = \min\{50x_1, 50x_2\}$ (Fig. 5b).

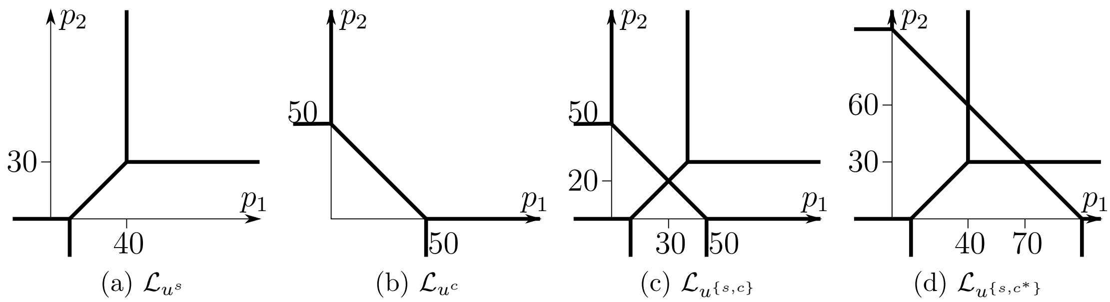


Figure 5: The LIPs of (a) a simple substitutes valuation; (b) a simple complements valuation; (c) any aggregate valuation of the substitutes and complements valuations shown; (d) any aggregate valuation of the substitutes valuation shown and a complements valuation with a higher value for the bundle of both rooms together.

It is easy to see that an aggregate demand set consists of a unique bundle iff all the individual demand sets do (and so to prove Lemma 3.11(2)).[24] Thus Fig. 5c shows the aggregate LIP, $\mathcal{L}_{u^{\{s,c\}}}$, for the valuations u^s and u^c. It is obvious that a demand type contains the individual valuations iff it contains any aggregate valuation (Cor. 3.12).

From the aggregate LIP we can obtain a polyhedral complex, Π, in the usual way (Prop. 2.7 and Lemma 2.11). Its cells are either cells, or subsets of cells, of the individual

[23]An implication of quasi-linear preferences is that $\max\left\{\sum_{j \in J} u^j(\mathbf{x}^j) : \mathbf{x}^j \in A^j, \sum_{j \in J} \mathbf{x}^j = \mathbf{y}\right\}$ is an aggregate valuation, as is well-known. Mathematically, this aggregate valuation is the tropical product of the tropical polynomials that are the individual valuations. Economically, it says that the aggregate value of a bundle is the maximum sum of agents' values that can be obtained by apportioning the bundle among the agents.

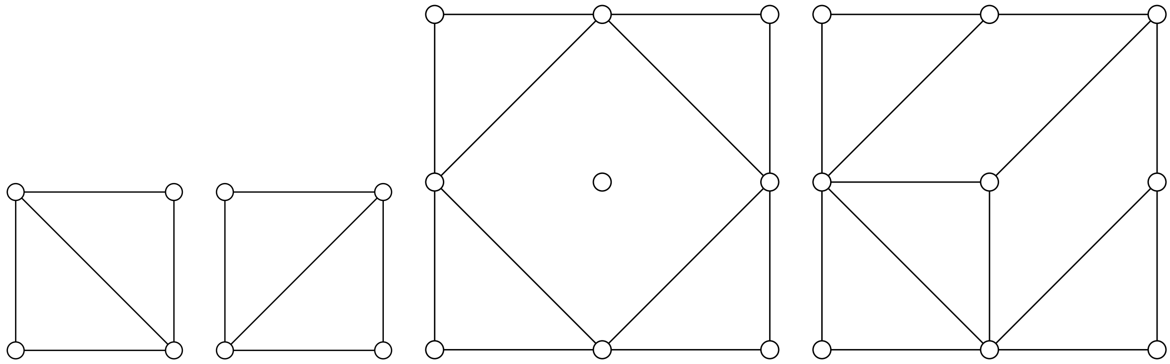
Adding a constant to the valuation, and/or changing the value of never-demanded bundles to leave them never demanded, also provide aggregate valuations according to Definition 3.10.

[24]This has been previously observed in pictures of demand correspondences, see for example Murota, 2003, Section 11.2.

LIPs: cells of individual LIPs that intersect in their interiors are split up into new, smaller cells in the aggregate LIP. Thus the price $(30, 20)$ in Fig. 5c is a 0-cell, on the boundary of four distinct 1-cells.

The change in aggregate demand between any pair of prices is just the sum of the changes of the individual demands. So the weight of any facet F of the aggregate LIP is just the sum of the weights of all the facets F' of the individual LIPs for which $F \subseteq F'$ (which proves Lemma 3.11(3)). And since the weighted polyhedral complex $(\Pi, \mathbf{w})$ is derived from balanced complexes, it is itself balanced, and so (using Thm. 2.10) it is the LIP of some valuation (so Lemma 3.11(1) holds).

Notice, however, that we *cannot* find the demand complex of an aggregate valuation using only the individual demand complexes–because a demand complex does not correspond to a unique valuation, and different valuations may aggregate in different ways.



(a) Σ_{u^s} (b) Σ_{u^c}, $\Sigma_{u^{c*}}$ (c) $\Sigma_{u^{\{s,c\}}}$ (d) $\Sigma_{u^{\{s,c*\}}}$

Figure 6: Demand complexes dual to the LIPs in Figs. 5a-d, when every facet has weight 1. (We also show grids of relevant integer bundles; we do not distinguish which bundles are never demanded.)

For example, the demand complexes corresponding to the LIPs of Figs. 5a-b are shown in Figs. 6a-b. The demand complex corresponding to their aggregate LIP (Fig. 5c) is shown in Fig. 6c; its domain is $\{0,1\}^2 + \{0,1\}^2 = \{0,1,2\}^2$. If Paul's valuation increases to $u^{c*}(x_1, x_2) = \min\{100x_1, 100x_2\}$, then his demand complex remains that of Fig. 5a. However, the LIP $\mathcal{L}_{u^{\{s,c*\}}}$ is shown in Fig. 5d, and its demand complex is that of Fig. 6d. So there is no unique aggregate demand complex corresponding to the demand complexes of Fig. 6a and Fig. 6b.

4 The Unimodularity Theorem–when does Equilibrium *always* exist for a "Demand Type"?

This section shows that our "demand types" classification yields a powerful theorem about when competitive equilibrium is and is not guaranteed.

This theorem requires much weaker assumptions about agents' preferences than used in the existing leading economics literature (though we retain the standard assumption of quasilinear preferences). So our condition for equilibrium is correspondingly much more general. It immediately generalises, for example, equilibrium results in Kelso and Crawford (1982), Gul and Stacchetti (1999), Sun and Yang (2006), Milgrom and

Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014). In particular it is *not* necessary for all agents to have substitutes valuations (or some basis change thereof) for equilibrium to always exist; complements valuations guaranteeing equilibrium are easy to find.

We state the Unimodularity Theorem and some immediate corollaries in the next subsection; give intuition and the proof in Section 4.2; and explain the close connections with Danilov et al. (2001) in Section 4.3.

A variety of applications of the theorem are in Section 6.

4.1 Statement of Results

We are interested in the standard notion of competitive equilibrium:

Definition 4.1. An *equilibrium* exists, for a market supply $\mathbf{x} \in \mathbb{Z}^n$ and a finite set of valuations, if $\mathbf{x}$ is in the valuations' aggregate demand set for some price.

It is standard (Lemma 2.13) that concavity of an aggregate valuation u^J is necessary and sufficient for equilibrium to exist for all integer bundles in the convex hull of the domain of u^J. We therefore refer to these bundles as the *relevant supply bundles*: equilibrium will clearly never exist for other bundles, as they are the wrong "size".

Concavity of individual valuations is therefore necessary even for all one-agent economies to have equilibrium, so our results will also restrict attention to concave valuations.

With *in*divisible goods (unlike with divisible goods), individual concavity is *not* sufficient to guarantee aggregate concavity (and so, for example, supporting hyperplanes do not necessarily exist). However, our geometric approach provides a simple additional condition that *is* sufficient to guarantee equilibrium. First we define:

Definition 4.2. A set of vectors in $\mathbb{Z}^n$ is *unimodular* if every linearly independent subset can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1.

By "the *determinant*" of n vectors we mean the determinant of the $n \times n$ matrix which has them as its columns.[25] If the set of vectors spans $\mathbb{R}^n$, then there exist sets of n of them that are linearly independent; it is therefore, of course, sufficient to check that all n-element sets have determinant ± 1 or 0. Alternative equivalent conditions for unimodularity are given by Facts 4.8 and A.5, so unimodularity of a demand type's vectors is not too hard to check (see also Remark A.24). We refer to "unimodular demand types" in the obvious way. We can now state:

Theorem 4.3 (The Unimodularity Theorem). *An equilibrium exists for* every pair *of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles,* iff *$\mathcal{D}$ is unimodular.*

We will prove Thm. 4.3 as the combination of Props. 4.10 and 4.17, below.

It follows that if the demand type is unimodular, then a valuation obtained by aggregating any two concave valuations is also a concave valuation, so we can apply the result repeatedly:

[25] We ignore the order of the vectors since we are only interested in determinants' absolute values.

Corollary 4.4. *An equilibrium exists for* every finite set *of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles,* iff *$\mathcal{D}$ is unimodular.*

It is also immediate from the discussion above that:

Corollary 4.5. *With n goods, if the vectors of $\mathcal{D}$ span $\mathbb{R}^n$, then an equilibrium exists for* every *finite set of concave valuations of demand type $\mathcal{D}$, for all relevant supply bundles,* iff *every subset of n vectors from $\mathcal{D}$ has determinant* 0 *or* ± 1.

Many standard results are immediate special cases. For example, it is well-known that "strong substitute" vectors form a unimodular set (Poincaré, 1900), and these valuations are (by definition) concave, so:

Proposition 4.6 (Danilov et al., 2001, 2003, and Milgrom and Strulovici, 2009)**.** *An equilibrium exists for* every *finite set of strong substitutes valuations, for all relevant supply bundles.*

Other familiar equilibrium results, such as those of Kelso and Crawford (1982) and Gul and Stacchetti (1999), of course also immediately follow.

Section 6 gives many other applications.

4.2 Intuition and Proof for the Unimodularity Theorem

4.2.1 The Role of Intersections

The first critical observation is that we can determine whether equilibrium exists by focusing only on intersections of individual LIPs: we know equilibrium always exists, that is, every relevant bundle is demanded at some price, *iff* any aggregate valuation is concave *iff* the aggregate demand set is discrete-convex at every price (Lemma 2.13). But if all but one of the agents have unique demand at some price, the aggregate demand set is simply the shift of the remaining agent's demand set by the other agents' (unique) demands. And this set must be discrete-convex, since we assumed that every individual valuation is concave. So we only need to check prices at which two or more agents have non-unique demand. This proves:

Lemma 4.7. *For concave valuations $u^1, \ldots, u^s$, an equilibrium exists for every relevant supply bundle* iff *the aggregate demand set is discrete-convex at every price in every* intersection $\mathcal{L}_{u^j} \cap \mathcal{L}_{u^{j'}}$ *for $j, j' = 1 \ldots, s$, $j \neq j'$.*

So, for example, for the case of our "hotel rooms" example in the introduction which had simple two-goods substitutes and complements valuations $u^s(x_1, x_2) = \max\{40x_1, 30x_2\}$ (Fig. 5a) and $u^c(x_1, x_2) = \min\{50x_1, 50x_2\}$ (Fig. 5b), whose aggregate LIP is shown in Fig. 5c, the *only* price we need to analyse is the intersection $(30, 20)$.

4.2.2 Unimodularity

Our "hotel rooms" example also illustrates the role of unimodularity.

The aggregate demand of u^s and u^c at the (only) price we need to check, the intersection price $(30, 20)$, is the sum of the individual demands, $D_{u^s}(30, 20) = \{(1, 0), (0, 1)\}$ and $D_{u^c}(30, 20) = \{(0, 0), (1, 1)\}$. So $D_{u^{\{s,c\}}}(30, 20)$ consists of the bundles at the corners of a square, $\{(1,0),(0,1),(2,1),(1,2)\}$ (the diamond at the centre of Fig. 6c). However, this square also contains the non-vertex integer bundle $(1,1)$, and since $(1,1)$ is not in the demand set at this price, it is not demanded at any price (by Lemma 2.20). So equilibrium fails; although both u^s and u^c are concave, any aggregate valuation for them is *not*. (Note we did *not* need to actually aggregate the valuations u^s and u^c to determine this. However, Fig. 13 and Ex. A.4 show directly that $u^{\{s,c\}}$ is indeed non-concave.)

To see the relevance of unimodularity to the example and also, we will see, more generally, note two equivalent conditions to unimodularity:

Fact 4.8. A set of vectors in $\mathbb{Z}^n$ is unimodular iff the following equivalent conditions hold for every linearly independent subset $\{\mathbf{v}^1,\ldots,\mathbf{v}^s\}$ of this set:

(1) the parallelepiped whose edges are these vectors, that is, $\left\{\sum_{j=1}^{s}\lambda^j\mathbf{v}^j : \lambda^j\in[0,1]\right\}$, contains no non-vertex integer point;

(2) if $\mathbf{x}\in\mathbb{Z}^n$ and $\mathbf{x}=\sum_{j=1}^{s}\alpha^j\mathbf{v}^j$ with $\alpha^j\in\mathbb{R}$, then $\alpha^j\in\mathbb{Z}$ for all j.

Fact 4.8(1) says that a parallelepiped (that is, an s-dimensional parallelogram) whose vertices are integer points contains a non-vertex integer point iff the vectors along its edges do not form a unimodular set. Now recall two geometric facts for the case $s=n$: that the volume of a parallelepiped is given by the (absolute value of the) determinant of its edge vectors; and that a parallelepiped whose vertices are integer points contains a non-vertex integer point iff its volume strictly exceeds 1. In our example, the edges of the demand complex cell (the diamond in Fig. 6c) are in directions $(1,1)$ and $(-1,1)$. Their determinant is 2, so the area of the cell is 2, and it therefore contains a non-vertex integer point at which equilibrium may fail. (Thm. 4.3 warned of this, since the determinant being 2 directly reveals a failure of unimodularity.)

So equivalent condition 4.8(1) will help to demonstrate the necessity of Thm. 4.3's condition for equilibrium.[26]

A second way to see the relevance of unimodularity is given by Fact 4.8(2), which tells us that any vector in the space spanned by a given set of vectors can be created as an integer combination of the set iff the set is unimodular. The four corners of the square are vertices of the aggregate demand complex cell, and are therefore dual to UDRs that each contain the price $(30,20)$ in their boundary (Prop. 2.21). If we move between these UDRs, around the price $(30,20)$, aggregate demand changes by the vector normal to any facet crossed (Prop. 2.5), that is, by the vector in the direction of one of the square's edges. Moreover, at $(30,20)$, the only possible changes in individual demand, and hence the only possible changes in aggregate demand, are made up of these vectors. So the impossibility of demanding (1,1) on aggregate at this price vector, and so (by Lemma 2.20) anywhere, is equivalent to the impossibility of obtaining (1,1) by starting at any of the four bundles at the corners of the square ($(1,0),(0,1),(2,1)$, and $(1,2)$), and adding integer combinations of the edge vectors (1,1) and (-1,1).

More generally if, by contrast, every demand complex cell's edge vectors were a unimodular set, this problem could not arise. So equivalent condition 4.8(2) will be useful for demonstrating the sufficiency of Thm. 4.3's condition for equilibrium.

[26] When the set of vectors is *not* unimodular, the number of non-vertex bundles in such a parallelepiped is one less than the determinant (Fact 5.15). So we expect these determinants should give bounds on the extent to which supply constraints need to be relaxed to achieve equilibrium. (Moreover, the extension of our results to matching theory–see Section 6.6–might then yield results related to Nguyen and Vohra, 2014 and Nguyen et al., 2016.)

4.2.3 Proof of Necessity Condition for the Unimodularity Theorem

It is now easy to prove the necessity of the condition of our Thm. 4.3 by replicating the situation of the simple example discussed in Sections 4.2.1-4.2.2 above.

Let agent j have a concave valuation, and be indifferent between (only) the bundles $\mathbf{x}^j, \mathbf{x}^j + \mathbf{v}^j$, at some price $\mathbf{p}$, for $j = J$, with $|J| \leq n$. (This requires that $\mathbf{v}^j$ are primitive integer vectors.) So $D_{u^J}(\mathbf{p}) = \left\{\sum_{j\in J}(\mathbf{x}^j + \delta_j \mathbf{v}^j) : \delta_j \in \{0,1\}; j \in J\right\}$. If the $\mathbf{v}^j$ are linearly independent, then this set is precisely the vertices of a $|J|$-dimensional parallelepiped in $\mathbb{Z}^n$ with edges $\mathbf{v}^j$. By Fact 4.8(1) there exists a non-vertex integer bundle in this parallelepiped iff the set $\{\mathbf{v}^j : j \in J\}$ is not unimodular. So, applying Lemma 2.20:

Lemma 4.9. *Under the hypotheses of the paragraph above, the set* $\{\mathbf{v}^j : j \in J\}$ *is unimodular iff every integer bundle in* $\operatorname{conv}(D_{u^J}(\mathbf{p}))$ *is demanded for some price.*

For Thm. 4.3, we need to show a failure of equilibrium with only two agents. So for any non-unimodular linearly independent set of primitive integer vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s \in \mathcal{D}$, find the k such that the set $\mathbf{v}^1, \ldots, \mathbf{v}^k$ is unimodular, but the set $\mathbf{v}^1, \ldots, \mathbf{v}^{k+1}$ is not.[27] Let agents $j = 1, \ldots, k+1$ be as above and specify that $\mathbf{x}^j, \mathbf{x}^j + \mathbf{v}^j$ are the *only* bundles in the domain of j's valuation. Write $J = \{1, \ldots, k\}$. By Lemma 4.7, equilibrium could possibly fail for $\{u^j : j \in J\}$ only at the intersection of two or more of their LIPs. But by Lemma 4.9, this does not happen, since the set $\{\mathbf{v}^j : j \in J\}$ (and any subset thereof) *is* unimodular, so u^J is concave (by Lemma 2.13). But if u^J is the valuation of a new agent k^*, then we can set $u^{\{k^*,k+1\}} = u^{J\cup\{k+1\}}$, for which equilibrium clearly fails, so we have proved:

Proposition 4.10. *If* $\mathcal{D}$ *is not unimodular, then there exists a pair of concave valuations of demand type* $\mathcal{D}$ *and a relevant supply bundle for which equilibrium fails.*

4.2.4 Transverse Intersections

If LIPs only had intersections of the simple form discussed above, a similar argument would demonstrate the sufficiency part of Thm. 4.3. For the general case, we need to consider more complex intersections, but we now show that we need only directly prove existence of equilibrium for generic intersections, more precisely, "transverse" LIP intersections:

Definition 4.11 (see e.g. Maclagan and Sturmfels, 2015, Defn. 3.4.9)**.**

(1) The intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is *transverse at* $\mathbf{p}$ if $\dim(C^1 + C^2) = n$, in which C^j is the minimal cell of $\mathcal{L}_{u^j}$ containing $\mathbf{p}$, for $j = 1, 2$.

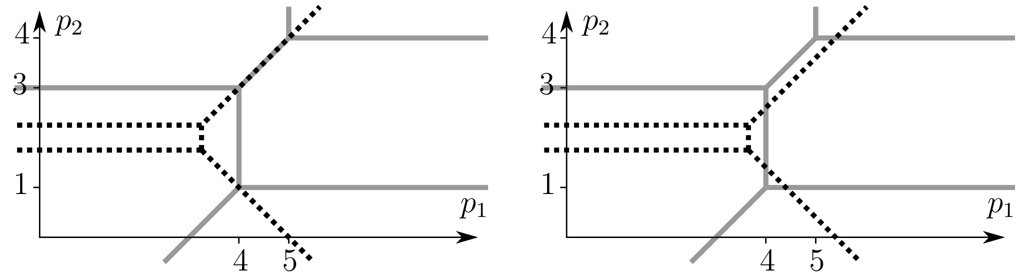
(2) The intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is *transverse* if they intersect transversally at every point of their intersection.

(3) The intersection of $\{\mathcal{L}_{u^j} : j \in \{1, \ldots, k\}\}$ is *transverse at* $\mathbf{p}$ if the intersection of $\mathcal{L}_{u^{\{1,\ldots,j\}}}$ and $\mathcal{L}_{u^{j+1}}$ is transverse at $\mathbf{p}$, for all $j = 1, \ldots, k-1$.[28]

[27] Clearly $1 \leq k \leq s-1$; a single primitive integer vector is a unimodular set, by Fact 4.8(2).

[28] This definition is independent of the order in which the LIPs are taken; see e.g. Lemma A.6(2).

For example, in two dimensions, two lines crossing at a single point are intersecting transversally. So the intersections in Figs. 5c and 5d are both transverse. However, two coincident lines do not intersect transversally, and nor does a line crossing through a 0-cell intersect transversally. So the grey LIP and the black dotted LIP of Fig. 7a, which intersect at $(4, 1)$ and along the line from $(4, 3)$ to $(5, 4)$, do not intersect transversally at *any* price. (For each of the three prices $(4, 1)$, $(4, 3)$ and $(5, 4)$, the minimal cell of



(a) The grey and black dotted LIPs do not intersect transversally at any price. (b) After a small translation, the intersection is transverse, cf. Prop. 4.12.

Figure 7: Illustration of intersections that are (a) non-transverse; (b) transverse.

the grey LIP containing the price is the 0-cell at the price itself.) In three dimensions, an intersection is transverse at all of the prices where a 1-cell meets a facet in a single point, or two facets meet along a line, or three facets meet in a single point.

We will make use of two important features of transverse intersections.

First, as is intuitive, the intersections of "tropical hypersurfaces", and therefore of LIPs, are generically transverse:

Proposition 4.12 (Maclagan and Sturmfels, 2015, Prop. 3.6.12)**.** *For any $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, and generic $\mathbf{v} \in \mathbb{R}^n$, the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2} + \{\epsilon\mathbf{v}\}$ is transverse for all sufficiently small $\epsilon > 0$.*

For example, a small translation of the black dotted LIP of Fig. 7a by $\{\epsilon(1, 0)\}$ yields the transverse intersection shown in Fig. 7b, which consists of the points $(4, 1+\epsilon)$, $(4+\epsilon, 1)$, $(4, 3-\epsilon)$ and $(5+\epsilon, 4)$.

The significance of this is that modifying a valuation from $u(\mathbf{x})$ to $u(\mathbf{x}) + \epsilon\mathbf{v} \cdot \mathbf{x}$ translates its LIP from $\mathcal{L}_u$ to $\mathcal{L}_u + \{\epsilon\mathbf{v}\}$, since any bundle demanded at $\mathbf{p}$ under valuation $u(\mathbf{x})$ is demanded at $\mathbf{p} + \epsilon\mathbf{v}$ under valuation $u(\mathbf{x}) + \epsilon\mathbf{v} \cdot \mathbf{x}$.

Furthermore, failure of equilibrium is preserved by a sufficiently small modification of this kind:

Proposition 4.13. *If equilibrium does* not *exist for valuations u^1 and u^2, and some relevant supply bundle, then for any $\mathbf{v} \in \mathbb{R}^n$, equilibrium also fails for valuations u^1 and $u^{2\epsilon}$, in which $u^{2\epsilon}(\mathbf{x}) = u^2(\mathbf{x}) + \epsilon\mathbf{v} \cdot \mathbf{x}$, for the same supply and all sufficiently small $\epsilon > 0$.*

To prove Prop. 4.13, consider an allocation of the supply bundle between agents with valuations u^1 and u^2. Failure of equilibrium means, by definition, that the two sets of prices at which these agents do demand their respective allocations, must be disjoint. But if the set of prices at which an agent demands a bundle is non-empty, it is either the closure of a UDR or a cell of the LIP, and so is a polyhedron (see Cor. 2.22). And if two polyhedra are disjoint, then there is a positive minimum Euclidean distance between any point in one and any point in the other (even if the polyhedra are not bounded; see, e.g., Gruber, 2007, p. 59). Let δ be the minimum such distance, over the pairs of polyhedra corresponding to the finite set of all possible allocations between the agents.

Now consider $u^{2\epsilon}(\mathbf{x}) = u^2(\mathbf{x}) + \epsilon\mathbf{v}\cdot\mathbf{x}$. Since $D_{u^{2\epsilon}}(\mathbf{p}+\epsilon\mathbf{v}) = D_{u^2}(\mathbf{p})$, the polyhedron of prices in which an agent with valuation $u^{2\epsilon}$ demands a bundle is shifted by $\{\epsilon\mathbf{v}\}$, compared with the corresponding prices for u^2. Suppose $\|\epsilon\mathbf{v}\| < \delta$. Then however we allocate the supply between agents with valuations u^1 and $u^{2\epsilon}$, the sets of prices at which these agents demand their allocations are still disjoint. So equilibrium will still fail.

Combining Props. 4.12 and 4.13, we see that we need only prove sufficiency in Thm. 4.3 for transverse intersections:

Corollary 4.14. *If equilibrium does* not *exist for a pair of concave valuations of demand type $\mathcal{D}$ and some relevant supply bundle, then it does not exist for a pair of concave valuations of demand type $\mathcal{D}$ whose LIP intersection is transverse and some relevant supply bundle.*

The second important fact about transverse intersections is that the changes in the bundles considered by agents at any prices in these intersections are in fundamentally different directions, in the sense that:

Definition 4.15. The *linear span of changes in demand*, K_σ, associated to a demand complex cell σ, is the set of linear combinations of vectors in $\{\mathbf{y}-\mathbf{x} : \mathbf{x},\mathbf{y}\in\sigma\}$.

Lemma 4.16. *Suppose $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ intersect at $\mathbf{p}$, and the two agents' individual demand complex cells at this price are σ^1 and σ^2, while the aggregate demand complex cell is $\sigma^{\{1,2\}}$. Then the intersection is transverse at $\mathbf{p}$ iff every vector in $\mathrm{K}_{\sigma^{\{1,2\}}}$ can be written* uniquely *as a sum of a vector in K_{σ^1} and a vector in K_{σ^2}.*

In our example of Sections 4.2.1-4.2.2, at the transverse intersection at price $(30, 20)$, we have $\mathrm{K}_{\sigma^{\{1,2\}}} = \mathbb{R}^2$, while $\mathrm{K}_{\sigma^s} = \{\lambda(1,-1) : \lambda\in\mathbb{R}\}$ and $\mathrm{K}_{\sigma^c} = \{\lambda(1,1) : \lambda\in\mathbb{R}\}$. And any vector in $\mathbb{R}^2$ can be uniquely written as $\lambda_1(1,-1)+\lambda_2(1,1)$ for some $\lambda_1,\lambda_2\in\mathbb{R}$.

The next subsection will use this second fact to complete the proof of the Unimodularity Theorem.

4.2.5 Proof of Sufficiency Condition for the Unimodularity Theorem

Cor. 4.14 shows that we need only prove sufficiency of the condition of Thm. 4.3 for concave valuations whose LIP intersection is transverse:

So let u^1 and u^2 be such valuations of a unimodular demand type. By Lemma 4.7 it suffices to show that the demand set is discrete convex at any price $\mathbf{p}$ in their intersection. Write σ^1 and σ^2 respectively for the individual demand complex cells at $\mathbf{p}$, and let $\sigma^{\{1,2\}}$ be the aggregate demand complex cell at $\mathbf{p}$. We want to show that any integer supply $\mathbf{y}\in\sigma^{\{1,2\}} = \mathrm{conv}\,(D_{u^{\{1,2\}}}(\mathbf{p}))$ is demanded, that is, also satisfies $\mathbf{y}\in D_{u^{\{1,2\}}}(\mathbf{p})$.

To do this, consider any vertex, $\mathbf{x}$, of $\sigma^{\{1,2\}}$. By Defn. 4.15, the change in demand, $\mathbf{y}-\mathbf{x}$, is in $\mathrm{K}_{\sigma^{\{1,2\}}}$. Fix a basis for $\mathrm{K}_{\sigma^{\{1,2\}}}$, composed of edge vectors of $\sigma^{\{1,2\}}$.[29] These

[29]That such a basis exists follows from, e.g., combining Gruber (2007) Thms. 14.2 and 15.8.

vectors are, equivalently, the normals to the facets of $\mathcal{L}_{u^{\{1,2\}}}$ which contain $\mathbf{p}$ (see Prop. 2.21). So this set is unimodular and, by Fact 4.8(2), $\mathbf{y}-\mathbf{x}$ can therefore be written as an *integer* combination of these vectors. Furthermore, since $\mathcal{L}_{u^{\{1,2\}}} = \mathcal{L}_{u^1} \cup \mathcal{L}_{u^2}$, each of these vectors is normal to a facet of $\mathcal{L}_{u^1}$ or of $\mathcal{L}_{u^2}$. So we can separate the basis vectors into two sets, correspondingly. Then our presentation of $\mathbf{y}-\mathbf{x}$, in terms of this basis, splits as $\mathbf{y}-\mathbf{x} = \mathbf{z}^1 + \mathbf{z}^2$, in which $\mathbf{z}^j \in \mathrm{K}_{\sigma^j}$, for $j = 1, 2$, and the $\mathbf{z}^j$ are integer bundles.

On the other hand, since $\sigma^{\{1,2\}} = \sigma^1 + \sigma^2$ (see e.g. Cox et al 2005, Section 7.4, Ex. 3) there exist $\mathbf{y}^j \in \sigma^j$, $j = 1, 2$, such that $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$. And since $\mathbf{x}$ is a vertex of $\sigma^{\{1,2\}}$, it is demanded in a UDR adjacent to $\mathbf{p}$, so $\mathbf{x} \in D_{u^{\{1,2\}}}(\mathbf{p})$, i.e. $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$ in which $\mathbf{x}^j \in D_{u^j}(\mathbf{p})$ (and so are integer bundles). So $\mathbf{y}-\mathbf{x} = (\mathbf{y}^1 - \mathbf{x}^1) + (\mathbf{y}^2 - \mathbf{x}^2)$, and we also have $\mathbf{y}^j - \mathbf{x}^j \in \mathrm{K}_{\sigma^j}$, $j = 1, 2$.

So, by transversality (Lemma 4.16), $\mathbf{y}^j - \mathbf{x}^j = \mathbf{z}^j$. And, since we already showed $\mathbf{x}^j$ and $\mathbf{z}^j$ are integer, it follows that $\mathbf{y}^j$ are also integer, for $j = 1, 2$.

So we have $\mathbf{y}^j \in \sigma^j = \operatorname{conv}(D_{u^j}(\mathbf{p}))$ and $\mathbf{y}^j \in \mathbb{Z}^n$. But since u^j is concave, its demand sets are all discrete convex (Lemma 2.13) and so $\mathbf{y}^j \in D_{u^j}(\mathbf{p})$, for $j = 1, 2$. Therefore, since $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$, we can conclude $\mathbf{y} \in D_{u^{\{1,2\}}}(\mathbf{p})$. So we have proved:[30]

Proposition 4.17. *If $\mathcal{D}$ is unimodular, then an equilibrium exists for every pair of concave valuations which are of demand type $\mathcal{D}$, for all relevant supply bundles.*

4.3 Related Work

Danilov et al. (2001) have developed results that are very closely related to our Thm. 4.3. In particular, their Thms. 3 and 4 together provide a *sufficient* condition for equilibrium, which is analogous to our condition on demand types.[31] (Howard's (2007, Thm. 1) subsequent work is equivalent to Thm. 4 of Danilov et al.)

However, the economic interpretation or usefulness of this condition is not clear. By contrast, our Thm. 4.3 both demonstrates the applicability of the result, and clarifies the connections to existing economic results. We will see in Section 6 that our Thm. 4.3 generalises many results in well-known work *subsequent to* Danilov et al.'s, including results in Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014).[32]

[30]Ex. A.7 in Appendix A.3 provides additional intuition for the sufficiency condition by illustrating the failure of this argument for the simple two-goods substitutes and complements example discussed in Section 4.2.2.

[31]Their Thm. 3 shows that equilibrium is guaranteed if the valuations are "$\mathscr{D}$-concave" for some "class of discrete convexity" $\mathscr{D}$. (This notation $\mathscr{D}$ is *not* connected with our use of $\mathcal{D}$ to represent demand types.) "$\mathscr{D}$-concave" valuations are concave valuations such that every demand set $D_u(\mathbf{p})$ belongs to a specified set "$\mathscr{D}$" of subsets of $\mathbb{Z}^n$. A collection of such sets is a "class of discrete convexity" if every set it contains, and every sum and every difference of these sets, is discrete convex.

Their Thm. 4 (which is proved by Danilov and Koshevoy, 2004, Thm. 2) is that $\mathscr{D}$ is a class of discrete convexity if the edges of the convex hulls of the sets in $\mathscr{D}$ form a unimodular set of vectors. But this is true if, in our language, all the valuations are of the demand type defined by this unimodular set. So the sufficiency part of our Thm. 4.3 follows from combining their two results.

[32]Danilov et. al.'s lack of our notion of demand types or of any economic interpretation of their "$\mathscr{D}$-concavity", and the presentation of their work in relatively unfamiliar terms (namely the relationships between sets of primitive integer vectors which are parallel to edges of specific collections of integral pointed polyhedra and their "classes of discrete convexity") seem to have resulted in leading economists being unaware of their work or of its implications. (We were also unaware of their work until after we had developed our own results.)

Danilov et al. also prove no *necessity* result. Because they have no concept of "demand types", and have not developed their definition as a taxonomy of demand, there is no natural result for them to give. But once our concept of demand types is introduced, a necessity result can easily be developed from their work (using our Lemma 2.20).

However, our methods seem simpler and more accessible to economists than Danilov et al.'s *extremely* advanced mathematics. So we prefer the proof we have given above (which we developed before we knew of their work). Tran and Yu (2015) provide another proof, via integer programming, in their recent exposition of our work.

Danilov et al. also state their results under different assumptions from ours. They assume the domain, A, of every agent's valuation is contained in $\mathbb{Z}^n_{\geq 0}$, which precludes, for example, the application to agents who both buy *and* sell which our more general assumption permits. For example, our model, unlike theirs, applies to (and extends) Hatfield et al. (2013)–see Section 6.1.

Finally, our techniques lead to an additional set of results about when equilibrium exists for *specific* valuations; the next section turns to these.

5 The Intersection Count Theorem–when does Equilibrium exist for *Specific* Valuations?

The Unimodularity Theorem (Thm. 4.3) tells us exactly which demand types *always* have a competitive equilibrium. But even when equilibrium is not guaranteed to exist for every concave valuation, it of course exists for many *specific* valuations. Our Intersection Count Theorem (Thm. 5.12) gives results about which these valuations are.

The key to the Unimodularity Theorem was to understand which demand types permitted "over-large" volumes to *potentially* arise in the demand complex; thus the key to the Intersection Count Theorem is to understand whether such large volumes *in fact* arise for agents' actual valuations.

As in Section 4.2's development of the Unimodularity Theorem, we need only analyse certain isolated points in the LIP intersection. Furthermore, tropical intersection theory bounds the number of such points; remarkably, a simple count of them often suffices to tell us whether there are "over-large" volumes, and hence demonstrate the existence or failure of equilibrium.

Section 5.1 provides a preview and explanation of the theorem. Although much of the intuition is clear from the two-dimensional case, the analysis for higher dimensions requires additional techniques. So Section 5.2 develops the necessary machinery, and provides a technical theorem–the "Subgroup Indices Theorem"(Thm. 5.16)–which is the backbone of all our results. Section 5.3 can then state, and sketch the proof of, the general Intersection Count Theorem (Thm. 5.12). Section 5.4 explains the limitations of the Theorem, but gives a small extension for transverse cases.

5.1 Preview, and Explanation, of the Theorem

5.1.1 The simple "Hotel Rooms" example

Recall the introduction's "hotel rooms" example. As discussed in Section 4.2.2, equilibrium fails when the supply is $(1,1)$ for the valuations u^s and u^c, illustrated in

Figs. 5a-b. This is shown by the fact that, if we look at the only price in the intersection of these LIPs, the dual demand complex cell has area 2, which exceeds 1 (see Fig. 6c).

But equilibrium exists for valuations u^s and u^{c*}, which have the same aggregate demand type. The two demand complex cells dual to the two prices at which their LIPs intersect both have area 1 (see Fig 6d). It is not a coincidence that the number of the points in the intersection, weighted by the corresponding areas, is constant: this follows from the tropical version of the Bernstein-Kouchnirenko-Khovanskii (BKK) Theorem.[33]

Specifically, in 2-dimensions, consider any two valuations with individual domains, A^1 and A^2, whose LIPs intersect transversally. The tropical BKK Theorem tells that *the number of price points at which the LIPs intersect, weighting each point by the* area *of the dual demand-complex cell is a constant,* $\Gamma^2(A^1, A^2)$.

Fig. 6 illustrates why: the total area of any 2-dimensional demand complex is the sum of the areas of its 2-cells. These 2-cells are dual, of course, to the 0-cells of its LIP. Furthermore, the 0-cells of the aggregate LIP correspond either to the 0-cells of the individual LIPs, or to points in the intersection between these LIPs (which categories are mutually exclusive when the intersection is transverse). But the 0-cells of the individual LIPs are dual in turn to the 2-cells of their individual demand complexes. That is, the triangular 2-cells in Figs. 6c and 6d are exactly the collection of 2-cells in Figs. 6a and 6b. So the total area of *these* aggregate demand complex 2-cells is the sum of the areas of the individual demand complexes, that is the sum of the areas of $\text{conv}(A^1)$ and $\text{conv}(A^2)$. Furthermore, the total area of the aggregate demand complex is the area of $\text{conv}(A^1 + A^2)$. And, by definition, $\Gamma^2(A^1, A^2)$ is the total area of all the aggregate demand complex cells that are dual to intersection prices. So $\Gamma^2(A^1, A^2)$ *equals the area of* $\text{conv}(A^1 + A^2)$ *minus the sum of the areas of* $\text{conv}(A^1)$ *and* $\text{conv}(A^2)$.

Moreover, in two dimensions, a price at which the intersection of two LIPs is transverse is dual to a parallelepiped in the aggregate demand complex, which must have area at least 1. It follows that the number of points in the intersection is bounded by $\Gamma^2(A^1, A^2)$. Furthermore, Sections 4.2.2-4.2.3 showed that equilibrium exists at a supply corresponding to such an intersection of facets of weight 1, if and only if the corresponding area is 1 (combine the discussion immediately below Fact 4.8 with Lemma 4.9). So, in two dimensions, if the valuations u^1 and u^2 are concave, all facets are weight 1, and the LIP intersection is transverse: (i) there are at most $\Gamma^2(A^1, A^2)$ points in the LIP intersection, and (ii) equilibrium exists for every possible supply if and only if there are exactly $\Gamma^2(A^1, A^2)$ points in this intersection.

In the hotel example, the domain of each individual valuation is $\{0, 1\}^2$, so the domain of the aggregate valuation is $\{0, 1, 2\}^2$, and $\Gamma^2(\{0, 1\}^2, \{0, 1\}^2) = 4 - 1 - 1 = 2$. Moreover, the intersection is transverse and the facets are weight 1. So the previous paragraph tells us that there are at most two points in the intersection, and equilibrium exists for every supply if and only if there are exactly two points in the intersection

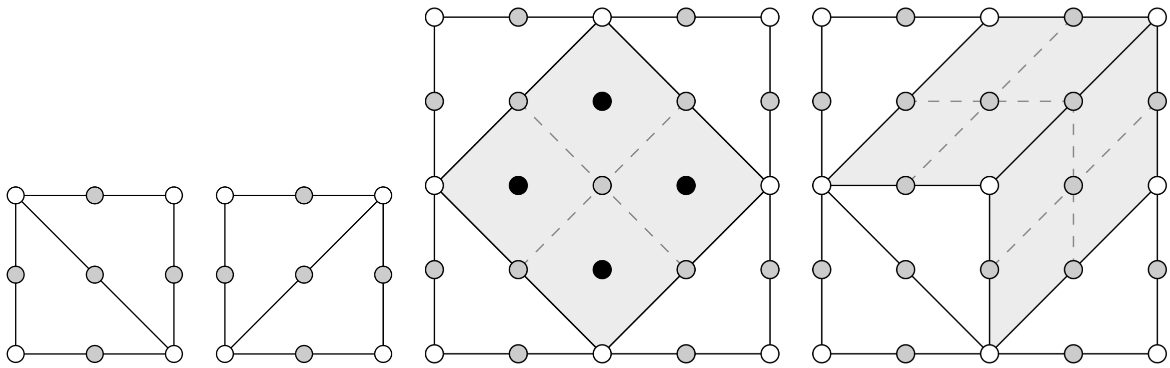
[33]As noted in the introduction, Bézout's (1779) theorem tells us that in two dimensions the number of intersection points of two (ordinary) geometric curves equals the product of their degrees, if we weight the intersection points by appropriate "multiplicities" (e.g., a tangency between a line and a parabola has multiplicity 2). Bernstein (1975), Kouchnirenko (1976) and Khovanskii (1978) extended the theorem, including to higher dimensions. LIPs are particular limits of logarithmic transformations of algebraic hypersurfaces (in complex projective space) (see, e.g., Maclagan and Sturmfels, 2015), and similar intersection theorems hold. Moreover, because a LIP is in real (not complex) space, we can "see" the intersection points.

(which results are consistent with the cases analysed above).

Our full Intersection Count Theorem develops these ideas in several directions:

5.1.2 Facet Weights

Reconsider the hotel example, but with weight 2 on every facet: let concave valuations u^{2s}, u^{2c} and u^{2c*}, have the same LIPs as u^s, u^c and u^{c*}, respectively, but with weight 2 on every facet. (So, for example u^{2s} is equivalent to an aggregate valuation of two identical copies of Elizabeth.) The demand complexes of u^{2s}, u^{2c}, $u^{\{2s,2c\}}$ and $u^{\{2s,2c*\}}$ are pictured in Fig. 8. As in Section 2.4, bundles are coloured white if they are uniquely demanded for some price, black if they are never demanded, and grey if they are demanded, but never uniquely.



(a) $\Sigma_{u^{2s}}$ (b) $\Sigma_{u^{2c}}$, $\Sigma_{u^{2c*}}$ (c) $\Sigma_{u^{\{2s,2c\}}}$ (d) $\Sigma_{u^{\{2s,2c*\}}}$

Figure 8: Demand complexes dual to the LIPs in Figs. 5a-d, when every facet has weight 2. The cells dual to intersection prices of the LIPs are shaded. (The dashed lines show they are grids of copies of cells from Figs. 6c-d.) Bundles uniquely demanded for some price are white; those never demanded are black; the remainder are grey. (As usual, we present the bundles increasing from top to bottom, and from right to left.)

The demand set $D_{u^{2s}}(30, 20)$ is $\{(2, 0), (1, 1), (0, 2)\}$: the non-vertex bundle in the diagonal demand complex 1-cell in Fig. 8a *is* demanded at this price, because the valuation is concave–indeed it is locally linear. Similarly, $D_{u^{2c}}(30, 20) = \{(0, 0), (1, 1), (2, 2)\}$. So aggregate demand at the intersection price of $\mathcal{L}_{u^{2s}}$ and $\mathcal{L}_{u^{2c}}$ is $D_{u^{\{2s,2c\}}}(30, 20) = \{(2, 0), (1, 1), (0, 2)\} + \{(0, 0), (1, 1), (2, 2)\}$. Thus the vertices of the cell highlighted in Fig. 8c, $(2, 0), (0, 2), (2, 4), (4, 2)$, the bundles on the mid-points of its edges, $(1, 1)$, $(1, 3)$, $(3, 1)$, $(3, 3)$, and its central bundle, $(2, 2)$, are all demanded at $(30, 20)$, while the cell's remaining four bundles are never demanded.

Observe that this cell is therefore just a grid of $2 \times 2 = 4$ copies of the central cell of Fig. 6c, as shown by the dashed lines in Fig. 8. This corresponds to the fact that the relevant 1-cells of the individual demand complexes have "length" 2 (Defn. 2.16(4)), that is, each of the facets of the individual LIPs have weight 2. The intuition for which bundles of the central cell of Fig. 8c are demanded is exactly as for the central cell of Fig. 6c in Section 4.2.2–the issue is which bundles can be reached from a vertex by an integer combinations of vectors that are (primitive) edge vectors of the cell.

Likewise, at the two prices at which the LIPs $\mathcal{L}_{u^{2s}}$ and $\mathcal{L}_{u^{2c*}}$ meet, the dual cells of the aggregate demand complex $\Sigma_{u^{\{2s,2c*\}}}$ (shaded in Fig. 8d) are each $2 \times 2 = 4$ copies of their corresponding cell in $\Sigma_{u^{\{s,c*\}}}$ (see Fig. 6d).

Ex. A.8 gives further details on both these cases.

This result is general: if the intersection is transverse, then multiplying any one facet weight multiplies the area of the demand complex cell by the same factor, without affecting the existence or otherwise of equilibrium for concave valuations.[34] So, applying this to the discussion of the previous subsection, in the two-dimensional transverse intersection case: (i) the weighted count of points in the LIP intersection, where each point is weighted by *the product of the weights of the facets passing through it* is bounded above by $\Gamma^2(\cdot,\cdot)$; and (ii) equilibrium exists for every relevant supply for two concave valuations if and only if this bound holds with equality.

It is easy to check for the example of Fig. 8 that $\Gamma^2(A^{2s}, A^{2c}) = 8$, and that each intersection point has weight 4. So there are at most two points in the LIP intersection, and equilibrium is guaranteed if and only if there are *exactly* 2, just as before.

5.1.3 Non-Transverse Intersections

As in our development of the Unimodularity Theorem in Section 4, we can handle non-transverse intersections by considering the effects of small perturbations.

Return to the weight-one example of Figs. 5a-b, but modify the complements valuation to $u^{c\#}(x_1, x_2) = \min\{70x_1, 70x_2\}$. Then $\mathcal{L}_{u^{c\#}}$ intersects $\mathcal{L}_{u^s}$ non-transversely, exactly through its 0-cell at (40,30). This case is intermediate between those illustrated in Figs. 5c-d. So the aggregate demand complex is like that of Fig. 6c, but without the edge that includes the bundles (1,0) and (0,1); equivalently, the aggregate demand complex is like that of Fig. 6d, but without any of the three edges that include the bundle (1,1).

If we translate $\mathcal{L}_{u^{c\#}}$ by $\epsilon(1,1)$, for small $\epsilon > 0$, we return to the situation of Fig. 5d. So, by Prop. 4.13, equilibrium exists for all supplies: the bundle (1,1) is now "grey". So our count will give us the "right" result for this non-transverse case if we weight the intersection point by the sum of the weights that apply after *this* translation, that is, by 2: then the weighted count equals $\Gamma^2(\{0,1\}^2, \{0,1\}^2)$, as with the case of Fig. 5d.

But if we had translated in the other direction, and returned to the Fig. 5c case, the sum of the weights would have been only 1, that is "too low". We therefore choose the translation that yields the maximum possible sum of weights.

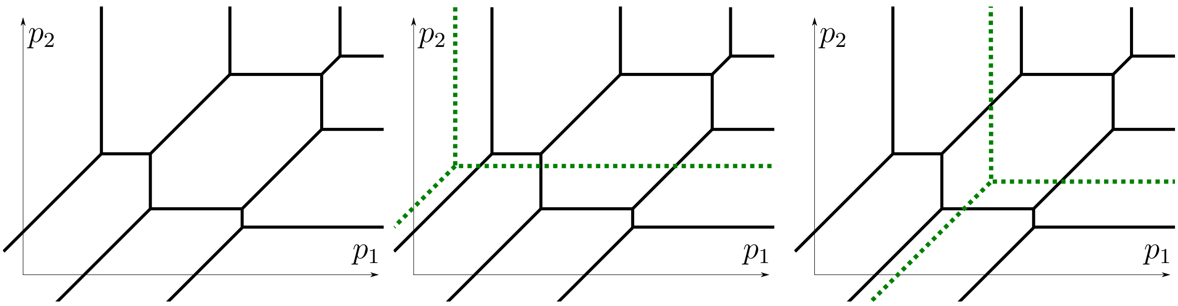
Weighting non-transverse intersection points in this way handles situations like this one. As in the transverse case, $\Gamma^2(\cdot,\cdot)$ is an upper bound on the possible count. That is, a weighted count equal to $\Gamma^2(\cdot,\cdot)$ remains sufficient, though, we will see, no longer necessary for equilibrium to exist for all supplies.

5.1.4 Illustration: the case of strong substitutes

An example the power of our approach is that it provides an elegant illustration of Prop. 4.6 that strong substitutes valuations always have equilibrium:

[34]A distinction is that bundles that are not at the vertices of the aggregate demand complex cells are not uniquely demanded, when they are demanded–that is, they are "grey", not "white". This is because concave valuations are locally linear at facets with weights greater than one.

Consider agents $j = 1, 2$ wanting up to d_j units, respectively, in total of two goods. Figs. 9b and 9c show two different cases for the LIPs of generic "strong substitutes" valuations for $d_1 = 1, d_2 = 3$. Observe that the LIP intersection contains $1 \times 3 = 3$ points in both cases.[35] It is not hard to check that this will remain true after any generic translation of either LIP. Moreover, since all "strong substitutes" facet normals for two goods are in $\pm\{(1,0), (0,1), (-1,1)\}$ (Prop. 3.9), all generic "strong substitutes" LIPs have similar "honeycomb" structures, so it is a general result that any intersection has $d_1 d_2$ points. It is also straightforward that the area of $\text{conv}(A^j)$ is $d_j^2/2$ and that of $\text{conv}(A^1 + A^2)$ is $(d_1 + d_2)^2/2$, so $\Gamma^2(A^1, A^2) = d_1 d_2$ (see also Fact A.11(3)). So equilibrium exists for all supplies in the generic case.



(a) LIP of a strong substitutes valuation for up to 3 units.

(b) The LIP from (a), and a LIP of a valuation for up to 1 unit.

(c) As (b), but with a different valuation for up to 1 unit.

Figure 9: The LIPs of two generic strong substitutes valuations, one for up to 3 units, and one for a single unit, always intersect exactly $3 \times 1 = 3$ times.

For non-generic cases, it is obvious from Figs. 9b and 9c, that any translation of a non-transverse LIP intersection to create a transverse intersection yields the count $d_1 d_2$. Furthermore if, e.g., all the facets of the LIP in Fig. 9a had weight w, we would obtain the "correct" count $w d_1 d_2$. So the argument can be extended to confirm equilibrium always exists in non-generic cases too.

5.1.5 Higher Dimensions

Handling $n > 2$ dimensions is harder. First, intersections have dimension at least $n - 2$, so do not consist of isolated points when $n > 2$. However, we will show it is sufficient to focus our analysis on "intersection 0-cells". Second, even if the intersection is transverse at such prices, the dual cell in the demand complex need not be a parallelepiped. Correspondingly, the individual LIP cells at an intersection point need not be facets, so we need to extend our definition of facet "weights" to cover all cells.

As in two dimensions, our results can be thought of in terms of whether or not there are problematic bundles within appropriate parallelepipeds. But we proceed in a similar way to Section 4.2.5. We there used an alternative equivalent definition of unimodularity (Fact 4.8(2)): whether all integer bundles in a demand complex cell can be reached by combinations of appropriate vectors. (We also used this for the second intuition of Section 4.2.2.) Our key tool here will be the "subgroup indices" that are exactly the

[35]The LIP of any valuation for up to d units in total is the tropical transformation of an "ordinary" polynomial of degree d. So Figs. 9a-c show tropical cubics and lines that intersect $3 \times 1 = 3$ times.

tool used by mathematicians to generalise the 2-dimensional tropical BKK Theorem to higher dimensions, but are also, as we show, intimately connected with existence of equilibrium (Thm. 5.16). Thus, for all n, there is an upper bound $\Gamma^n(A^1, A^2)$ (Defn. 5.11) for an appropriately weighted count of intersection 0-cells (in general, the domains $A^1, A^2 \subsetneq \mathbb{Z}^n$). And moreover, the weighted count equalling $\Gamma^n(A^1, A^2)$ is a sufficient, but for $n \geq 4$ not a necessary, condition for equilibrium to always exist–hence the general version of the Intersection Count Theorem (Thm. 5.12).

5.2 Definitions for the Theorem

5.2.1 Intersection 0-cells

Recall that the existence of equilibrium depends on whether the aggregate demand set is discrete convex at all prices in the LIP intersection (Lemma 4.7). For more than two goods, and for some non-transverse intersections, these prices are not a set of isolated points. However, it will suffice to focus our analysis on a particular (finite) set of 0-cells in the intersection:

Definition 5.1. An *intersection* 0*-cell for LIPs* $\mathcal{L}_{u^1}$ *and* $\mathcal{L}_{u^2}$ is a 0-cell of their aggregate LIP, $\mathcal{L}_{u^{\{1,2\}}}$, contained in the intersection $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$.[36]

Proposition 5.2. *If two individual valuations are concave, and their aggregate demand complex has dimension n, then equilibrium exists for every relevant supply bundle iff the aggregate demand set is discrete-convex at every intersection* 0*-cell.*

If the aggregate demand complex is n'-dimensional for $n' < n$, we can simply make a unimodular basis change so that its linear span is the span of the first n' coordinate directions. This transforms the problem to an equivalent one with n' new "goods", and the proposition can then be applied. (Otherwise we would have to analyse higher-dimensional cells.)

5.2.2 Parallel Lattices, and Cell Weights

As we saw in Section 5.1.2, the facets weights provide a measure of the "relative size" of the dual demand complex cell. To generalise this to lower-dimensional cells, we need a "lattice-volume" of a polytope analogous to the "length" of Defn. 2.16(4).

Definition 5.3.

(1) A *lattice* is a set $\Lambda \subseteq \mathbb{Z}^n$ such that $\mathbf{0} \in \Lambda$ and if $\mathbf{v}, \mathbf{v}' \in \Lambda$ then $\mathbf{v} - \mathbf{v}' \in \Lambda$.[37]
(2) Λ' is a *sublattice* of Λ if $\Lambda' \subseteq \Lambda$ and Λ' has the structure of a lattice.
(3) An *(integer) basis* for a lattice Λ is a set $\{\mathbf{v}^1, \ldots, \mathbf{v}^k\}$ such that any $\mathbf{v} \in \Lambda$ can be uniquely presented as $\mathbf{v} = \sum_j \alpha_j \mathbf{v}^j$ for $\alpha_j \in \mathbb{Z}$.[38]
(4) The *rank* of a lattice is the dimension of its basis.

[36] If the intersection is transverse, then the set of intersection 0-cells is exactly the obvious set of 0-cells of $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$. If the intersection is not transverse the set may contain additional points which are not 0-cells in the simplest structure of a polyhedral complex that one might impose on $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$.

[37] This is the group-theoretic meaning of "lattice" (see, e.g., Cassels, 1971), not the (completely different) order-theoretic meaning of, e.g., Milgrom and Shannon (1994).

[38] We will refer to "*integer* bases" rather than just "bases" when there is ambiguity.

Importantly, every lattice has an integer basis, and the rank is well-defined (see e.g. Cox et al., 2005, p. 334). The lattices important to us are (recalling–Defn. 4.15–that K_σ denotes the linear span of changes in demand associated with σ):

Definition 5.4. The *parallel lattice* to a demand complex cell σ is $\Lambda_\sigma := \mathrm{K}_\sigma \cap \mathbb{Z}^n$.

It is easy to see that if σ is a k-cell then Λ_σ has rank k.

For our "hotel room" example (Figs. 5 and 6), if we let σ^s, σ^c and $\sigma^{\{s,c\}}$ be the demand complex cells at $\mathbf{p} = (30, 20)$ of u^s, u^c and $u^{\{s,c\}}$ respectively, then $\Lambda_{\sigma^s} = \{m(-1, 1) : m \in \mathbb{Z}\}$, and $\Lambda_{\sigma^c} = \{m(1, 1) : m \in \mathbb{Z}\}$, while $\Lambda_{\sigma^{\{s,c\}}} = \mathbb{Z}^2$ (see Fig. 10 in Section 5.3.1). The parallel lattices are the same in the weight two version of this example (Section 5.1.2): at $\mathbf{p} = (30, 20)$ and cells σ^{2s}, σ^{2c} and $\sigma^{\{2s,2c\}}$ of $\Sigma_{u^{2s}}, \Sigma_{u^{2c}}$ and $\Sigma_{u^{\{2s,2c\}}}$ respectively, $\Lambda_{\sigma^{2s}} = \Lambda_{\sigma^s}$, $\Lambda_{\sigma^{2c}} = \Lambda_{\sigma^c}$ and $\Lambda_{\sigma^{\{2s,2c\}}} = \Lambda_{\sigma^{\{s,c\}}}$.

We now generalise the "weight" of Defn. 2.4. Given a rank-k lattice Λ, we can find a $k \times n$ matrix G_Λ such that $G_\Lambda \Lambda := \{G_\Lambda \mathbf{v} : \mathbf{v} \in \Lambda\} = \mathbb{Z}^k$.[39] We can use this identification to give volumes *relative* to the lattice Λ (as usual, the k-dimensional volume of $X \subsetneq \mathbb{R}^k$ is $\mathrm{vol}_k(X) := \int \cdots \int_X 1\, dp_1 \ldots dp_k$):

Definition 5.5. If $X \subsetneq \mathbb{R}^n$ is a polytope with vertices in Λ, define the *lattice-volume* of X in Λ as $\mathrm{vol}_\Lambda(X) := \mathrm{vol}_k(G_\Lambda X)$, where G_Λ is a $k \times n$ matrix such that $G_\Lambda \Lambda = \mathbb{Z}^k$.

It is standard that this is independent of the choice of G_Λ: if also $\hat{G}_\Lambda \Lambda = \mathbb{Z}^k$ then the images of X under the two transformations are related by a change of basis matrix on $\mathbb{Z}^k$, which must be unimodular, and so the volumes of the images are the same.

For example, given Λ_{σ^s} as above, we may set $G_{\Lambda_{\sigma^s}} = (-\frac{1}{2}, \frac{1}{2})$. Then $G_{\Lambda_{\sigma^s}}(-1, 1)' = 1$ and so, since $(-1, 1)$ is a basis for Λ_{σ^s}, it follows that $G_{\Lambda_{\sigma^s}} \Lambda_{\sigma^s} = \mathbb{Z}$. That is, operating with $G_{\Lambda_{\sigma^s}}$ on the lattice Λ_{σ^s} "rotates it" to "match it up" with $\mathbb{Z}$. So $G_{\Lambda_{\sigma^s}} \sigma^s$ is just the interval $[0, 1]$, which has length (1-dimensional volume) equal to 1, that is, its facet weight according to Defn. 2.4. In general, σ does not lie in K_σ, but for any $\mathbf{x} \in \sigma$, the shifted cell $\sigma' := \sigma + \{-\mathbf{x}\}$ lies in K_σ, and clearly has the same volume as σ.

Definition 5.6. Let C_σ be an $(n-k)$-cell of the LIP, dual to the demand complex k-cell σ. The *weight* of C_σ is $w_u(C_\sigma) := k!\mathrm{vol}_{\Lambda_\sigma}(\sigma + \{-\mathbf{x}\})$, where $\mathbf{x} \in \sigma$.

So if $k = 1$ the weight of a cell is just its "length" in terms of the primitive integer vector in the direction of its demand-complex cell. Thus this definition generalises Defn. 2.4. For example, $w_{u^c}(C_{\sigma^c}) = w_{u^s}(C_{\sigma^s}) = 1$ and $w_{u^{2c}}(C_{\sigma^{2c}}) = w_{u^{2s}}(C_{\sigma^{2s}}) = 2$.

The factor of $k!$ ensures that the cell weight is an integer (all cells are measured relative to a lattice simplex). So in our "hotel room" example, the weights of the 0-cells of $\mathcal{L}_{u^s}$ are both 1 (see Figs. 5a and 6a). Since the lattice-volume of the central cell in the aggregate demand complex, $\sigma^{\{s,c\}}$, is 2, this means that $w_{u^{\{s,c\}}}(C_{\sigma^{\{s,c\}}}) = 4$. Similarly, $w_{u^{\{2s,2c\}}}(C_{\sigma^{\{2s,2c\}}}) = 16$.

5.2.3 Naïve Multiplicities

Definition 5.7. If the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse at an intersection 0-cell C, define the *naïve multiplicity* $\hat{m}(C) := w_{u^1}(C^1) \cdot w_{u^2}(C^2)$, where C^j is the smallest cell of $\mathcal{L}_{u^j}$ containing C, for $j = 1, 2$.

[39] Specifically, if H_Λ is an invertible $n \times n$ matrix whose first k columns give a basis $\{\mathbf{v}^1, \ldots, \mathbf{v}^k\}$ for Λ, then $H_\Lambda \mathbf{e}^i = \mathbf{v}^i$ and so $H_\Lambda^{-1} \mathbf{v}^i = \mathbf{e}^i$ for $i = 1, \ldots, k$: we set G_Λ to be the first k rows of H_Λ^{-1}.

To extend this to non-transverse cases, we first write:

Definition 5.8. If C is an intersection 0-cell for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, and if C' is an intersection 0-cell for $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\}+\mathcal{L}_{u^2}$ (for any $\epsilon > 0$ and $\mathbf{v} \in \mathbb{R}^n$), then we say C' *emerges from* C if there exist cells C^j of $\mathcal{L}_{u^j}$ for $j = 1, 2$, such that $C^1 \cap C^2 = C$ and $C^1 \cap (\{\epsilon\mathbf{v}\} + C^2) = C'$.

There can be several intersection 0-cells emerging from C under the same translation; for example, the intersection 0-cells at $(4, 1 + \epsilon)$ and $(4 + \epsilon, 1)$ in Fig. 7b emerge from the intersection 0-cell at $(4, 1)$ in Fig. 7a. There is always at least one:

Lemma 5.9. *For every $\mathbf{v} \in \mathbb{R}^n$ and sufficiently small $\epsilon > 0$, there exists an intersection 0-cell for $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ emerging from every intersection 0-cell for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$.*

Now if $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ intersect transversally, each of their intersection 0-cells has a naïve multiplicity. Moreover, for fixed $\mathbf{v}$ and small enough $\epsilon > 0$, the set of these multiplicities, and therefore also the sum of these multiplicities, is independent of ϵ. But as there are only finitely many cells in each LIP, there are only finitely many different sums that can be obtained in this way.

Take, for example, the intersection 0-cell at $(4, 1)$ in Fig. 7a. When $\mathbf{v} = (1, 0)$, as shown in Fig. 7b, we obtain intersection 0-cells whose naïve multiplicities sum to 2. But for $\mathbf{v} = (-1, 0)$, only one intersection 0-cell, with naïve multiplicity 1, would have emerged. We will always want the *maximum* of the sums, so:

Definition 5.10. The *naïve multiplicity* $\hat{m}(C)$ at an intersection 0-cell C for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is the maximum number that can be obtained by adding the naïve multiplicities of 0-cells emerging from C under a small translation of $\mathcal{L}_{u^2}$ which makes the intersection transverse at C.

Note that this "naïve multiplicity" is *not* the "true multiplicity" of the tropical BKK Theorem (Defn. 5.18).

5.2.4 General definition of $\Gamma^n(\cdot,\cdot)$

Finally, we extend the definition of $\Gamma^2(\cdot,\cdot)$ to $n > 2$:

Definition 5.11. If $A^1, A^2 \subsetneq \mathbb{Z}^n$ are finite then, for $k = 1, \ldots, n-1$, define:

(1) $\Gamma^n_k(A^1, A^2) := \sum_{r=0}^{k} \sum_{s=0}^{n-k} (-1)^{n-r-s} \binom{k}{r} \binom{n-k}{s} \operatorname{vol}_n \operatorname{conv}(rA^1 + sA^2)$.

(2) $\Gamma^n(A^1, A^2) := \sum_{k=1}^{n-1} \Gamma^n_k(A^1, A^2)$

Thus $\Gamma^n(A^1, A^2)$ is a linear combination of ordinary volumes; $\Gamma^n_k(A^1, A^2)$ is the "mixed volume" of k copies of the convex hull of A^1 with $n-k$ copies of the convex hull of A^2 (Fact A.11 gives more details and special cases). Importantly, $\Gamma^n_k(A^1, A^2)$ is therefore a non-negative integer.

5.3 The Intersection Count Theorem

Theorem 5.12 (**The Intersection Count Theorem**). *For $j = 1, 2$, let u^j be concave valuations, on finite domains $A^j \subsetneq \mathbb{Z}^n$ such that* $\dim \operatorname{conv}(A^1 + A^2) = n$*. Then the number of intersection* 0*-cells for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, counted with naïve multiplicities, is bounded above by* $\Gamma^n(A^1, A^2)$*. If the number equals this bound, equilibrium exists for all relevant supplies.*

Suppose additionally that the intersection is transverse and that $n \leq 3$. The number of intersection 0*-cells for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, counted with naïve multiplicities, is equal to* $\Gamma^n(A^1, A^2)$ *iff equilibrium exists for all relevant supplies.*

The remainder of Section 5.3 develops the proof. Full details are in Appendix A.4.

5.3.1 Subgroup Indices

As usual, we will want to investigate whether, starting at a point at which we know equilibrium exists, a change in aggregate supply can be matched by a change in aggregate demand. Write σ^j and σ^J for the individual and aggregate demand complex cells that correspond to bundles in the convex hull of aggregate demand at some price $\mathbf{p}$. Then changes in aggregate supply are in the directions given by the lattice Λ_{σ^J}, while changes in aggregate demand are in the directions in $\sum_{j\in J} \Lambda_{\sigma^j}$. So we use a standard tool to compare these lattices:

Definition 5.13. Let Λ be a lattice, and $\Lambda' \subseteq \Lambda$ a sublattice of the same rank as Λ.

(1) A *fundamental parallelepiped* of Λ is a set $\Delta_\Lambda := \{\sum_j \lambda_j \mathbf{v}^j \in \mathbb{R}^n : 0 \leq \lambda_j < 1\}$, where $\{\mathbf{v}^1, \dots, \mathbf{v}^k\}$ are a basis for Λ.[40]

(2) The *subgroup index* $[\Lambda : \Lambda']$ is the lattice-volume in Λ of a fundamental parallelepiped of Λ', that is, $[\Lambda : \Lambda'] := \mathrm{vol}_\Lambda(\Delta_{\Lambda'})$.[41]

Lemma 5.14. *If $\{u^j : j \in J\}$ are a finite set of valuations with individual and aggregate demand complex cells σ^j and σ^J, respectively, at $\mathbf{p}$, then $\sum_{j\in J} \Lambda_{\sigma^j}$ is a sublattice of Λ_{σ^J} of the same rank as Λ_{σ^J}.*

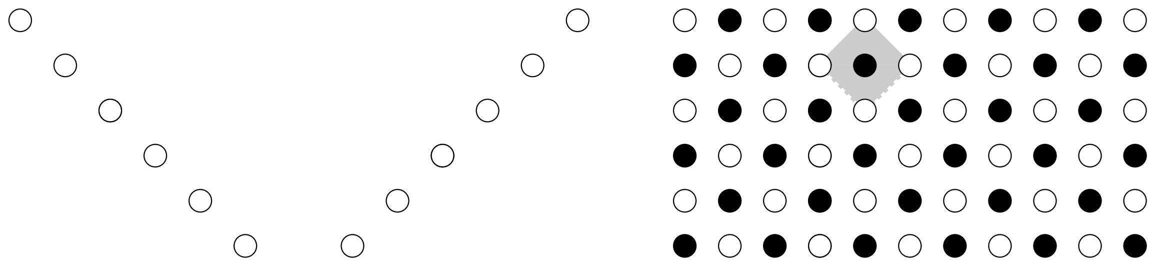
The parallel lattices for our "hotel room" example (as given in Fig. 5), with $\mathbf{p} = (30, 20)$, are given in Fig. 10. By considering Λ_{σ^s} and Λ_{σ^c}, we see that the sublattice $\Lambda_{\sigma^s} + \Lambda_{\sigma^c} = \{m(1, -1) + m'(1, 1) : m, m' \in \mathbb{Z}\}$. (That is, this sublattice comprises the white bundles in Fig. 10c, while the lattice $\Lambda_{\sigma^{\{s,c\}}} = \mathbb{Z}^2$ comprises all the bundles in Fig. 10c.) A fundamental parallelepiped, $\Delta_{\Lambda_{\sigma^s}+\Lambda_{\sigma^c}}$, of this sublattice is shown; its lattice-volume in $\Lambda_{\sigma^{\{s,c\}}}$ is 2. So the subgroup index $[\Lambda_{\sigma^{\{s,c\}}} : \Lambda_{\sigma^s} + \Lambda_{\sigma^c}] = 2$.

Observe in Fig. 10c that the subgroup index also corresponds to the ratio of the total number of bundles to the number of white bundles. We can in fact understand subgroup indices generally in terms of such ratios (see Fact A.13). Subgroup indices generalise unimodularity, as is seen by comparing Fact 5.15(1), below, with Fact 4.8(1), and Fact 5.15(3) with Defn. 4.2:

Facts 5.15. Let Λ be a lattice, and $\Lambda' \subseteq \Lambda$ a sublattice of the same rank as Λ.

[40] Different fundamental parallelepipeds are images of one another under unimodular basis changes.

[41] These are the ordinary group-theoretic subgroup indices (see the proof of Fact 5.15), and so are independent of the choice of parallelepiped.



(a) Λ_{σ^s} (b) Λ_{σ^c} (c) $\Lambda_{\sigma^{\{s,c\}}}$, with $\Lambda_{\sigma^s}+\Lambda_{\sigma^c}$ in white, and an example of $\Delta_{\Lambda_{\sigma^s}+\Lambda_{\sigma^c}}$ shaded.

Figure 10: The parallel lattices corresponding to demand complex cells σ^s, σ^c and $\sigma^{\{s,c\}}$, of respectively, u^s, u^c and $u^{\{s,c\}}$ (as shown in Figs. 5a-c and Figs. 6a-c) at $\mathbf{p} = (30, 20)$.

(1) $[\Lambda : \Lambda'] - 1$ is equal to the number of elements of Λ in $\Delta_{\Lambda'}$ which are not vertices.[42]
(2) $[\Lambda : \Lambda'] = 1$ iff $\Lambda = \Lambda'$.
(3) If $\Lambda = \mathbb{Z}^n$, and $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ are a basis for Λ', then $[\Lambda : \Lambda'] = |\det(\{\mathbf{v}^1, \ldots, \mathbf{v}^n\})|$.[43]
(4) Suppose $\Lambda = \mathrm{K} \cap \mathbb{Z}^n$ for some linear subspace K of $\mathbb{R}^n$. Then $[\Lambda : \Lambda'] = 1$ iff any basis for Λ' is unimodular.

In our example, there is one element of $\Lambda_{\sigma^{\{s,c\}}}$ in $\Delta_{\Lambda_{\sigma^s}+\Lambda_{\sigma^c}}$ which is not at a vertex (see Fig. 10c). This illustrates Fact 5.15(1). As $\Lambda_{\sigma^{\{s,c\}}} = \mathbb{Z}^2$ we can use Fact 5.15(3) to calculate $[\Lambda_{\sigma^{\{s,c\}}} : \Lambda_{\sigma^s}+\Lambda_{\sigma^c}]$ as the determinant of the matrix with columns $(1, -1), (1, 1)$ (cf. Section 4.2.2). This basis is not unimodular, and so Fact 5.15(4) verifies for us again that $\Lambda_{\sigma^s} + \Lambda_{\sigma^c} \neq \Lambda_{\sigma^{\{s,c\}}}$.

5.3.2 The Relationship Between Subgroup Indices and Equilibrium

Subgroup indices allow us to determine whether the aggregate demand set is discrete convex at a particular intersection price–and hence identify the bundles for which equilibrium fails. The logic of Section 4.2.5 shows that it is sufficient that the facet normals at this price form a unimodular set. However, the following theorem gives a weaker sufficient condition than this. Although the Intersection Count Theorem only requires the $r = 2$ case of Thm. 5.16, we give it in a more general form:

Theorem 5.16 (The Subgroup Indices Theorem). *Let u^j be concave for j in a finite set J, and suppose the intersection of those LIPs which contain* $\mathbf{p}$ *is transverse at* $\mathbf{p}$. *Write σ^j, σ^J for the demand complex cell at* $\mathbf{p}$ *of respectively u^j (where $j \in J$), u^J.*

(1) If $\left[\Lambda_{\sigma^J} : \sum_{j\in J} \Lambda_{\sigma^j}\right] = 1$ then $D_{u^J}(\mathbf{p})$ is discrete-convex.
(2) If $\left[\Lambda_{\sigma^J} : \sum_{j\in J} \Lambda_{\sigma^j}\right] > 1$ and if also $\exists j_0 \in J$ with $\dim \sigma^{j_0} \leq 2$, while $\dim \sigma^j \leq 1$ for $j \in J \setminus \{j_0\}$, then $D_{u^J}(\mathbf{p})$ is not discrete-convex.

To translate part (1) back to the familiar terms of Section 4.2, first suppose $|J| = 2$ and suppose both LIPs contain $\mathbf{p}$. Recall that $\Lambda_\sigma = \mathrm{K}_\sigma \cap \mathbb{Z}^n$ (Defn. 5.4). So, by Fact

[42] Since fundamental parallelepipeds are not *closed*, we do not count points on the "upper" boundary.
[43] When $\Lambda \neq \mathbb{Z}^n$, we can calculate $[\Lambda : \Lambda']$ by fixing a matrix G_Λ identifying Λ with $\mathbb{Z}^k$ for some k, as in Footnote 39, and then applying G_Λ to Λ'.

5.15(4), $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] = 1$ iff any basis for $\Lambda_{\sigma^1} + \Lambda_{\sigma^2}$ is unimodular. Moreover, the combination of a basis for Λ_{σ^1} with a basis for Λ_{σ^2} gives a basis for $\Lambda_{\sigma^1} + \Lambda_{\sigma^2}$ (see Fact A.12(3) for more details). The proof in Section 4.2.5 that $D_{u^{\{1,2\}}}(\mathbf{p})$ is discrete-convex depended on a unimodular basis for $\mathrm{K}_{\sigma^{\{1,2\}}}$, consisting of integer vectors in either K_{σ^1} or K_{σ^2}. Exactly the same arguments prove Thm. 5.16(1).

But this result is more powerful than showing only that equilibrium exists if the normals to agents' facets at $\mathbf{p}$ are a unimodular set. Thm. 5.16(1) shows that all that is required for equilibrium existence is an integer basis for the changes in demand that each agent considers, such that the combination of these integer bases forms a unimodular set. When the dimension of an individual agent's demand complex cell exceeds 1, equilibrium does *not* require that the changes in demand that the agent considers can be broken down as integer combinations of vectors along the *edges* of this cell.

For example, if Figs. 5c and 6c correspond to an *individual* agent with a concave valuation, the bundle $(1, 1)$ is demanded at price $(30, 20)$. In this case the edges to this demand complex cell do not provide a basis for its parallel lattice. By combining two such agents in 4-dimensional space, Ex. A.14 shows that Thm. 5.16(1) demonstrates equilibrium in situations in which Thm. 4.3 does not.

Now we consider Thm. 5.16(2). If $\dim \sigma^j \leq 1$ for $j \in J$, then the LIP cells are all facets. If additionally they have weight 1, then the situation at the price $\mathbf{p}$ is exactly that described in Lemma 4.9. So, just as in Lemma 4.9, unimodularity is necessary (as well as sufficient) for equilibrium; now apply apply Fact 5.15(4). But, for any weights, σ^J is a grid of copies of a "small parallelepiped", as we saw in Section 5.1.2. This "small parallelepiped" is (the closure of) a fundamental parallelepiped of $\sum_{j\in J} \Lambda_{\sigma^j}$. So equilibrium fails for some supply if and only if such a fundamental parallelepiped contains a non-vertex integer point. Applying Fact 5.15(1) therefore yields Thm. 5.16(2) if $\dim \sigma^j \leq 1$ for all $j \in J$.

But if $\dim \sigma^j > 1$ for some j, then σ^J need not be a parallelepiped. In such cases, even if any fundamental parallelepiped of $\sum_{j\in J} \Lambda_{\sigma^j}$ contains a non-vertex integer point, it does not necessarily follow that a corresponding point lies in $\sum_{j\in J} \sigma^j = \sigma^J$ itself: it can fall outside the aggregate demand set at this price.

However, when we have one 2-cell, along with 1-cells, then we *can* capture this point. The reason is that a 2-cell will always contain a triangle, two of whose edges give a basis for its parallel lattice (see Appendix A.4). And this triangle is a copy of *half* of a fundamental parallelepiped. Meanwhile, each 1-cell contains an entire fundamental parallelepiped of its parallel lattice. The sum of these, σ^J, therefore contains at least *half* of a fundamental parallelepiped of $\sum_{j\in J} \Lambda_{\sigma^j}$. So by symmetry, if, in this case, $\left[\Lambda_{\sigma^J} : \sum_{j\in J} \Lambda_{\sigma^j}\right] > 1$, the aggregate demand complex cell σ^J *does* contain a bundle not in $\sum_{j\in J} \Lambda_{\sigma^j}$, i.e., one that cannot be reached via changes in demand among the agents.

Examples A.15-A.18 show that this is the furthest we can go:

Proposition 5.17. *For both the case in which* $\dim \sigma^1 = \dim \sigma^2 = 2$, *and the case in which* $\dim \sigma^1 = 3, \dim \sigma^2 = 1$, *there exist examples in which* $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] > 1$ *and in which* $D_{u^{\{1,2\}}}(\mathbf{p})$ *is discrete-convex, and other examples in which* $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] > 1$ *and in which* $D_{u^{\{1,2\}}}(\mathbf{p})$ *is* not *discrete-convex.*

5.3.3 The Tropical Bernstein-Kouchnirenko-Khovanskii (BKK) Theorem

We now define the "true multiplicity", $\text{mult}(C)$, of an intersection 0-cell, C:[44]

Definition 5.18 (Bertrand and Bihan, 2013, Defn. 5.2)**.** For valuations u^1, u^2 whose LIPs have an intersection 0-cell at $\mathbf{p}$, write $\sigma^1, \sigma^2, \sigma^{\{1,2\}}$ for their individual and aggregate demand complex cells at $\mathbf{p}$; write $C_{\sigma^1}, C_{\sigma^2}, C_{\sigma^{\{1,2\}}}$ for the dual cells in $\mathcal{L}_{u^1}, \mathcal{L}_{u^2}, \mathcal{L}_{u^{\{1,2\}}}$.

(1) If the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse at $C_{\sigma^{\{1,2\}}}$, then the *true multiplicity* of $C_{\sigma^{\{1,2\}}}$ is $\text{mult}(C_{\sigma^{\{1,2\}}}) := [\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] \cdot w_{u^1}(C_{\sigma^1}) \cdot w_{u^2}(C_{\sigma^2})$.

(2) In general, the *true multiplicity*, $\text{mult}(C_{\sigma^{\{1,2\}}})$, of $C_{\sigma^{\{1,2\}}}$, is the sum of the multiplicities at all the intersection 0-cells emerging from $C_{\sigma^{\{1,2\}}}$ after any small translation of $\mathcal{L}_{u^2}$ which makes the intersection transverse.

So the distinction between "true" and "naïve" multiplicities is embodied in the subgroup indices that capture our results on equilibrium (Thm. 5.16). Thus, for example, the intersection 0-cell at $(30, 20)$ in Fig. 5c has naïve multiplicity 1 but (see Section 5.3.1) true multiplicity 2.

The use of subgroup indices means that true multiplicities, unlike naïve multiplicities, are unchanged by small translations in any directions. Note that both multiplicities are positive integers, since this is true of cell weights (and by Lemma 5.9 and Fact 5.15(1)).

A version of the "tropical BKK Theorem" is now:

Theorem 5.19 (Bertrand and Bihan, 2013, Thm. 6.1)**.**

(1) The number of intersection 0*-cells for* $\mathcal{L}_{u^1}$ *and* $\mathcal{L}_{u^2}$*, counting with their true multiplicities, is equal to* $\Gamma^n(A^1, A^2)$.

(2) If the intersection is transverse whenever a k*-cell of* $\mathcal{L}_{u^1}$ *meets an* $(n-k)$*-cell of* $\mathcal{L}_{u^2}$*, then the number of such intersection* 0*-cells, counting with their true multiplicity, is equal to* $\Gamma^n_k(A^1, A^2)$.

5.3.4 Proving the Intersection Count Theorem

It is now easy to prove the sufficiency result for the transverse case. From Defns. 5.7 and 5.18, and Fact 5.15(1) (which implies that the subgroup index is always at least 1) we know that $\hat{m}(C) \leq \text{mult}(C)$ for any intersection 0-cell C. Equality holds iff the subgroup index at this price is 1. So, by Thm. 5.19(1), the number of intersection 0-cells, counted with naïve multiplicities, is bounded above by $\Gamma^n(A^1, A^2)$, and this bound holds with equality iff the subgroup index at every intersection 0-cell is 1. If this is true, then the aggregate demand is discrete-convex at every intersection 0-cell (Thm. 5.16(1)) and so equilibrium exists for every relevant supply (Prop. 5.2).

For necessity, we additionally assume $n \leq 3$. Every intersection 0-cell is the intersection of a 2-cell and a 1-cell of the two individual LIPs. So apply Thm. 5.16(2): if the number of intersection 0-cells, counted with naïve multiplicities, is strictly below $\Gamma^n(A^1, A^2)$ then the subgroup index is greater than 1 for some intersection 0-cell, and this implies a failure of equilibrium for some relevant supply.

When the intersection is not transverse, the logic is similar, but we need a little more care. We need to translate in the "right direction" to find the naïve multiplicity for each

[44] We follow the conventions of Bertrand and Bihan (2013) although these ideas and theorems are older (see, e.g., McMullen, 1993, Huber and Sturmfels, 1995 and Fulton and Sturmfels, 1996).

intersection 0-cell, but these directions need not be the same across all intersection 0-cells. However, we can combine Prop. 4.13 with Thms. 5.16(1) and 5.19(1) by working *locally* at each 0-cell. See Appendix A.4 for details.

5.4 Limitations, and Extension, of the Theorem

5.4.1 Necessity

The condition of the Intersection Count Theorem is necessary for equilibrium only when the intersection is transverse and $n \leq 3$.

If the intersection is not transverse, it is possible for equilibrium to exist for every relevant supply, but to fail for some relevant supply after *any* small translation of the LIPs (see Ex. A.20). However, if an intersection is not transverse, our results relating unimodularity and subgroup indices to equilibrium fail to hold (recall, for example, the proof given in Section 4.2.5). So we cannot hope to identify these "fragile" equilibria with these tools, and so cannot give a necessary result in the non-transverse case.

The failure of our condition to give necessity for $n \geq 4$ is immediate from Prop. 5.17. (Recall we needed Thm. 5.16(2) to prove necessity for $n \leq 3$.)

Note, however, that our upper bound $\Gamma^n(\cdot, \cdot)$ is tight (strong substitutes valuations illustrate this for both non-transverse cases and arbitrary n; see Prop. 4.6 and Section 5.1.4) so this bound cannot be improved on.

5.4.2 Sufficiency

If the bound in Thm. 5.12 is *not* met with equality, the combination of Thm. 5.19(2) with Thm. 5.16 provides an alternative sufficient condition for equilibrium existing at particular prices. (Note that an intersection is transverse at a 0-cell iff the minimal cells of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ that contain that 0-cell have dimensions k and $(n-k)$ for some k.) An argument analogous to that in Section 5.3.4 (for the transverse case of Thm. 5.12) shows:

Proposition 5.20. *Let $u^j : A^j \to \mathbb{R}$ be concave valuations for $j = 1, 2$, suppose $\dim \operatorname{conv}(A^1 + A^2) = n$, and fix $k \in \{1, \ldots, n-1\}$. Suppose the intersection is transverse at every intersection 0-cell C such that the minimal cell of $\mathcal{L}_{u^1}$ containing C has dimension k. Then the number of these intersection 0-cells, counted with naïve multiplicities, is bounded above by $\Gamma^n_k(A^1, A^2)$. If the number equals this bound, then equilibrium exists for all supplies in the convex hull of demand at each of these intersection 0-cells.*

6 Applications

6.1 Interpreting Classic Models in a Unified Framework

Our model encompasses some classic studies as special cases, so clarifies connections between them. It also facilitates our understanding of them. In particular, it makes many of their equilibrium-existence results straightforward.

It is straightforward that Bikhchandani and Mamer (1997)'s model is the special case of ours in which each agent's domain is restricted to $\{0, 1\}^n$.

Kelso and Crawford's (1982) seminal analysis of n_1 firms, each of which is interested in hiring some of n_2 workers, can be understood as a model with $n_1 n_2$ distinct "goods", each of which is the "transfer of labour" by a specified worker (a "seller") to a specified firm (a "buyer"); the "price" of a good is the salary to be paid. So the full set of bundles is $\{-1, 0, 1\}^n$, in which $n = n_1 n_2$. However, each worker's valuation is defined only over a subset of this domain of the form $\{-1, 0\}^{n_1}$ (that is, only over the n_1 goods that correspond to its own labour), and only over the subset of these vectors that have at most one non-zero entry (it can work for at most one firm). Obviously, their only possible demand complex edges are the strong substitute vectors (non-zero vectors with at most one $+1$ entry, at most one -1 entry, and no other non-zero entries, see Prop. 3.9). Similarly, each firm's valuation is defined only over a subset of the form $\{0, 1\}^{n_2}$ (that is, it has preferences only over the n_2 goods that correspond to workers it can employ). Kelso and Crawford assume firms have ordinary substitutes preferences over workers but, since there is only one unit of each good, the only substitute changes of demand are the strong substitutes vectors, so all valuations are of this type.[45]

It may be less obvious that Hatfield et al.'s (2013) model of networks of trading agents, each of whom can both buy and sell, both fits into our framework, and is also closely related to Kelso and Crawford's model. To show this, we again treat each transfer of a product from a specified seller to a specified buyer as a distinct good, so each agent again has preferences over a subset of $\{-1, 0, 1\}^n$, where n is the number of distinct goods.

Since Hatfield et al. restrict each agent to be either a seller or a buyer (or neither) on any one good, an agent j which is the specified seller in n_1^j potential trades and is the specified buyer in n_2^j potential trades simply has preferences over a subset of the domain which, after an appropriate re-ordering of the goods for that agent, is of the form $\{-1, 0\}^{n_1^j} \times \{0, 1\}^{n_2^j}$. (As in Hatfield et al., we can restrict an agent's domain of preferences further so that, e.g., it cannot sell good 1 unless it also buys one of goods 2 or 3. They do this by using "$-\infty$" valuations, while we simply exclude bundles from the domain, but the effect is the same–see Section 2.1.) Furthermore, although Hatfield et al. describe goods to be sold as complements of goods to be bought, this is because they measure both buying and selling as non-negative quantities. So, since in our framework selling is just "negative buying", the "complementarities" disappear and it is clear that the condition they impose is exactly ordinary substitutes.[46] Just as for Kelso and Crawford's model, the only demand complex edges of such a domain that are vectors of the ordinary substitutes demand type are also vectors of the strong substitutes demand type.

Trivially, any valuation over any subset of $\{-1, 0\}^{n_1}$ or $\{0, 1\}^{n_2}$ or $\{-1, 0\}^{n_1^j} \times \{0, 1\}^{n_2^j}$ is concave so, in both Kelso and Crawford's and Hatfield et al.'s models, the existence of equilibrium follows immediately from Section 4.1's discussion.

Another model that is easy to analyse using our methods is a "circular ones" model (cf. Bartholdi et al., 1980) in which each of n kinds of agent is only interested in a single,

[45] Kelso and Crawford make a superficially more restrictive assumption than ordinary substitutes, but it is equivalent in their context: see Danilov et al. (2003), Baldwin and Klemperer (2014), and Baldwin, Klemperer and Milgrom (in preparation).

[46] Their "choice language" definition differs superficially from Defn. (1), but Hatfield et al., 2015, Thm. B.1 confirms the equivalence.

specific, pair of goods, and these pairs form a cycle. The Unimodularity Theorem immediately tells us equilibrium is guaranteed to exist if and only if n is even. Furthermore, we can use our method of proof of Prop. 4.10 to find examples of equilibrium failure if n is odd; Ex. A.22 in Appendix A.5 gives details. More generally, our proof of Prop. 4.10 shows how to easily construct an explicit example of failure of equilibrium for *any* model that permits valuations from a non-unimodular demand type.

Reformulating models in our framework also shows clearly how we can generalise them. It is immediate, for example, that as long as we retain concavity and the strong substitutes demand type, we can remove Hatfield et al.'s restriction that an agent cannot be both a buyer and a seller on any one good (by simply extending their domain to be *any* subset of $\{-1, 0, 1\}^n$) and can also permit their agents to trade multiple units of the same products (by enlarging the domain to any subset of $\mathbb{Z}^n$).

6.2 Basis Changes of LIPs and Demand Types

Obviously, many properties are preserved if we re-package goods so that any integer bundle can still be obtained by buying and selling an integer selection of the new packages. This corresponds, of course, to a unimodular change of basis;[47] it simply distorts the LIP by a linear transformation which leaves its important structure unaffected.

Specifically, for a unimodular $n \times n$ matrix G, it is standard to define the "pullback" of a valuation $u : A \to \mathbb{R}$ to be $G^*u : G^{-1}A \to \mathbb{R}$ via $G^*u(\mathbf{x}) := u(G\mathbf{x})$. Then:

Proposition 6.1 (cf. e.g. Gorman, 1976, p. 219-220)**.**

(1) $\mathbf{x} \in D_u(\mathbf{p}) \iff G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p})$.

(2) $\mathcal{L}_{G^*u} = G^T\mathcal{L}_u := \{G^T\mathbf{p} : \mathbf{p} \in \mathcal{L}_u\}$*;*

(3) $u(\cdot)$ *is of demand type* $\mathcal{D}$ *iff* $G^*u(\cdot)$ *is of demand type* $G^{-1}\mathcal{D} := \{G^{-1}\mathbf{v} : \mathbf{v} \in \mathcal{D}\}$.

Ex. A.23 gives an illustration.

Baldwin and Klemperer (2014, especially Section 5; and in preparation-b) use the fact that important properties of demand are unaffected by unimodular basis changes in their analysis of the comparative statics of individual valuations. But the most immediate pay-off from this and our method of categorising valuations into "demand types" is that it is straightforward (see Appendix A.5) that:

Proposition 6.2. *"Having equilibrium for every finite set of concave valuations, for all relevant supply bundles" is a property of a demand type that is preserved under unimodular basis changes.*

So we can find demand types which always have equilibrium. For example, it is immediate that equilibrium is guaranteed for the following demand types:

"Consecutive Games" (see Greenberg and Weber, 1986, and also Danilov et al., 2013). Premultiplying the strong substitutes vectors, $\mathbf{e}^i$ and $(\mathbf{e}^i - \mathbf{e}^j)$, by the upper triangular matrix of 1s (of the appropriate dimension) yields the vectors $\sum_{k=1}^{i} \mathbf{e}^k$ and $\sum_{k=j+1}^{i} \mathbf{e}^k$ for $i > j$, respectively (and their negations). This is the demand type for

[47] A unimodular matrix is an integer matrix with integer inverse, or, equivalently, an integer matrix with determinant ± 1.

goods which have a natural fixed order, and for which any contiguous collection of goods may be considered as complements by any agent. For example, valuations for bands of radio spectrum, or for "lots" of sea bed to be developed for offshore wind (see Ausubel and Cramton, 2011) may be of this form.

"Generalised Gross Substitutes and Complements Valuations". Premultiplying the strong substitutes vectors by a matrix formed of $\{\mathbf{e}^i : i \leq k\} \cup \{-\mathbf{e}^i : i > k\}$, for some k, yields the demand type for which goods can be separated into two groups, with goods within the same group being strong substitutes, and each good also may exhibit 1:1 complementarities with any good in the other group.[48]

6.3 New Demand Types which Guarantee Equilibrium

Our Unimodularity Theorem helps identify new demand types of economic interest which are *not* unimodular basis changes of ordinary substitutes, or any subset thereof, for which equilibrium is guaranteed. For example, consider the demand type whose vectors are the columns of:

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This might, for example, model a firm's demand for "bundles of" four kinds of workers–three sorts of specialist (the first three goods) and a supervisor (the fourth good). The first three columns of D show that any one of the three kinds of specialist has value on his own; the middle three columns of D show that a supervisor increases the value of any specialist (that is, there are pairwise complementarities between any one of the first three "goods" together with the fourth); the last three columns of D show that there are also complementarities between any pair of different specialists if (but only if) a supervisor is also present; but a supervisor on her own is worthless.[49]

This demand type is pure complements, and is not a basis change of ordinary substitutes, or any subset thereof, as we show in the supplementary material to this paper (O'Connor, 2015; Baldwin and Klemperer (2014) give a mathematical proof that it is not a basis change of strong substitutes). It is routine to check that it is unimodular. In fact, it is a basis change of a "cographic" unimodular set (see Seymour 1980) but we think it has not previously been presented in this way as an example of purely complementary preferences. Cographic matrices are *mathematically* closely related to matrices formed of the strong substitute vectors (also known as "graphic", or "network" matrices) but are *not* in general related by basis changes, and so Prop. 6.1 cannot be

[48]The demand type's vectors are $\{\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^i - \mathbf{e}^{i'}, \mathbf{e}^i + \mathbf{e}^j, \mathbf{e}^j - \mathbf{e}^{j'} : i, i' \in \{1, \ldots, k\}, j, j' \in \{k+1, \ldots, n\}\}$. This extends the valuations introduced by Sun and Yang (2006, 2009) to permit multiple units of goods, and sellers as well as buyers. Shioura and Yang (2015) have independently made the same extension, and shown that equilibrium always exists for it.

[49]The reason is that $\mathbf{e}^4 \notin \mathcal{D}$; perhaps each firm's owner is a supervisor, and an additional supervisor without any workers would merely "spoil the broth". (There are, of course, many basis changes of D–and of any unimodular demand type–that include all the coordinate vectors.)

used to link valuations of these types. Indeed very little appears to be known on the relationship between "graphic" and "cographic" valuations. Their *economic* properties are clearly very different.

6.4 Relationships between Equilibrium and Complements; and between Equilibrium and Substitutes

Understanding the relationship between equilibrium and unimodularity shows that much conventional wisdom is mistaken in connecting the existence of equilibrium to substitutabilities: taking advantage of existing mathematical work on unimodular sets, demonstrates a link between equilibrium and *complements* that is, if anything, stronger.[50]

First, we can easily extend the previous section's example to obtain:

Proposition 6.3. *With $n > 3$, there exist demand types which are* not *a unimodular basis change of ordinary substitutes, or a subset thereof, and for which an equilibrium exists for every finite set of concave valuations of the demand type, for all relevant supply bundles. For $n \leq 3$ there exist no such demand types.*

To see the result for $n > 3$, consider the demand type defined by the matrix D of Section 6.3, with its vectors extended by $n-4$ zeros, and with the coordinate vectors $\mathbf{e}^i$, $i = 5, \ldots, n$ appended. Any basis change taking such a demand type to a substitutes demand type would restrict to a basis change taking D to substitutes, contradicting the result given in Section 6.3. The result for $n \leq 3$ follows from Seymour's (1980) characterisation.

We can also make use of mathematical results from Grishukhin et al. (2010) that imply that every unimodular set of vectors is a unimodular basis change of a set that contains only vectors in $\pm\{0,1\}^n$. Since demand types containing these vectors contain only complements valuations, Prop. 6.2 tells us:

Proposition 6.4. *Every demand type for which an equilibrium is guaranteed (that is, exists for every finite set of concave valuations of the demand type, for all relevant supply bundles) is a unimodular basis change of a demand type which contains* only *complements valuations and for which equilibrium is guaranteed.*

Observe that Prop. 6.3 shows the corresponding statement *cannot* be made about substitutes.[51] It *is* true (and easy to show) that the strong substitutes vectors are *maximal* as a unimodular set of vectors. So, given any one valuation *not* for strong substitutes, we can find valuations which are strong substitutes such that equilibrium fails (see Section 4.2.3). Some have interpreted this result as a necessity of substitutes, but this overlooks the fact that not all strong substitutes valuations need be within the demand type in question.[52]

[50] Seymour (1980) developed a characterisation of unimodular sets of vectors; Danilov and Grishukhin (1999) extended this to give a full characterisation of all maximal such sets.

[51] For two goods (but not more–see Section 3.2), substitutes are a basis change of complements via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see Section 6.2) so (only) for two goods, equilibrium fails "as often" for substitutes as for complements.

[52] Gul and Stacchetti demonstrate this maximality result using n-dimensional variants of our "simple

6.5 An Algorithm to Check for Existence of Equilibrium

The following algorithm checks for (and summarises our results on) existence of equilibrium:

Algorithm 6.5. Given concave valuations $u^j : A^j \to \mathbb{R}$ on finite $A^j \subsetneq \mathbb{Z}^n$ for $j = 1, \ldots, m$: apply steps (1)-(8), below, to $u^{k*} := u^{\{1,\ldots,k-1\}}$ and u^k, for each of $k = 2, \ldots, m$. If, for each k, equilibrium exists for u^k and u^{k*} for all relevant supplies, then equilibrium exists for $u^1, \ldots, u^m$, for all relevant supplies.

(1) Are u^k, u^{k*} of the same unimodular demand type?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies. End here.
  (ii) If no, continue.

(2) If $\dim \operatorname{conv}(A^k + A^{k*}) = n' < n$, make a unimodular change of basis so that $A^k, A^{k*} \subsetneq \mathbb{Z}^{n'}$, and use n' instead of n in the following steps.
(3) Calculate $\Gamma^n_k(A^k, A^{k*})$ for $k = 0, \ldots, n$ and $\Gamma^n(A^k, A^{k*})$. Find the intersection 0-cells.
(4) Does the *number* of intersection 0-cells equal $\Gamma^n(A^k, A^{k*})$?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies. End here.
  (ii) If no, continue.

(5) Find the naïve multiplicity of each intersection 0-cell C; note if the intersection is transverse at C, and the dimension of the minimal cell of $\mathcal{L}_{u^k}$ containing C.
(6) Does the naïvely-weighted count of all intersection 0-cells equal $\Gamma^n(A^k, A^{k*})$?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies. End here.
  (ii) If no, if $n \leq 3$, and if the intersection is transverse, then equilibrium fails for some relevant supply; end here. Otherwise continue.

(7) Identify all intersection 0-cells C such that the dimension k of the minimal cell of $\mathcal{L}_{u^k}$ containing C does not satisfy *both:*
  (i) the intersection of $\mathcal{L}_{u^k}$ and $\mathcal{L}_{u^{k*}}$ is transverse at every intersection 0-cell, C', for which the minimal cell of $\mathcal{L}_{u^k}$ containing C' has dimension k,
and (ii) the number of all intersection 0-cells C' as in (7)(i), counted with naïve multiplicities, is equal to $\Gamma^n_k(A^k, A^{k*})$.

(8) Is $D_{u^{\{k,k*\}}}(\mathbf{p})$ discrete-convex for all $\mathbf{p}$ at intersection 0-cells identified in Step (7)?
  (i) If yes, equilibrium exists for u^k and u^{k*} for all relevant supplies.
  (ii) If no, equilibrium fails for some relevant supply.

Step (1) summarises the Unimodularity Theorem (Thm. 4.3). Step (4) uses a special case of the Intersection Count Theorem (Thm. 5.12): if all naïve multiplicities equal 1,

---

substitutes" valuation (Fig. 5a). Such valuations are not, of course, in (e.g.) any purely-complements demand type, but Gul and Stacchetti's description of them as "unit demand" might suggest they should be in any set of interest (cf., the paper's remark (p. 96) "in a sense, the GS [gross substitutes] condition is necessary to ensure existence of a Walrasian equilibrium"). See also Kelso and Crawford (1982). And Azevedo et al.'s (2013, p.286) remark "adding a continuum of consumers ... eliminates the existence problems created by complementarities" can also be misinterpreted. (However, Bikhchandani and Mamer, 1997, Prop. 3, previously noted the existence of non-substitute preferences which guarantee equilibrium if just two agents each value at most one unit of each good.)

then the weighted count of intersection 0-cells is just the number of these cells. (This can save computations.) Step (6) is the Intersection Count Theorem itself. Finally, Steps (7)-(8) use Prop. 5.20 (with Prop. 5.2).

It is not surprising that we cannot rule out needing Step (8). The question of equilibrium with indivisible goods is closely related to a long-studied problem in mathematics: the question of when the integer points in the Minkowski sum of two polytopes, is equal to the Minkowski sum of the integer points in those polytopes (see, e.g., Haase et al., 2007). However, we know (Thm. 5.12 with Prop. 5.2) that we have to check at most $\Gamma^n(A^1, A^2)$ prices.

Appropriate tools exist for each step of this algorithm. These tools often behave well in "usual" conditions, but may have bad worst-cases. (See Remark A.24.) So it seems hard to give clean results of computational complexity, or to compare the efficiency of using earlier stages of our algorithm with simply resorting to many "brute force" calculations, as in Step (8). And, because we may often need to resort to Step (8), it is also hard to compare with Bikhchandani and Mamer's (1997) algorithm which checks for equilibrium by the alternative "brute force" approach of comparing the solution of an integer programming problem with the corresponding linear programming problem for every relevant supply.

Of course, an important virtue of the Unimodularity and Intersection Count Theorems is that, when the earlier Steps do suffice to determine existence, they increase our understanding and intuition as to why equilibrium exists or fails.

6.6 Multi-Party Matching

Section 6.1 showed Kelso and Crawford's (1982) model of bipartite matching was a special case of our model.

We can also apply our model to other forms of matching and coalition-formation. The example in Section 6.3 can be interpreted as a multi-player matching problem in which the columns of D are the coalitions of workers that create value: Baldwin and Klemperer (2014, Thm. 6.7) show that, assuming perfectly transferable utility, a stable matching in which no subset of workers can gain from re-matching (that is, an allocation in the core of the game among the workers) corresponds exactly to an equilibrium allocation of workers in our model (in which every worker receives its competitive wage, and no further gains from trade are possible). So, since the demand type is unimodular, it describes a class of multi-player matching problems for which a stable match always exists.

More generally, Baldwin and Klemperer (2014 Sections 2.5, 6.2, 6.3.1) show that any model of coalition formation with perfectly transferable utility can be modelled using our tools; it corresponds to a demand type containing some subset of vectors in $\pm\{0,1\}^n$. And a stable match always exists iff this demand type is unimodular (Baldwin and Klemperer, 2014, Cor. 6.11), as it is in the case of the "workers" example above. If the demand type is not unimodular, the methods of Section 5 tell us for what coalitions' valuations there are stable matchings. Moreover, applying Prop. 6.3 to matching problems shows that stable matchings can be guaranteed for a broader class of preferences than many people assume.[53]

[53] Consider, e.g., Hatfield and Milgrom's (2005) statement that "preferences that do not satisfy the

Baldwin and Klemperer (in preparation-a) develops the applications to matching and coalition-formation in more detail.[54]

6.7 Understanding Individual Demand

Our techniques are powerful tools for understanding individual demand. In particular, Thm. 2.10 tells us that *any* balanced rational polyhedral complex is the LIP of some quasilinear valuation *and conversely.* This allows us to explore properties of valuations by drawing and analysing appropriate geometric diagrams *without* needing to undertake the typically much more challenging task of constructing valuations that generate these diagrams.

Baldwin and Klemperer (2014, especially Section 5; and in preparation-b) further explores the comparative statics of individual demand, in order to better understand demand changes at non-UDR prices.

Unimodularity turns out to have important implications for the structure of *individual* demand, as well as (as we saw in our discussion of the existence of equilibrium) for aggregate demand. This work also leads to a generalisation of Gul and Stacchetti's (1999) "Single Improvement Property".

Related work (joint with Paul Milgrom) uses our framework to help understand implications of different notions of substitutability for indivisible goods that have been suggested in the literature.[55]

6.8 Auction Design

Practical auctions need to restrict the kinds of bids that can be made, thus restricting the preferences that bidders can express. Restricting to a demand type is often natural, since the economic context often suggests appropriate trade-offs between goods. For example, the Bank of England expected bidders to have £1:£1 trade-offs between any pair of the several different "kinds" of money it loaned in the financial crisis.[56] Such trade-offs represent a form of strong substitutes preferences. So the Bank chose auction rules that made it easy for bidders to communicate such preferences, and was also unconcerned about ruling out the expression of other preferences.[57]

Knowing that the bids in an auction must all express preferences of a demand type also clarifies the meaning, and the implications, of the restrictions that have been im-

substitutes condition cannot be guaranteed always to select a stable allocation", though the Proposition (p.921) that their introductory remark loosely summarises is, of course, correct in its context.

[54]Since our framework allows us to consider multiple players of each kind, it easily yields results along the lines of Chiappori, Galichon, and Salanié (2014, see also Balinski, 1970).

[55]Baldwin, Klemperer, and Milgrom (in preparation). This paper also develops the relationship between the existence of equilibrium for substitutes and properties of the Vickrey auction and the core.

[56]The different "goods" were long-term loans (repos) against different qualities of collateral.

[57]*Any* strong substitutes preference could be expressed if the Bank's "Product-Mix" Auctions were augmented by permitting "negative" bids (see Klemperer, 2010, and Baldwin and Klemperer, in preparation-c). Product-mix auctions are "one-shot" auctions for allocating heterogeneous goods. Their equilibrium allocations and prices are similar to those of clock or Simultaneous Multiple-Round Auctions in private value contexts, but they permit the bid-taker to express richer preferences; they are more robust against collusive and/or predatory behaviour; and they are, of course, much faster.

posed on the bidders.[58] In particular, the motivation of the Product-Mix Auction is to find competitive equilibrium, given bidders' and the bid-taker's reported preferences. Since the Bank of England's implementation of the Product-Mix auction allows rationing (which makes "goods" divisible) ensuring the existence of equilibrium is not too hard.[59] But in many contexts rationing is less sensible. For example, a too-small piece of radio spectrum may not be useful. Similarly, a government may be interested in offers to build gas-fired plants, nuclear-power stations, wind farms, etc., and these may be indivisible. So results about equilibrium with indivisibilities tell us when Product-Mix Auctions can easily be used.

Our techniques also facilitate the analysis of Product-Mix Auctions. Individuals' bids in these auctions are aggregated in exactly the same simple way that (weighted) LIPs are combined to find aggregate demand. This also makes the auctions more "user-friendly", which is critical for getting them implemented in practice. Moreover, geometric analysis can develop methods for finding equilibrium in new versions of the Product-Mix Auction; this may help resolve problems currently facing regulators such as the U.S. Federal Communications Commission, the U.K.'s Ofcom and the U.K. Department for Energy and Climate Change.

7 Conclusion

An agent's demand is completely described by its choices at all possible price vectors. So it can also be described by the divisions between the regions of price space in which the agent demands different bundles, and hence by the vectors that define these divisions. This suggests a natural way of classifying valuations into "demand types".

Using this classification, together with the duality between the geometric representations of valuations in price space and in quantity space, yields significant new insights into when competitive equilibrium exists.

A demand type's vectors also encode the possible comparative statics of demand, and we expect many other results can be understood more readily, and developed more efficiently, using our geometric perspective.

Companion papers[60] use our framework and tools to obtain new results about the existence of stable matchings in multiple-agent matching models; about individual demand; and further develop the Product-Mix Auction implemented by the Bank of England in response to the 2007 Northern Rock bank run and the subsequent financial crisis.

---

[58]Restricting to a demand type also permits relatively complex "bids" while still checking that they satisfy the restrictions, since there are easy software solutions to calculate the normal vectors of the LIP for any valuation and so reveal the demand "type" (see Remark A.24).

[59]So the updated (2014) implementation of the Bank's auction also permitted some complements preferences while maintaining the existence of equilibrium.

[60]See Baldwin and Klemperer (in preparation-a and b) and Baldwin, Goldberg and Klemperer (in preparation). Preliminary work is in Baldwin and Klemperer (2014).

References

L. Ausubel and P. Cramton. Auction design for wind rights. Report to Bureau of Ocean Energy Management, Regulation and Enforcement, 2011.

L. M. Ausubel and P. Milgrom. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1(1):1–42, 2002.

E. M. Azevedo, E. G. Weyl, and A. White. Walrasian equilibrium in large, quasi-linear markets. *Theoretical Economics*, 8(2):281–290, May 2013.

E. Baldwin and P. Klemperer. "Demand types", and stable multi-agent matching. In preparation, a.

E. Baldwin and P. Klemperer. "Demand types", and comparative statics with indivisibilities. In preparation, b.

E. Baldwin and P. Klemperer. Tropical geometry to analyse demand. Mimeo., May 2012.

E. Baldwin and P. Klemperer. Tropical geometry to analyse demand. Mimeo. Available from elizabeth-baldwin.me.uk and www.paulklemperer.org, May 2014.

E. Baldwin, P. Goldberg, and P. Klemperer. The multi-dimensional product-mix auction. In preparation.

E. Baldwin, P. Klemperer, and P. Milgrom. Substitute goods, competitive equilibrium, and Vickrey outcomes, with indivisible goods. In preparation.

M. Balinski. On maximum matchings, minimum coverings, and their connections. In H. Kuhn, editor, *Proceedings of the Princeton Symposium on Mathematical Programming*. Princeton University Press, 1970.

Bank of England. The Bank's new indexed long-term repo operations. *Bank of England Quarterly Bulletin*, 50(2):90–91, 2010.

Bank of England. The Bank's indexed long-term repo operations. *Bank of England Quarterly Bulletin*, 51(2):93, 2011.

C. B. Barber, D. P. Dobkin, and H. Huhdanpaa. The quickhull algorithm for convex hulls. *ACM Transactions on Mathematical Software*, 22(4):469–483, December 1996.

J. J. Bartholdi, J. B. Orlin, and H. D. Ratliff. Cyclic scheduling via integer programs with circular ones. *Operations Research*, 28(5):1074–1085, 1980.

D. Bernstein. The number of roots of a system of equations. *Functional Analysis and Its Applications*, 9(3):183–185, 1975.

B. Bertrand and F. Bihan. Intersection multiplicity numbers between tropical hypersurfaces. In E. Brugalle, M. A. Cueto, A. Dickenstein, E.-M. Feichtner, and I. Itenberg, editors, *Algebraic and Combinatorial Aspects of Tropical Geometry*, volume 589. American Mathematical Soc., 2013.

E. Bézout. *Théorie générale des équations algébriques.* de l'imprimerie de Ph.-D. Pierres, rue S. Jacques, 1779.

S. Bikhchandani and J. Mamer. Competitive equilibrium in an exchange economy with indivisibilities. *Journal of Economic Theory*, 74(2):385–413, 1997.

J. W. S. Cassels. *An introduction to the geometry of numbers*, volume 99 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1971.

T. Chen, T.-L. Lee, and T.-Y. Li. Mixed volume computation in parallel. *Taiwanese Journal of Mathematics*, 18(1):93–114, 2014.

P.-A. Chiappori, A. Galichon, and B. Salanié. The roommate problem is more stable than you think. Working Paper Series 4676, CESifo, February 2014.

D. A. Cox, J. B. Little, and D. O'Shea. *Using algebraic geometry*, volume 185 of *Graduate Texts in Mathematics.* Springer, 2005.

R. A. Crowell and N. M. Tran. Tropical geometry and mechanism design. Preprint arXiv:1606.04880, arXiv, June 2016.

V. Danilov and V. Grishukhin. Maximal unimodular systems of vectors. *European Journal of Combinatorics*, 20(6):507–526, 1999.

V. Danilov and G. Koshevoy. Discrete convexity and unimodularity–I. *Advances in Mathematics*, 189(2):301–324, 2004.

V. Danilov, G. Koshevoy, and C. Lang. Gross substitution, discrete convexity, and submodularity. *Discrete Applied Mathematics*, 131(2):283–298, 2003.

V. Danilov, G. Koshevoy, and C. Lang. Equilibria in markets with indivisible goods. *Journal of the New Economic Association*, 18(2):10–34, 2013. In Russian.

V. Danilov, G. Koshevoy, and K. Murota. Discrete convexity and equilibria in economies with indivisible goods and money. *Mathematical Social Sciences*, 41:251–273, 2001.

Economist, The. A golden age of micro. *The Economist*, October 19, 2012. Free Exchange.

P. Fisher. Recent developments in the sterling monetary framework. In *Manchester Economics Seminar, University of Manchester*, 2011.

T. Frost, N. Govier, and T. Horn. Innovations in the Bank's provision of liquidity insurance via Indexed Long-Term Repo (ILTR) operations. *Bank of England Quarterly Bulletin*, 55(2):181–188, 2015.

E. Gawrilow and M. Joswig. *polymake: a Framework for Analyzing Convex Polytopes*, pages 43–73. Birkhäuser Basel, Basel, 2000. Polymake is available from polymake.org.

J. K. Goeree and A. Kushnir. A geometric approach to mechanism design. Working Paper 56, University of Zurich, Department of Economics, June 2013.

W. M. Gorman. Tricks with utility functions. In M. J. Artis and A. R. Nobay, editors, *Essays in economic analysis*, pages 211–243. Cambridge: Cambridge University Press, 1976.

J. Greenberg and S. Weber. Strong tiebout equilibrium under restricted preferences domain. *Journal of Economic Theory*, 38(1):101–117, 1986.

V. Grishukhin, V. Danilov, and G. Koshevoy. Unimodular systems of vectors are embeddable in the (0,1)-cube. *Mathematical Notes*, 88(6):891–893, 2010.

P. M. Gruber. *Convex and discrete geometry*, volume 336. Springer, 2007.

F. Gul and E. Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87(1):95–124, 1999.

C. Haase, T. Hibi, and D. Maclagan. Mini-workshop: Projective normality of smooth toric varieties. Report 39/2007, Mathematisches Forschungsinstitut Oberwolfach, August 2007.

S. Hampe. a-tint: A polymake extension for algorithmic tropical intersection theory. *European Journal of Combinatorics*, 36:579–607, Feb. 2014a.

S. Hampe. *Algorithmic aspects of tropical intersection theory.* PhD thesis, Technischen Universität Kaiserslautern, June 2014b.

J. W. Hatfield and P. R. Milgrom. Matching with contracts. *The American Economic Review*, 95(4):913–935, 2005.

J. W. Hatfield, S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp. Stability and competitive equilibrium in trading networks. *Journal of Political Economy*, 121 (5):966–1005, 2013.

J. W. Hatfield, S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp. Full substitutability. Mimeo., February 2015.

B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. *Mathematics of computation*, 64(212):1541–1555, 1995.

A. N. Jensen. *Gfan version 0.5: A User's Manual*, January 2011. Gfan is available at: http://home.math.au.dk/jensen/software/gfan/gfan.html.

A. N. Jensen. Tropical homotopy continuation. Preprint arXiv:1601.02818, arXiv, 2016.

A. S. Kelso and V. P. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.

A. G. Khovanskii. Newton polyhedra and the genus of complete intersections. *Functional Analysis and Its Applications*, 12(1):38–46, 1978.

P. Klemperer. A new auction for substitutes: Central bank liquidity auctions, the U.S. TARP, and variable product-mix auctions. Working paper, Oxford University, 2008.

P. Klemperer. The product-mix auction: A new auction design for differentiated goods. *Journal of the European Economic Association*, 8(2-3):526–536, 2010.

A. Kouchnirenko. Polyèdres de Newton et nombres de Milnor. *Inventiones mathematicae*, 32(1):1–31, 1976.

J. Ma. Competitive equilibrium with indivisibilities. *Journal of Economic Theory*, 82 (2):458–468, 1998.

D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

A. Mas-Colell, M. D. Whinston, and J. R. Green. *Microeconomic Theory*. Oxford University Press, June 1995.

V. Matveenko. Tropical support sets in analysis of weak links and complementarity. *Tropical and idempotent mathematics and applications. Contemporary Mathematics*, 616:211–220, 2014.

R. D. McKelvey and A. McLennan. The maximal number of regular totally mixed Nash equilibria. *Journal of Economic Theory*, 72(2):411–425, 1997.

A. McLennan. The expected number of real roots of a multihomogeneous system of polynomial equations. *American Journal of Mathematics*, 124(1):49–73, 2002.

A. McLennan. The expected number of Nash equilibria of a normal form game. *Econometrica*, 73(1):141–174, 2005.

A. McLennan and J. Berg. Asymptotic expected number of Nash equilibria of two-player normal form games. *Games and Economic Behavior*, 51(2):264–295, 2005. Special Issue in Honor of Richard D. McKelvey.

P. McMullen. Separation in the polytope algebra. *Contributions to Algebra and Geometry*, 34(1):15–30, 1993.

G. Mikhalkin. Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *Topology*, 43(5):1035–1065, 2004.

P. Milgrom and C. Shannon. Monotone comparative statics. *Econometrica*, 62(1):157–180, 1994.

P. Milgrom and B. Strulovici. Substitute goods, auctions, and equilibrium. *Journal of Economic Theory*, 144(1):212–247, 2009.

A. Milnes. Creating confidence in cash. *Blueprint*, page 14, October 2010.

K. Murota. *Discrete Convex Analysis*, volume 10 of *Monographs on Discrete Mathematics and Applications*. SIAM, Philadelphia, 2003.

K. Murota and A. Shioura. M-convex function on generalized polymatroid. *Mathematics of Operations Research*, 24(1):95–105, 1999.

K. Murota and A. Tamura. New characterizations of m-convex functions and their applications to economic equilibrium models with indivisibilities. *Discrete Applied Mathematics*, 131(2):495–512, 2003.

T. Nguyen and R. Vohra. Near feasible stable matchings with complementarities. Working Paper 14-028, PIER, 2014.

T. Nguyen, A. Peivandi, and R. Vohra. Assignment problems with complementarities. *Journal of Economic Theory*, 165:209–241, 2016.

T. O'Connor. A unimodular demand type which is not a basis change of substitutes. supplementary material to Baldwin and Klemperer (2016): "Understanding Preferences: Demand Types, and the Existence of Equilibrium with Indivisibilities". Matlab code, available from elizabeth-baldwin.me.uk and www.paulklemperer.org, July 2015.

H. Poincaré. Second complément à l'analysis situs. *Proceedings of the London Mathematical Society*, 32(1):277–308, 1900.

M. Pycia. Non-existence result for matching games with transfers. Mimeo., March 2008.

M. Richter and A. Rubinstein. Back to fundamentals: Equilibrium in abstract economies. *American Economic Review*, 105(8):2570–94, 2015.

A. Schrijver. *Theory of linear and integer programming.* Interscience series in discrete mathematics and optimisation. Wiley-Interscience, 2000.

P. Seymour. Decomposition of regular matroids. *Journal of Combinatorial Theory, Series B*, 28(3):305–359, 1980.

A. Shioura and A. Tamura. Gross substitutes condition and discrete concavity for multi-unit valuations: A survey. *Journal of the Operations Research Society of Japan*, 58 (1):61–103, January 2015.

A. Shioura and Z. Yang. Equilibrium, auction, and generalized gross substitutes and complements. *Journal of the Operations Research Society of Japan*, 58(4):410–435, 2015.

N. Sun and Z. Yang. Equilibria and indivisibilities: Gross substitutes and complements. *Econometrica*, 74(5):1385–1402, 2006.

N. Sun and Z. Yang. A double-track adjustment process for discrete markets with substitutes and complements. *Econometrica*, 77(3):933–952, 2009.

N. Sun and Z. Yang. A general competitive trading network with indivisibilities and gross substitutes and complements. Private Communication, 2011.

A. Teytelboym. Gross substitutes and complements: A simple generalization. *Economics Letters*, 123(2):135–138, 2014.

N. M. Tran and J. Yu. Product-mix auctions and tropical geometry. Preprint arxiv:1505.05737, arXiv, May 2015.

R. V. Vohra. *Mechanism design: a linear programming approach*, volume 47. Cambridge University Press, 2011.

J. A. Weymark. A geometric approach to dominant strategy implementation. Presentation at Hausdorff Institute Workshop on Tropical Mathematics and Economics, May 2016.

A Proofs and Additional Examples

Results are given below in the order in which it is most convenient to prove them.

A.1 Proofs for Section 2

Proof of Prop. 2.7. Write Π for the set containing all cells of the LIP as well as closures of its UDRs. There are finitely many objects in Π, as $A \subsetneq \mathbb{Z}^n$ is finite. Each is defined by a collection of equalities $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ and inequalities $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') \geq u(\mathbf{x}) - u(\mathbf{x}')$, so it follows (since $A \subsetneq \mathbb{Z}^n$) that each is a rational polyhedron. Conversely, a set defined by equalities and inequalities of this form is in Π. A face of a polyhedron C in Π is the subset of C on which at least one of these inequalities holds with equality: it is therefore a cell of the LIP. To consider $C \cap C'$, where $C \neq C'$ in Π, suppose X, X' are such that $C = \{\mathbf{p} \in \mathbb{R}^n : X \subseteq D_u(\mathbf{p})\}$ and $C' = \{\mathbf{p} \in \mathbb{R}^n : X' \subseteq D_u(\mathbf{p})\}$. Then $X \neq X'$ and $C \cap C' = \{\mathbf{p} \in \mathbb{R}^n : X \cup X' \subseteq D_u(\mathbf{p})\}$. If this is non-empty, it is itself a cell of the LIP and a face of both C and C'. Thus Π is a polyhedral complex.

If more than one bundle is demanded then $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for some $\mathbf{x} \neq \mathbf{x}'$; as there are only finitely many bundles it is thus generic for only one bundle to be demanded, and hence the closures of UDRs are all n-dimensional. So the LIP is equal to the boundary of the closures of the UDRs, and thus is contained in its $(n-1)$-dimensional components, and any cell of Π of dimension $k < n$ is a cell of the LIP. Thus the set of cells of the LIP forms an $(n-1)$-dimensional rational polyhedral complex. □

Proof of Lemma 2.8. It is easier to prove these statements in the opposite order. The set $\{\mathbf{p} \in \mathbb{R}^n : D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})\}$ is non-empty (it contains $\mathbf{p}^\circ$ itself) and so defines a cell, C'. If $X \subseteq A$ is such that $C = \{\mathbf{p} \in \mathbb{R}^n : X \subseteq D_u(\mathbf{p})\}$ then $X \subseteq D_u(\mathbf{p}^\circ)$ and so any $\mathbf{p} \in C'$ satisfies $X \subseteq D_u(\mathbf{p})$, so $C' \subseteq C$. From Prop. 2.7, we know that $C \cap C'$ is a face of C and of C'. However $C \cap C'$ contains a point, $\mathbf{p}^\circ$, that is in the interior of C. This is only consistent if $C \cap C' = C$. As $C' \subseteq C$ we conclude $C' = C$.

So $D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})$ iff $\mathbf{p} \in C$. But since this holds for any $\mathbf{p}^\circ \in C^\circ$, we may reverse the roles of $\mathbf{p}$ and $\mathbf{p}^\circ$ to see that $D_u(\mathbf{p}^\circ) = D_u(\mathbf{p})$ if $\mathbf{p} \in C^\circ$. □

Proof of Lemma 2.11. As in the proof of Prop. 2.7 above, the set of all cells of the LIP together with the closures of UDRs forms a polyhedral complex, and in particular therefore the intersection of two or more distinct closures of UDRs is a cell of the LIP.

Conversely, given any cell C of the LIP and $\mathbf{p}^\circ \in C^\circ$, let X be the set of all bundles uniquely demanded at some price in a small neighbourhood of $\mathbf{p}^\circ$. Then $C' = \{\mathbf{p} \in \mathbb{R}^n : X \subseteq D_u(\mathbf{p})\}$ is a cell of the LIP. By continuity of the indirect utility in $\mathbf{p}$, that is, of $\mathbf{p} \mapsto u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$, every bundle in X is demanded at $\mathbf{p}^\circ$, so $\mathbf{p}^\circ \in C'$ and hence (since $\mathbf{p}^\circ$ is in the interior of C) $C \subseteq C'$. So C is a face of C': there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{p}' \cdot \mathbf{v} \leq \mathbf{p} \cdot \mathbf{v}$ for all $\mathbf{p}' \in C'$, $\mathbf{p} \in C$. Suppose for a contradiction that $C \subsetneq C'$, so there exists $\mathbf{p}' \in C'$ such that $\mathbf{p}' \cdot \mathbf{v} < \mathbf{p}^\circ \cdot \mathbf{v}$. But because the UDRs are dense in $\mathbb{R}^n$, for any $\epsilon > 0$ there exists a UDR price $\mathbf{p} \in \mathbb{R}^n$ such that $0 < \mathbf{v} \cdot (\mathbf{p} - \mathbf{p}^\circ) < \epsilon$. Thus, if $\{\mathbf{x}\} = D_u(\mathbf{p})$, we must have $\mathbf{x} \in X$. However, again by continuity of indirect utility, $\mathbf{x}$ is only demanded in the closure of the UDR containing $\mathbf{p}$, and by construction, $\mathbf{p}'$ is not in this set. Thus, by contradiction, $C = C'$.

By construction, C' is the intersection of the closures of a set of UDRs. This completes the proof. □

Proof of Lemma 2.18. Let G be the upper boundary, with respect to the final coordinate, of $\text{conv}(\{(\mathbf{x}, u(\mathbf{x})) : \mathbf{x} \in A\})$. Then G is the graph of a weakly-concave function on $\text{conv}(A)$, which clearly satisfies the definition of $\text{conv}(u)$. By the supporting hyperplane theorem, for every $\mathbf{x} \in \text{conv}(A)$ there exists a supporting hyperplane, H, to G, at $(\mathbf{x}, \text{conv}(u)(\mathbf{x}))$. Both $\text{conv}(D_u(\mathbf{p}))$ and $D_{\text{conv}(u)}(\mathbf{p})$ are equal to the projection of $H \cap G$ to its first n coordinates, and thus are equal. □

Proof of Lemma 2.13. Suppose that A is discrete-convex. It is a standard application of the supporting hyperplane theorem that a function $u : A \to \mathbb{R}$ is concave iff, for all $\mathbf{x} \in A$, there exists $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} \in D_u(\mathbf{p})$. As we have defined concavity to require discrete-convexity of domain, this provides the first equivalence. Moreover, since the intersection between a supporting hyperplane and a convex set is always itself convex, discrete-convexity of $D_u(\mathbf{p})$ also follows.

Conversely, suppose $D_u(\mathbf{p})$ is discrete-convex for all $\mathbf{p}$. Let $u' : \text{conv}(A) \cap \mathbb{Z}^n \to \mathbb{R}$ be the restriction of $\text{conv}(u)$ to $\text{conv}(A) \cap \mathbb{Z}^n$. Consider any $\mathbf{x} \in \text{conv}(A) \cap \mathbb{Z}^n$. By the previous paragraph, there exists $\mathbf{p}$ such that $\mathbf{x} \in D_{u'}(\mathbf{p})$. As $\text{conv}(u) = \text{conv}(u')$, it follows by Lemma 2.18 that $\text{conv}(D_u(\mathbf{p})) = \text{conv}(D_{u'}(\mathbf{p}))$. But then, by assumption, $\mathbf{x} \in D_u(\mathbf{p})$. So the second property holds (and in particular A is discrete-convex). □

Proof of Lemma 2.14. Restrict u to $\text{conv}(D_u(\mathbf{p})) \cap \mathbb{Z}^n$ to see that Part (1) follows from Lemma 2.13. For (2), recall that $\text{conv}(u) = \text{conv}(u')$ and so, by Lemma 2.18, $D_u(\mathbf{p}) = \{\mathbf{x}\}$ iff $D_{u'}(\mathbf{p}) = \{\mathbf{x}\}$, for any bundle $\mathbf{x}$. So $|D_u(\mathbf{p})| = 1$ iff $|D_{u'}(\mathbf{p})| = 1$. □

Proof of Prop. 2.17. The "roof" is $\{(\mathbf{x}, \text{conv}(u)(\mathbf{x})) \in \mathbb{R}^{n+1} : \mathbf{x} \in \text{conv}(A)\}$ (it is the graph of $\text{conv}(u)$). It is clearly the upper boundary, with respect to the final coordinate, of the convex hull of the points $\{(\mathbf{x}, u(\mathbf{x})) \in \mathbb{Z}^n \times \mathbb{R} : \mathbf{x} \in A\}$. The set of faces of the "roof" thus has the structure of a polyhedral complex (see e.g. Grünbaum and Shephard, 1969). Moreover, there is a clear bijection between $\text{conv}(A)$ and the "roof", which is linear on each of these faces. So the projections of these faces to their first n coordinates also form a polyhedral complex. Now the result follows from Lemma 2.19 (which is shown in the text to be a consequence of Lemma 2.18, which is proved above). □

Proof of Prop. 2.21. By Lemma 2.8, the demand set is constant in a cell interior; it is clearly also constant in a UDR. So the correspondence in (1) is well-defined. Moreover, the affine span of C_σ is given by the set of prices $\mathbf{p}'$ such that $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = u(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}'$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, i.e. all prices such that $\mathbf{p}' \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all such $\mathbf{x}, \mathbf{x}'$. If $\sigma = \text{conv}(D_u(\mathbf{p}))$ is k-dimensional, there are k linearly independent vectors of the form $\mathbf{x} - \mathbf{x}'$, and so k linearly independent constraints $\mathbf{p}'$. So $\dim C_\sigma = n - k$.

For (2), recall from Lemma 2.8 that $C_\sigma = \{\mathbf{p} \in \mathbb{R}^n : D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})\}$, where $\mathbf{p}^\circ \in C_\sigma^\circ$. But for such $\mathbf{p}^\circ$, we show $D_u(\mathbf{p}^\circ) \subseteq D_u(\mathbf{p})$ iff $\sigma = \text{conv}\, D_u(\mathbf{p}^\circ) \subseteq \text{conv}(D_u(\mathbf{p}))$. Necessity is obvious, and sufficiency follows from Lemma 2.20: if we assume $\text{conv}(D_u(\mathbf{p}^\circ)) \subseteq \text{conv}(D_u(\mathbf{p}))$, then any $\mathbf{x} \in D_u(\mathbf{p}^\circ) \subsetneq \text{conv}(D_u(\mathbf{p}))$ must satisfy $\mathbf{x} \in D_u(\mathbf{p})$. So $\mathbf{p} \in C_\sigma$ iff $\sigma \subseteq \text{conv}(D_u(\mathbf{p}))$. Now (3) follows from the combination of (1) and (2).

For (4), we saw already that $\mathbf{p}' \cdot (\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$ and all $\mathbf{p}' \in C_\sigma$ where $\sigma = \text{conv}(D_u(\mathbf{p}))$. Thus $(\mathbf{p}'' - \mathbf{p}') \cdot (\mathbf{x}' - \mathbf{x}) = 0$ for all $\mathbf{p}', \mathbf{p}'' \in C_\sigma$, also for any $\mathbf{x}, \mathbf{x}' \in \text{conv}(D_u(\mathbf{p})) = \sigma$. Now (5) is immediate from Defns. 2.4 and 2.16(4). □

Corollary A.1. *If the demand complex is n-dimensional then every k-cell C_σ of $\mathcal{L}_u$ has some 0-cell C_τ in its boundary, with $\sigma \subseteq \tau$. Moreover if $\mathbf{x} \in \sigma$ but $\mathbf{x} \notin D_u(\mathbf{p}^\circ_\sigma)$ for $\mathbf{p}^\circ_\sigma \in C^\circ_\sigma$, then also $\mathbf{x} \notin D_u(\mathbf{p}^\circ_\tau)$ for $\mathbf{p}^\circ_\tau \in C^\circ_\tau$.*

Proof. If σ is an $(n-k)$-cell of an n-dimensional demand complex, then σ is contained in an n-cell τ. So C_τ is a 0-cell of the LIP and $C_\tau \subseteq C_\sigma$ (Prop. 2.21(3)). By Prop. 2.21(1) we know $\sigma = \operatorname{conv}(D_u(\mathbf{p}^\circ_\sigma))$, so by Lemma 2.20, $\mathbf{x} \notin D_u(\mathbf{p}^\circ_\tau)$. □

A.2 Examples for Sections 2 and 3

Additional Discussion of Figs. 2 and 3. In Figs. 3b-3c, the bundles demanded in the UDRs are (0,0), (0,1), (0,2), (1,2), (2,2), (2,1), and (2,0), clockwise from the top right of the LIP in Fig. 3a.

Observe that if the "black bundles"'s value was greater, so the corresponding bar in Figs. 2b and 2c just touched the roof, then it would still not be at a vertex of Fig. 3a but it would be demanded at the price corresponding to the wavy-shaded 0-cell (that is, $(1, 2)$) that is at a vertex of the LIP. And if it had an (even) higher valuation (so "poked through" the current roof), then the corresponding demand complex point would become a vertex, and the corresponding LIP 0-cell would "open up" to form a new UDR corresponding to the range of prices at which the bundle $(1, 1)$ would then be demanded.

To find the exact LIP of Fig. 2a's valuation using the demand complex of Fig. 3a, compare the values of bundles in adjacent UDRs: the valuations of bundles $(1, 0)$ and $(0, 1)$ show that the dotted 0-cell of the LIP is at $\mathbf{p} = (4, 8)$, since 4 and 8 are the prices below which the agent will first buy any of goods 1 and 2, respectively, when the other good's price is very high. And the wavy-shaded 0-cell must be at (1,2) since $10 - 8 = 2$ is the incremental value of a second unit of good 2, when the agent has none of good 1, and $11 - 10 = 1$ would be the incremental value from then adding a unit of good 1, etc.

So constructing the LIP via the demand complex separates the questions "in what directions are there line segments?" and "where in space are they?", and clarifies which bundle values have to be compared to fix the precise locations of the LIP's cells.

Example A.2. For $A = \{0, 1\}^2$ it is easy to draw every possible demand complex and so obtain every possible combinatorial type of weighted LIP–see Fig. 11. It is clear that

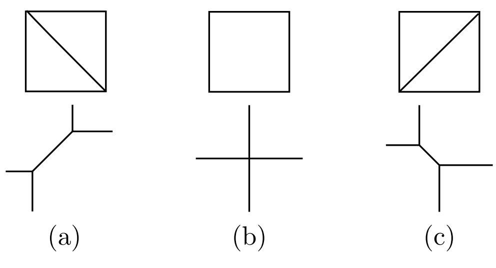


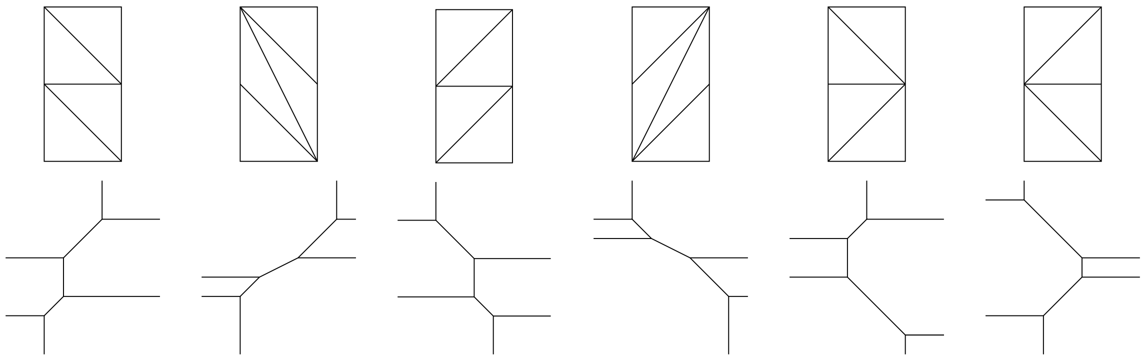
Figure 11: All possible demand complexes, and examples of their dual weighted LIPs, giving all the combinatorial types when $A = \{0, 1\}^2$.

Fig. 11a applies when $u(0, 0) + u(1, 1) < u(1, 0) + u(0, 1)$, so represents substitutes; Fig. 11b applies when $u(0, 0) + u(1, 1) = u(1, 0) + u(0, 1)$, so is additively separable demand; and Fig. 11c applies when $u(0, 0) + u(1, 1) > u(1, 0) + u(0, 1)$, so is complements. (See

Section 3.2 for these distinctions). Importantly, it is also clear that these are the *only* possibilities.

Observe that Fig. 11b can be seen as a limit of Fig. 11a (or Fig. 11c). In the LIP, the two 0-cells become arbitrarily close and then coincide in the limit; in quantity space, the faces of the roof tilt until they are coplanar, so that a demand complex edge disappears.

More generally, any demand complex in which the subdivision is not maximal (that is, additional valid $(n-1)$-faces could be added) can be recovered by deleting $(n-1)$-faces from some demand complex whose subdivision *is* maximal. For example, Fig. 12 shows all the demand complexes with maximal subdivision, and examples of their dual weighted LIPs, for $A = \{0,1\} \times \{0,1,2\}$; we can then easily recover the remaining combinatorial types of weighted LIP if desired.



(a) (b) (c) (d) (e) (f)

Figure 12: All possible demand complexes with maximal subdivision, and examples of their dual weighted LIPs, giving all such combinatorial types when $A = \{0,1\} \times \{0,1,2\}$.

Example A.3. To illustrate why the condition for indivisible goods to be substitutes is so restrictive, consider three trips: trip A can be made only by bus or train; trip B only by car or train; and trip C only by car or bus. As divisible goods, the three modes of transport are all mutual substitutes. But if the price of *either* bus tickets or train tickets is raised, a consumer might buy a car and use less of *both* forms of public transport, which are therefore locally complements–the car takes the role of good 2 in Fig. 4.[61]

A.3 Proofs and Additional Examples for Section 4

Example A.4 (Failure of aggregate concavity in "hotel rooms" example). Figs. 13a-c show the valuations u^s, u^c, and an aggregate valuation for them, $u^{\{s,c\}}(\mathbf{y}) = \max\{u^s(\mathbf{x}^s) + u^c(\mathbf{x}^c) : \mathbf{x}^s, \mathbf{x}^c \in \{0,1\}^2,\ \mathbf{x}^s + \mathbf{x}^c = \mathbf{y}\}$, respectively. The failure of aggregate concavity at $(1,1)$ is clear from the fact that $\frac{1}{4}(u^{\{s,c\}}(1,0)+u^{\{s,c\}}(0,1)+u^{\{s,c\}}(2,1)+$

[61]Even if all goods are mutual substitutes, there can never be trade-offs between more than two of them across the whole of a facet. One mutual substitute might trade-off against two others at prices where more than one facet meet, if at least one of those facets has weight greater than 1. For example, an agent might switch 2 units of some good A for 1 each of two other goods, B and C, which it treats as indistinguishable, in the intersection of all three weight-2 facets where the agent switches between two of the three goods.

$u^{\{s,c\}}(1,2)) = 60 > 50 = u^{\{s,c\}}(1,1)$. This is illustrated in Fig. 13d, which shows $u^{\{s,c\}}$ together with the cell of its roof that corresponds to the price vector $(30, 20)$; the bundles $(1,0)$, $(0,1)$, $(2,1)$, and $(1,2)$ are all demanded at this price, but $(1,1)$ is not demanded at any price.

x_1			
1	*0*	$u^s(\mathbf{x})$	
40	0	*0*	x_2
40	30	*1*	

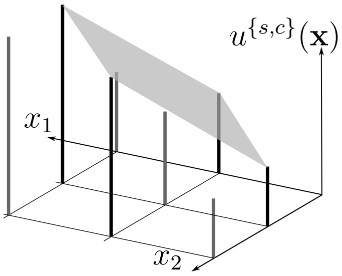
(a) Valuation $u^s(\mathbf{x})$.

x_1			
1	*0*	$u^c(\mathbf{x})$	
0	0	*0*	x_2
50	0	*1*	

(b) Valuation $u^c(\mathbf{x})$

x_1				
2	*1*	*0*	$u^{\{s,c\}}(\mathbf{x})$	
40	40	0	*0*	
90	50	30	*1*	x_2
90	80	30	*2*	

(c) Valuation $u^{\{s,c\}}(\mathbf{x})$.



(d) The $\mathbf{p} = (30, 20)$ cell of the "roof".

Figure 13: Illustration of an aggregate valuation $u^{\{s,c\}}$ which is not concave.

Proof of Fact 4.8. That (2) is equivalent to unimodularity follows from Cassels (1971) Lemma I.1 and Cor. I.3. For (1)$\Leftrightarrow$(2) consider a parallelepiped P whose vertices are $\mathbf{y} + \sum_{j=1}^{s} a_j \mathbf{v}^j$ for $a_j \in \{0, 1\}$. If $\mathbf{z}$ is a non-vertex integer point in P, then $\mathbf{z} - \mathbf{y}$ exhibits the failure of (2). Conversely, if failure of (2) is exhibited by an integer vector $\sum_{j=1}^{s} b_j \mathbf{v}^j$ where b_j are not all integers, then $\mathbf{y} + \sum_{j=1}^{s} a_j \mathbf{v}^j$ exhibits failure of (1), where a_j is the non-integer part of b_j in each case. $\square$

Fact A.5 (see, e.g. Cassels, 1971, Lemma I.2)**.** A set of $s \leq n$ linearly independent vectors in $\mathbb{Z}^n$ are unimodular iff, among the determinants of all the $s \times s$ matrices consisting of s rows of the $n \times s$ matrix whose columns are these s vectors, the greatest common factor is 1.

Lemma A.6. *For a finite set of valuations $\{u^j : j \in J\}$, and $\mathbf{p} \in \mathbb{R}^n$, write $\sigma^j = \operatorname{conv}(D_{u^j}(\mathbf{p}))$ and $\sigma^J = \operatorname{conv}(D_{u^J}(\mathbf{p}))$.*

(1) $\sigma^J = \sum_{j \in J} \sigma^j$ and $\mathrm{K}_{\sigma^J} = \sum_{j \in J} \mathrm{K}_{\sigma^j}$.

(2) If the LIPs intersect at $\mathbf{p}$, they are transverse at $\mathbf{p}$ iff $\mathrm{K}_{\sigma^J} = \bigoplus_{j \in J} \mathrm{K}_{\sigma^j}$.

Proof. (1): by definition $D_{u^J}(\mathbf{p}) = \sum_{j \in J} D_{u^j}(\mathbf{p})$, from which $\sigma^J = \sum_{j \in J} \sigma^j$ follows (see e.g. Cox et al 2005, Section 7.4, Exercise 3). Thus, by Defn. 4.15, $\mathrm{K}_{\sigma^J} = \sum_{j \in J} \mathrm{K}_{\sigma^j}$.

(2): First suppose $r = 2$ and write $k^j = \dim \sigma^j = \dim K_{\sigma^j}$, and $k^J = \dim \sigma^J = \dim K_{\sigma^J}$. By Part (1) it follows that $k^J = k^1 + k^2 - \dim(\mathrm{K}_{\sigma^1} \cap \mathrm{K}_{\sigma^2})$. But also, $\dim(C_{\sigma^1} + C_{\sigma^2}) = \dim C_{\sigma^1} + \dim C_{\sigma^2} - \dim C_{\sigma^J} = (n - k^1) + (n - k^2) - (n - k^J)$. By Defn. 4.11, the intersection is transverse at $\mathbf{p}$ iff this is equal to n, that is, iff $k^1 + k^2 = k^J$. So we conclude that the intersection is transverse at $\mathbf{p}$ iff $\mathrm{K}_{\sigma^1} \cap \mathrm{K}_{\sigma^2} = \{\mathbf{0}\}$, which, together with (1), is the definition of $\mathrm{K}_{\sigma^J} = \mathrm{K}_{\sigma^1} \oplus \mathrm{K}_{\sigma^2}$. The $r \geq 3$ case now follows, as we check for transversality incrementally using the $r = 2$ case (Defn. 4.11). $\square$

Proof of Lemma 4.16 This is a standard consequence of Lemma A.6(2). Every vector $\mathbf{x}$ in $\mathrm{K}_{\sigma^{\{1,2\}}}$ can be written as a sum of a vector $\mathbf{x}^1 \in \mathrm{K}_{\sigma^1}$ and a vector $\mathbf{x}^2 \in \mathrm{K}_{\sigma^2}$. If we also write $\mathbf{x} = \mathbf{x}^{1\prime} + \mathbf{x}^{2\prime}$, where $\mathbf{x}^{j\prime} \in \mathrm{K}_{\sigma^j}$, $j = 1, 2$, then $\mathbf{x}^1 - \mathbf{x}^{1\prime} = \mathbf{x}^2 - \mathbf{x}^{2\prime} \in \mathrm{K}_{\sigma^1} \cap \mathrm{K}_{\sigma^2} = \{\mathbf{0}\}$ and so $\mathbf{x}^1 - \mathbf{x}^{1\prime} = \mathbf{x}^2 - \mathbf{x}^{2\prime} = \mathbf{0}$. $\square$

Example A.7. (**The "hotel rooms" example, and Section 4.2.5's argument.**) It helps intuition to see which parts of the argument of Section 4.2.5 are, and are not, valid for our simple two-goods substitutes, $u^s(x_1, x_2) = \max\{40x_1, 30x_2\}$ (Fig. 5a), and complements, $u^c(x_1, x_2) = \min\{50x_1, 50x_2\}$ (Fig. 5b), valuations, for which the failure of equilibrium was discussed in Section 4.2.2.

Consider the (transverse) intersection price $\mathbf{p} = (30, 20)$, and the bundle $\mathbf{y}$ =$(1, 1) \in \sigma^{\{s,c\}} = \text{conv}\,(D_{u^{\{s,c\}}}(\mathbf{p}))$. So $\sigma^{\{s,c\}}$ is the central square 2-cell in Fig. 6c. In this example it is clear that the corresponding cells, $\sigma^s = \text{conv}(D_{u^s}(\mathbf{p}))$ and $\sigma^c = \text{conv}(D_{u^c}(\mathbf{p}))$, of Figs. 6a and 6b, respectively, are the two diagonal 1-cells. (The general procedure for identifying σ^s and σ^c is to examine the LIP intersection in Fig. 5c, hence to identify the relevant cells of the LIPs in Figs. 5a-b, and so the relevant cells in Figs. 6a-b.)

We have $\mathrm{K}_{\sigma^{\{s,c\}}} = \mathbb{R}^2$, and the edges of $\sigma^{\{s,c\}}$ are $\{(1, -1), (1, 1)\}$ (see Fig. 6c), and in this case the whole set of edges is needed to form a basis of $\mathrm{K}_{\sigma^{\{s,c\}}}$. And (see Figs. 5a, 5b) K_{σ^s} and K_{σ^c} are the sets of scalar multiples of $(1, -1)$ and $(1, 1)$, respectively (and these vectors give a basis for each).

Now say, for example, $\mathbf{x} = (1, 0) \in \sigma^{\{s,c\}}$. We can write $\mathbf{y} - \mathbf{x}$ =$(0, 1)$ as $\lambda_1(1, -1) + \lambda_2(1, 1)$ by choosing $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. So $\mathbf{y} - \mathbf{x} = \mathbf{z}^s + \mathbf{z}^c$ in which $\mathbf{z}^s = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\mathbf{z}^c = \left(\frac{1}{2}, \frac{1}{2}\right)$. (These are *not* integer bundles because unimodularity fails in this example.)

On the other hand, we use $\sigma^{\{s,c\}} = \sigma^s + \sigma^c$ to write $\mathbf{y} = \mathbf{y}^s + \mathbf{y}^c$ with $\mathbf{y}^s \in \sigma^s$ and $\mathbf{y}^c \in \sigma^c$. We can see from Figs. 6a, 6b that the unique way to do this is $\mathbf{y}^s = \mathbf{y}^c = \left(\frac{1}{2}, \frac{1}{2}\right)$. Also, since $\mathbf{x} \in D_{u^{\{s,c\}}}(\mathbf{p})$, we can write $\mathbf{x} = \mathbf{x}^s + \mathbf{x}^c$ in which $\mathbf{x}^s$ and $\mathbf{x}^c$ are integer bundles. In fact, $\mathbf{x}^s$=$(1, 0) \in D_{u^s}(\mathbf{p})$ and $\mathbf{x}^c$=$(0, 0) \in D_{u^c}(\mathbf{p})$, as can be seen by using Fig. 5c to see that the UDR of $\mathbf{x}$ is the region above the price $(30, 20)$, hence the relevant UDRs in Figs. 5a and 5b are $(1, 0)$ and $(0, 0)$, respectively. So $\mathbf{y}^s - \mathbf{x}^s = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\mathbf{y}^c - \mathbf{x}^c = \left(\frac{1}{2}, \frac{1}{2}\right)$.

The intersection is transverse at $\mathbf{p}$ and so, as predicted, $\mathbf{y}^s - \mathbf{x}^s = \mathbf{z}^s$, and $\mathbf{y}^c - \mathbf{x}^c = \mathbf{z}^c$. However, in this example, the set of edges of $\sigma^{\{s,c\}}$, that is, $\{(1, -1), (1, 1)\}$ is *not* unimodular, so $\mathbf{z}^s$ and $\mathbf{z}^c$, and hence also $\mathbf{y}^s$ and $\mathbf{y}^c$, are *not* integer bundles, so the method does *not* demonstrate equilibrium. (Indeed, in this case it demonstrates the failure of equilibrium.)

A.4 Proofs and Additional Examples for Section 5

Example A.8 (**The "hotel rooms" example with weight 2 facets–further discussion of Section 5.1.2**). We can understand "mid-point bundles" such as (1,1) and (1,3) in Fig. 8c as being reached by starting from a vertex bundle, and then changing demand to move one "step" along an edge. These bundles correspond to demand changing part-way along a diagonal edge in Figs. 8a and 8b. Likewise the "central bundle" $(2, 2)$ can be reached by a combination of these part-way diagonal steps, as shown by the dashed grey lines. Meanwhile the remaining four bundles (the black grid points in Fig. 8c) cannot be reached by any such moves, so are not demanded.

As in the discussion of Fig. 6c in Section 4.2.2, this illustrates the relevance of Fact 4.8(2). Unimodularity is the ability to create any vector in the space spanned by the vectors of the demand type as an integer combination of any spanning set of vectors of the demand type. $u^{\{2s,2c\}}$ is not of a unimodular demand type; there is therefore the possibility that some bundles cannot be reached using combinations of the edge vectors of the relevant demand complex cell; and indeed these four bundles are unreachable.

Fig. 8d can be understood similarly to Fig. 8c. Now all bundles can be reached via changes of demand that take the agents part-way along their individual demand complex cells (so are edge vectors of the relevant aggregate demand complex cell; these are again denoted by dashed lines). So equilibrium always exists in this case.

Proof of Prop. 5.2. Necessity of this condition follows from Lemma 2.13.

Conversely, suppose that equilibrium fails for some relevant supply. By Lemma 4.7, there exists a price $\mathbf{p}$ in $\mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$ such that $D_{u^{\{1,2\}}}(\mathbf{p})$ is not discrete-convex. So there exists $\mathbf{x} \in \mathbb{Z}^n$ such that $\mathbf{x} \in \sigma^{\{1,2\}} := \operatorname{conv}(D_{u^{\{1,2\}}}(\mathbf{p}))$, but $\mathbf{x} \notin D_{u^{\{1,2\}}}(\mathbf{p})$. By Lemma 2.8, $D_{u^{\{1,2\}}}(\mathbf{p}')$ is constant for $\mathbf{p}' \in C^{\circ}_{\sigma^{\{1,2\}}}$, and thus both individual demand sets must be constant in $C^{\circ}_{\sigma^{\{1,2\}}}$. So $C_{\sigma^{\{1,2\}}} \subseteq \mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$.

Since the aggregate demand complex is n-dimensional, we may apply Cor. A.1: there exists a 0-cell C_τ of $\mathcal{L}_{u^{\{1,2\}}}$ such that $\sigma \subseteq \tau$ and which also satisfies $\mathbf{x} \notin D_{u^{\{1,2\}}}(\mathbf{p}_\tau)$ for $\mathbf{p}_\tau \in C_\tau$. Moreover, $C_\tau \subseteq C_\sigma$ (by Prop. 2.21(3)) and so $C_\tau \subseteq \mathcal{L}_{u^1} \cap \mathcal{L}_{u^2}$. □

Lemma A.9. *Fix $\mathbf{p} \in \mathbb{R}^n$ and write u^* for the restriction of u to $D_u(\mathbf{p})$. Then there exists $\delta > 0$ such that $D_u(\mathbf{p}') = D_{u^*}(\mathbf{p}')$ for all $\mathbf{p}' \in B_\delta(\mathbf{p}) = \{\mathbf{p}' \in \mathbb{R}^n : \|\mathbf{p}' - \mathbf{p}\| < \delta\}$. In particular the LIPs coincide for all such $\mathbf{p}'$.*

Proof. There exists some $\epsilon > 0$ such that $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} > u(\mathbf{y}) - \mathbf{p} \cdot \mathbf{y} + \epsilon$ for every $\mathbf{x} \in D_u(\mathbf{p})$, $\mathbf{y} \notin D_u(\mathbf{p})$. So if we choose $\delta < \frac{\epsilon}{\|\mathbf{x}\|}$ for all $\mathbf{x} \in D_u(\mathbf{p})$ then it is easy to show that $u(\mathbf{y}) - \mathbf{p}' \cdot \mathbf{y} < u(\mathbf{x}) - \mathbf{p}' \cdot \mathbf{x}$ for all $\mathbf{p}' \in B_\delta(\mathbf{p})$, $\mathbf{x} \in D_u(\mathbf{p})$, $\mathbf{y} \notin D_u(\mathbf{p})$. Thus, at prices $\mathbf{p}' \in B_\delta(\mathbf{p})$, only bundles in $D_u(\mathbf{p})$ might be demanded; $D_u(\mathbf{p}') = D_{u^*}(\mathbf{p}')$. □

Lemma A.10. *Suppose $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ have an intersection 0-cell C at $\mathbf{p}$. Suppose $\mathbf{v} \in \mathbb{R}^n$ is such that $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ intersect transversally for sufficiently small $\epsilon > 0$, and write $\mathcal{C}$ for the set of their intersection 0-cells emerging from C. Write $\mathbf{x}^1$, $\mathbf{x}^2$ for the bundles such that $\{\mathbf{x}^1\} = D_{u^1}(\mathbf{p} + \epsilon\mathbf{v})$, $\{\mathbf{x}^2\} = D_{u^2}(\mathbf{p} - \epsilon\mathbf{v})$. Then the demand complex cell $\sigma_C \in \Sigma_{u^{\{1,2\}}}$ dual to C is equal to $(\{\mathbf{x}^2\} + \sigma^1) \cup (\{\mathbf{x}^1\} + \sigma^2) \cup \bigcup\{\sigma_{C'} : C' \in \mathcal{C}\}$, where $\sigma^j = \operatorname{conv}(D_{u^j}(\mathbf{p}))$ and $\sigma_{C'} \in \Sigma_{u^{\{1,2_\epsilon\}}}$ is dual to $C' \in \mathcal{C}$.*

Proof. Use the notation of Lemma A.9; using that lemma, choose $\delta > 0$ sufficiently small so that, for all $\mathbf{p}' \in B_\delta(\mathbf{p})$, *both* $D_{u^1}(\mathbf{p}') = D_{u^{1*}}(\mathbf{p}')$ *and* $D_{u^2}(\mathbf{p}') = D_{u^{2*}}(\mathbf{p}')$.

We write u^{2*_ϵ} for the valuation given by $u^{2*_\epsilon}(\mathbf{x}) = u^{2*}(\mathbf{x}) + \epsilon\mathbf{v} \cdot \mathbf{x}$. Since $B_\delta(\mathbf{p})$ is topologically open, and since $\mathcal{L}_{u^{2_\epsilon}} = \{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$, we can choose ϵ sufficiently small that $D_{u^{2_\epsilon}}(\mathbf{p}') = D_{u^{2*_\epsilon}}(\mathbf{p}')$ for all $\mathbf{p}' \in B_\delta(\mathbf{p})$. Moreover, by Defn. 5.8, emerging 0-cells get arbitrarily close to C for arbitrarily small ϵ. So we can choose ϵ sufficiently small that all intersection 0-cells for $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^{2_\epsilon}}$, emerging from C, are contained in $B_\delta(\mathbf{p})$. Now, it is sufficient to prove the result for $\mathcal{L}_{u^{1*}}$, $\mathcal{L}_{u^{2*}}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^{2*}} = \mathcal{L}_{u^{2*_\epsilon}}$.

The cell σ_C is, by definition, the convex hull of the domain of $u^{\{1*,2*\}}$, and hence also convex hull of the domain of $u^{\{1*,2*_\epsilon\}}$. So σ_C is equal to the union of the top-dimensional cells in the demand complex $\Sigma_{u^{\{1*,2*_\epsilon\}}}$, i.e., the demand complex cells dual to 0-cells of $\mathcal{L}_{u^{\{1*,2*_\epsilon\}}}$. But the 0-cells of $\mathcal{L}_{u^{\{1*,2*_\epsilon\}}}$ are: those in $\mathcal{C}$; and, if they exist, 0-cells at $\mathbf{p}$ of $\mathcal{L}_{u^{1*}}$, $\mathcal{L}_{u^{2*}}$. Take duals, and in the latter cases account for the demand of the other agent. Since in every case $\{\mathbf{x}^1\} + \sigma^2 \subseteq \sigma_C$ and $\{\mathbf{x}^2\} + \sigma^1 \subseteq \sigma_C$, we obtain the result. □

Proof of Lemma 5.9. Write σ^1, σ^2 and $\sigma^{\{1,2\}}$, respectively, for the demand complex cells at $\mathbf{p}$ of valuations u^1, u^2 and $u^{\{1,2\}}$. For any $\mathbf{v}$ and ϵ, if there exist no intersection

0-cells emerging from C then, by Lemma A.10, $\mathrm{vol}_n(\sigma^1) + \mathrm{vol}_n(\sigma^2) \geq \mathrm{vol}_n(\sigma^{\{1,2\}})$. But this is impossible since $\sigma^{\{1,2\}} = \sigma^1 + \sigma^2$, since both σ^1 and σ^2 contain more than one element, and since $\dim \sigma^{\{1,2\}} = n$, which implies $\mathrm{vol}_n(\sigma^{\{1,2\}}) \neq 0$. □

Fact A.11. The quantity we have defined as $\Gamma^n_k(A^1, A^2)$ is the n-dimensional mixed volume of k copies of $\mathrm{conv}(A^1)$ with $(n-k)$ copies of $\mathrm{conv}(A^2)$. In particular:

(1) $\Gamma^n_k(A^1, A^2) \in \mathbb{Z}_{\geq 0}$ for finite $A^1, A^2 \subsetneq \mathbb{Z}^n$.
(2) $\Gamma^n_k(A, A) = n!\mathrm{vol}_n \,\mathrm{conv}(A)$ for $k = 0, \ldots, n$.
(3) If $A^j = \{\mathbf{x} \in \mathbb{Z}^n_{\geq 0} : \sum_i x_i \leq d_j\}$ for $j = 1, 2$ then $\Gamma^n_k(A^1, A^2) = d_1^k d_2^{n-k}$ (and so $\Gamma^2(A^1, A^2) = d_1 d_2$).
(4) $\Gamma^n(A^1, A^2) = 0$ if $\dim \mathrm{conv}(A^1 + A^2) < n$.

Proof. Cox et al. (2005, Thm. 7.4.12.d) shows that $\Gamma^n_k(A^1, A^2)$ is the mixed volume described. (1) is now Cox et al. (2005, Exercise 7.9.a). (2) is Cox et al. (2005, Exercise 7.7.b). (3) is an elementary calculation. (4) is clear by Defn. 5.11. □

Facts A.12. Write K_Λ for the *linear span* in $\mathbb{R}^n$ of a lattice $\Lambda \subseteq \mathbb{Z}^n$, so K_Λ is the set of all linear combinations of finite subsets of Λ.

(1) K_Λ is the minimal vector subspace of $\mathbb{R}^n$ containing Λ.
(2) If $\Lambda' \subseteq \Lambda$ is a sublattice, then $\mathrm{K}_{\Lambda'} = \mathrm{K}_\Lambda$ iff $\mathrm{rank}(\Lambda') = \mathrm{rank}(\Lambda) = \dim \mathrm{K}_\Lambda$.
(3) If $\Lambda', \Lambda \subseteq \mathbb{Z}^n$ are lattices then $\Lambda' + \Lambda$ is a lattice. If additionally $\mathrm{K}_{\Lambda'} \cap \mathrm{K}_\Lambda = \{\mathbf{0}\}$ then a basis for Λ' and a basis for Λ together give a basis for $\Lambda' + \Lambda$.

Proof. (1) is clear. An integer basis for Λ provides a set spanning K_Λ, and these are linearly independent by Cassels (1971, Thm. III.VI; our definition of lattices is in line with the characterisation given there), so (2) follows. (3) is also easy to see. □

Proof of Lemma 5.14. From Lemma A.6(1), $\sum_{j \in K} \mathrm{K}_{\sigma^j} = \mathrm{K}_{\sigma^J}$. It follows that $\mathrm{K}_{\sigma^j} \subseteq \mathrm{K}_{\sigma^J}$, and so $\Lambda_{\sigma^j} \subseteq \Lambda_{\sigma^J}$, for all $j \in J$. Thus, as Λ_{σ^J} is additively closed, $\sum_{j \in J} \Lambda_{\sigma^j} \subseteq \Lambda_{\sigma^J}$. Moreover $\sum_{j \in J} \Lambda_{\sigma^j}$ is a sublattice by Fact A.12(3).

The linear span of Λ_{σ^J} is K_{σ^J}. The linear span of $\sum_{j \in J} \Lambda_{\sigma^j}$ contains K_{σ^j} for all $j \in J$, and so it contains their sum; on the other hand $\sum_{j \in J} \Lambda_{\sigma^j} \subseteq \sum_{j \in J} \mathrm{K}_{\sigma^j}$ and the latter is linear; so $\sum_{j \in J} \mathrm{K}_{\sigma^j}$ is the linear span of $\sum_{j \in J} \Lambda_{\sigma^j}$. Since, again, $\sum_{j \in J} \mathrm{K}_{\sigma^j} = \mathrm{K}_{\sigma^J}$, the result follows by Fact A.12(2). □

Fact A.13. $[\Lambda : \Lambda']$ is the number of disjoint "cosets" $\{\mathbf{v}\} + \Lambda'$ where $\mathbf{v} \in \Lambda$.

Proof of Facts 5.15 and A.13. Cassels (1971) assumes that all lattices $\Lambda \subseteq \mathbb{Z}^n$ have rank n. To adapt his results to our conventions, first fix a $k \times n$ matrix G_Λ such that $G_\Lambda \Lambda = \mathbb{Z}^k$. Then $G_\Lambda \Lambda'$ is a rank-k sublattice of $\mathbb{Z}^k$. From our definitions it is clear that $[G_\Lambda \Lambda : G_\Lambda \Lambda'] = [\Lambda : \Lambda']$. Now Cassels (1971) Lemma I.1 gives Fact A.13. And Fact 5.15(1) follows: if $\mathbf{v}, \mathbf{w} \in \Delta_{\Lambda'} \cap \Lambda$ but $\{\mathbf{v}\} + \Lambda' = \{\mathbf{w}\} + \Lambda'$ then $\mathbf{v} - \mathbf{w} \in \Lambda'$, so $\mathbf{v} - \mathbf{w} = \sum_i \alpha_i \mathbf{v}'^i$ where $\mathbf{v}'^i$ are our basis for Λ', and $\alpha_i \in \mathbb{Z}$; then $\mathbf{w} \in \Delta_{\Lambda'}$ and $\mathbf{v} = \mathbf{w} + (\mathbf{v} - \mathbf{w}) \in \Delta_{\Lambda'}$ are consistent only if $\mathbf{v} = \mathbf{w}$; the converse is argued similarly. Fact 5.15(2) follows immediately from Fact A.13. Fact 5.15(3) is given by Cassels (1971, p. 69).Fact 5.15(4) follows from (2) and Fact 4.8(1). □

Proof of Thm. 5.16. Part (1) is very similar to the proof of "sufficiency" for Thm. 4.3, given in Section 4.2.5. Let $\mathbf{x} \in \sigma^J \cap \mathbb{Z}^n$. By Lemma A.6, $\sigma^J = \sum_{j \in J} \sigma^j$ and so

$\mathbf{x} = \sum_{j\in J} \mathbf{x}^j$, where $\mathbf{x}^j \in \sigma^j$ for $j \in J$. We wish to show that each $\mathbf{x}^j \in \mathbb{Z}^n$; then $\mathbf{x}^j \in D_{u^j}(\mathbf{p})$ by concavity of u^j (Lemma 2.13) and so $\mathbf{x} \in D_{u^J}(\mathbf{p})$.

Fix $\mathbf{y} \in D_{u^J}(\mathbf{p})$, so $\mathbf{y} = \sum_{j\in J} \mathbf{y}^j$ where $\mathbf{y}^j \in D_{u^j}(\mathbf{p}) \subsetneq \mathbb{Z}^n$ for $j \in J$. Then $\mathbf{y} - \mathbf{x} \in \Lambda_{\sigma^J}$. But $\Lambda_{\sigma^J} = \sum_{j\in J} \Lambda_{\sigma^j}$ (by Fact 5.15(2)) and so $\mathbf{y} - \mathbf{x} = \sum_{j\in J} \mathbf{z}^j$, where $\mathbf{z}^j \in \Lambda_{\sigma^j} \subseteq \mathbb{Z}^n$. But now $\sum_{j\in J} \mathbf{z}^j = \sum_{j\in J}(\mathbf{y}^j - \mathbf{x}^j)$. By Lemma 4.16, $\mathbf{y}^j - \mathbf{x}^j = \mathbf{z}^j$ for $j \in J$. Thus $\mathbf{x}^j = \mathbf{y}^j - \mathbf{z}^j \in \mathbb{Z}^n$ for $j \in J$, as required.

We prove Part (2) in the case when $\dim \sigma^{j_0} = 2$. The case in which $\dim \sigma^{j_0} = 1$ may be seen from the following argument by ignoring the role of σ^{j_0}. So suppose $\dim \sigma^{j_0} = 2$.

Assume that $\mathbf{0} \in \sigma^j$ for $j \in J$ (otherwise the following arguments are simply augmented by a fixed shift). If $\dim \sigma^j = 0$ for any $j \in J$ then $\Lambda_{\sigma^j} = \{\mathbf{0}\}$ and inclusion of σ^j has no effect on σ^J. So we assume $\dim \sigma^j = 1$ for all $j \in J \setminus \{j_0\}$.

For $j \in J \setminus \{j_0\}$, fix a minimal integer non-zero vector $\mathbf{v}^j \in \sigma^j$. In each case this vector then forms a basis for the corresponding lattice Λ_{σ^j}.

We also need a basis for $\Lambda_{\sigma^{j_0}}$ consisting of vectors $\mathbf{v}^0, \mathbf{v}^1$ contained inside σ^{j_0}. Start by taking $\mathbf{w}^0$,$\mathbf{w}^1 \in \sigma^{j_0}$ which are linearly independent integer vectors. If these are a basis for $\Lambda_{\sigma^{j_0}}$, we are done. If not, they span a sublattice Λ'_1 of $\Lambda_{\sigma^{j_0}}$, such that $[\Lambda_{\sigma^{j_0}} : \Lambda'_1] > 1$, and so there must exist $\mathbf{w} \in \Lambda_{\sigma^{j_0}}$ which is a non-vertex point of $\Delta_{\Lambda'_1}$. Then $\mathbf{w} = \alpha^0\mathbf{w}^0 + \alpha^1\mathbf{w}^1$ with $\alpha^0, \alpha^1 \in [0, 1)$. If $\alpha^0 + \alpha^1 \leq 1$ then we fix $\mathbf{w}^2 := \mathbf{w}$; if $\alpha^0 + \alpha^1 > 1$ then let $\mathbf{w}^2 = \mathbf{w}^0 + \mathbf{w}^1 - \mathbf{w}$. In either case now $\mathbf{w}^2 = \beta^0\mathbf{w}^0 + \beta^1\mathbf{w}^1$ with $\beta^0 + \beta^1 \leq 1$. As σ^{j_0} is convex we conclude that $\mathbf{w}^2 \in \sigma^{j_0}$.

Recalling by definition that $\mathbf{w} \notin \Lambda'_1$, we know $\mathbf{w}^2$ is distinct from $\mathbf{w}^0, \mathbf{w}^1, \mathbf{0}$. So $\mathbf{w}^2$ is a non-vertex point of the convex hull Δ^0 of $\mathbf{0}, \mathbf{w}^0, \mathbf{w}^1$. Hence the convex hull Δ^1 of $\mathbf{0}, \mathbf{w}^1, \mathbf{w}^2$ has strictly smaller area than Δ^0. Moreover, the parallelepipeds spanned by $\mathbf{w}^0, \mathbf{w}^1$ and by $\mathbf{w}^1, \mathbf{w}^2$ have areas equal to twice the areas of Δ^0, Δ^1, respectively. So, if Λ'_2 is the sublattice of $\Lambda_{\sigma^{j_0}}$ spanned by $\mathbf{w}^1, \mathbf{w}^2$, then $[\Lambda_{\sigma^{j_0}} : \Lambda'_2] < [\Lambda_{\sigma^{j_0}} : \Lambda'_1]$.

As all subgroup indices are positive-integer-valued, after a finite number of repetitions of this process, the subgroup index is 1, and hence (by Fact 5.15(2)) we have obtained vectors in σ^{j_0} which are a basis of $\Lambda_{\sigma^{j_0}}$, as required. Label these vectors $\mathbf{v}^{j_0}$ and $\mathbf{v}^{j_1}$ where $j_1 \notin J$ and write $J' = J \cup \{j_1\}$.

Now, by Fact 5.15(2), there exists $\mathbf{x} \in \Lambda^J_\sigma$, $\mathbf{x} \notin \sum_{j\in J} \Lambda_{\sigma^j}$. And by Fact A.12(3), our identified vectors $\{\mathbf{v}^j : j \in J'\}$ are an integer basis for $\sum_{j\in J} \Lambda_{\sigma^j}$. By Lemma 5.14 and Fact A.12(2), they are thus a vector space basis for K_{σ^J}, so we can write $\mathbf{x} = \sum_{j\in J'} \alpha^j\mathbf{v}^j$. Moreover, since subtracting integer multiples of the $\mathbf{v}^j$ from $\mathbf{x}$ yields a new element of Λ_{σ^J}, we can assume that $\alpha^j \in [0, 1)$ for $j = J'$. Additionally, we can assume that $\alpha^{j_0} + \alpha^{j_1} \leq 1$: if $\alpha^{j_0} + \alpha^{j_1} > 1$ then replace $\mathbf{x}$ with $\sum_{j\in J'} \mathbf{v}^j - \mathbf{x} \in \Lambda_{\sigma^J}$. Now $\alpha^{j_0}\mathbf{v}^{j+0} + \alpha^{j_1}\mathbf{v}^{j_1} \in \sigma^{j_0}$ and, for all other $j \in J$, also $\alpha^j\mathbf{v}^j \in \sigma^j$. So, $\mathbf{x} \in \sum_{j\in J} \sigma^j = \sigma^J = \mathrm{conv}\,(D_{u^J}(\mathbf{p}))$. Moreover, $\mathbf{x} \in \Lambda_{\sigma^J} \subseteq \mathbb{Z}^n$. But, by assumption, $\mathbf{x} \notin \sum_{j\in J} \Lambda_{\sigma^j}$, and so $\mathbf{x} \notin \sum_{j\in J} D_{u^j}(\mathbf{p}) = D_{u^J}(\mathbf{p})$. □

Example A.14 (Equilibrium demonstrated by the Subgroup Indices Theorem but not the Unimodularity Theorem). Suppose $n = 4$ and that $D_{u^1}(\mathbf{p}) = \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^1 + \mathbf{e}^2, 2\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 + 2\mathbf{e}^2\}$ while $D_{u^2}(\mathbf{p}) = \{\mathbf{e}^3, \mathbf{e}^4, \mathbf{e}^3 + \mathbf{e}^4, 2\mathbf{e}^3 + \mathbf{e}^4, \mathbf{e}^3 + 2\mathbf{e}^4\}$ (Prop. 2.21 and Thm. 2.10 show how to construct u^1, u^2 with these properties). Write $\sigma^j = \mathrm{conv}(D_{u^j}(\mathbf{p}))$ for $j = 1, 2, \{1, 2\}$. Then $\Lambda_{\sigma^1} = \{\mathbf{v} \in \mathbb{Z}^4 : v_1 = v_2 = 0\}$ and $\Lambda_{\sigma^2} = \{\mathbf{v} \in \mathbb{Z}^4 : v_3 = v_4 = 0\}$, while $\Lambda_{\sigma^{\{1,2\}}} = \mathbb{Z}^4$, so $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] = 1$, and $D_{u^{\{1,2\}}}(\mathbf{p})$ is discrete-convex (as may also be checked directly).

However, the facets to Agent 1's demand at $\mathbf{p}$ have normal vectors $(1,1,0,0)$ and $(1,-1,0,0)$, while the facets normals for Agent 2 are $(0,0,1,1)$ and $(0,0,1,-1)$. This set is *not* unimodular, and so equilibrium is not assured by application of Thm. 4.3.

Example A.15. Let $n=4$. Agent 1 has valuation $u^1(0,0,0,0)=0$, $u^1(1,1,0,0)=6$, $u^1(0,0,1,1)=6$. So for prices in the LIP 2-cell $\{\mathbf{p}\in\mathbb{R}^4 : p_1+p_2=6,\ p_3+p_4=6\}$, Agent 1 is indifferent between these three bundles; the dual demand complex cell is $\sigma^1=\operatorname{conv}((0,0,0,0),(1,1,0,0),(0,0,1,1))$.

Agent 2 has valuation $u^2(0,0,0,0)=0$, $u^2(0,1,1,0)=9$, $u^2(4,0,0,1)=6$. So for prices in the LIP 2-cell $\{\mathbf{p}\in\mathbb{R}^4 : p_2+p_3=9,\ 4p_1+p_4=6\}$, Agent 2 is indifferent between these three bundles; the dual demand complex cell is $\sigma^2=\operatorname{conv}((0,0,0,0),(0,1,1,0),(4,0,0,1))$.

These two individual LIP 2-cells intersect at $\mathbf{p}=(1,5,4,2)$. At this price, the individual demand complex cells are σ^1 and σ^2 as above. The aggregate demand complex cell $\sigma^{\{1,2\}}$ is 4-dimensional and so $\Lambda_{\sigma^{\{1,2\}}}=\mathbb{Z}^4$. Meanwhile Λ_{σ^1} and Λ_{σ^2} are rank-2 lattices, and we check that the non-zero vectors we already know in each lattice do give a basis in each case, by checking that the sets $\{(1,1,0,0),(0,0,1,1)\}$ and $\{(0,1,1,0),(4,0,0,1)\}$ are unimodular (use Fact A.5). Thus the union of these sets is a basis for $\Lambda_{\sigma^1}+\Lambda_{\sigma^2}$ (use Fact A.12(3)). But this union is *not* unimodular, and calculating its determinant tells us (by Fact 5.15(3)) that $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1}+\Lambda_{\sigma^2}]=3$.

So, by Fact 5.15(1), we now know that there are exactly 2 non-vertex integer points to a fundamental parallelepiped of $\Lambda_{\sigma^1}+\Lambda_{\sigma^2}$. In terms of the basis vectors, these are:

$$\frac{2}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\frac{2}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix}=\begin{pmatrix}2\\1\\1\\1\end{pmatrix}\tag{1}$$

and

$$\frac{1}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}+\frac{2}{3}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+\frac{2}{3}\begin{pmatrix}4\\0\\0\\1\end{pmatrix}=\begin{pmatrix}3\\1\\1\\1\end{pmatrix}\tag{2}$$

These expressions show clearly that $(2,1,1,1)$ can be decomposed to give a part in σ^2 and a part *not* in σ^1, whereas $(3,1,1,1)$ can be decomposed to give a part in σ^1 and a part *not* in σ^2. Moreover, by linear independence of this set of four vectors, these are the *only* possible decompositions into sums of bundles in the affine spans of σ^1,σ^2. So neither is in $\sigma^1+\sigma^2=\sigma^{\{1,2\}}=\operatorname{conv} D_{u^{\{1,2\}}}(1,5,4,2)$. So the only integer vectors in $\operatorname{conv} D_{u^{\{1,2\}}}(1,5,4,2)$ are in fact in $D_{u^{\{1,2\}}}(1,5,4,2)$ itself: it is discrete-convex.

Examples A.16–A.18 (Modifications of Ex. A.15) In each of these examples we specify a demand complex containing a single maximal cell σ^j, for each of our two agents $j=1,2$. In each case a dual LIP is easy to find, and so (Thm. 2.15) there exists a concave valuation with these properties.[62] The analysis then proceeds as in Ex. A.15.

[62] In these examples it is not hard to find u^1 and u^2 once $\mathbf{p}$ has been chosen. For example, for Ex. A.16, let $\mathbf{p}=(1,5,4,2)$. Then $u^1(0,0,0,0)=0$, $u^1(1,1,0,0)=6$, $u^1(0,1,1,0)=9$ and $u^2(0,0,0,0)=0$, $u^2(0,0,1,1)=6$, $u^2(4,0,0,1)=6$ have the required properties.

Example A.16. Swap a pair of the bundles: $\sigma^1 = \mathrm{conv}((0,0,0,0),(1,1,0,0),(0,1,1,0))$ and let $\sigma^2 = \mathrm{conv}((0,0,0,0),(0,0,1,1),(4,0,0,1))$. In this case the decompositions (1) and (2) of Ex. A.15 show that both $(2,1,1,1)$ and $(3,1,1,1)$ *are* in $\sigma^1 + \sigma^2$. Thus they are in $\mathrm{conv}\,(D_{u^{\{1,2\}}}(\mathbf{p}))$ but not in $D_{u^{\{1,2\}}}(\mathbf{p})$: this set is in this case *not* discrete-convex.

Example A.17. Let $\sigma^1 = \mathrm{conv}((0,0,0,0),(1,1,0,0),(0,1,1,0),(0,0,1,1))$ while $\sigma^2 = ((0,0,0,0),(4,0,0,1))$. This time $\dim \sigma^1 = 3$ and $\dim \sigma^2 = 1$, and expressions (1) and (2) show that in neither case do these vectors decompose to give a part in σ^1, and so cannot be in $\sigma^1 + \sigma^2 = \sigma^{\{1,2\}}$. So in this case, $D_{u^{\{1,2\}}}(\mathbf{p})$ *is* discrete-convex.

Example A.18. Let σ^1 be as in Ex. A.17, but let $\sigma^2 = \mathrm{conv}((0,0,0,0),(4,1,1,1))$. By the same techniques as in Ex. A.15, see that $[\Lambda_{\sigma^{\{1,2\}}} : \Lambda_{\sigma^1} + \Lambda_{\sigma^2}] = 3$. A non-vertex integer point in the fundamental parallelepiped of $\Lambda_{\sigma^1} + \Lambda_{\sigma^2}$ is given by

$$\frac{1}{3}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}4\\1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\\1\end{pmatrix}$$

This point *does* lie in $\sigma^1 + \sigma^2$, and hence does demonstrate failure of discrete-convexity.

Lemma A.19. *$\hat{m}(C) \leq \mathrm{mult}(C)$ for an intersection 0-cell C, with equality holding iff there exists a small translation such that the subgroup indices are 1 at all emerging intersection 0-cells.*

Proof. If the intersection is transverse at C then $\mathrm{mult}(C)$ is the product of $\hat{m}(C)$ with a subgroup index (Defns. 5.7 and 5.18). Moreover, $\hat{m}(C) > 0$ by definition, and any subgroup index is at least 1 (by Fact 5.15(1)), so the result follows. The non-transverse case is similar; here $\hat{m}(C) > 0$ because emerging 0-cells always exist (Lemma 5.9). □

Proof of Thm. 5.12. It is clear from Thm. 5.19(1) and Lemma A.19 that $\Gamma^n(A^1, A^2)$ is an upper bound for intersection 0-cells counted with naïve multiplicities, and that the count equals this bound iff $\hat{m}(C) = \mathrm{mult}(C)$ for every intersection 0-cell C.

Suppose that equilibrium does not exist for all relevant supplies. By Prop. 5.2 there exists an intersection 0-cell C at price $\mathbf{p}$, and a bundle $\mathbf{y} \in \mathbb{Z}^n$, such that $\mathbf{y} \notin D_{u^{\{1,2\}}}(\mathbf{p})$, but $\mathbf{y} \in \sigma^{\{1,2\}} := \mathrm{conv}(D_{u^{\{1,2\}}}(\mathbf{p}))$. Let $\mathbf{v} \in \mathbb{R}^n$ and small $\epsilon > 0$ be such that $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ intersect transversally and the sum of the naïve multiplicities of intersection 0-cells emerging from C is equal to $\hat{m}(C)$. By Lemma A.10, there exists an intersection 0-cell C' at $\mathbf{p}'$ for $\mathcal{L}_{u^1}$ and $\{\epsilon\mathbf{v}\} + \mathcal{L}_{u^2}$ emerging from C, such that $\mathbf{y} \in \mathrm{conv}(D_{u^{\{1,2_\epsilon\}}}(\mathbf{p}'))$ (where $u^{2_\epsilon}(\mathbf{x}) = u^2(\mathbf{x}) + \mathbf{p}\cdot\mathbf{x}$); the fact $\mathbf{y} \notin D_{u^{\{1,2\}}}(\mathbf{p})$ rules out the other cases. By Prop. 4.13 we know $\mathbf{y} \notin D_{u^{\{1,2_\epsilon\}}}(\mathbf{p}')$. By Theorem 5.16(1) we conclude that the subgroup index for u^1, u^{2_ϵ} and $u^{\{1,2_\epsilon\}}$ at $\mathbf{p}'$ is greater than 1. By Lemma A.19, $\hat{m}(C) < \mathrm{mult}(C)$, so as argued above, the count with naïve multiplicities is strictly below $\Gamma^n(A^1, A^2)$.

"Necessity" with transverse intersection and $n \leq 3$ is presented in the text. □

Example A.20 (A "fragile" equilibrium). Consider two identical agents whose demand sets at price $(2,2)$ are the bundles $(0,0),(1,2),(2,1)$ and $(1,1)$. (For example, $u : \{0,1,2\}^2 \to \mathbb{R}$ and $u(x,0) = x; u(0,y) = y; u(1,1) = 4; u(1,2) = u(2,1) = 6; u(2,2) = 7$). Then the bundle $(2,2)$ is in the aggregate demand set at this price:

we assign bundle $(1,1)$ to both agents. But observe that bundle $(1,1)$ is an interior point of each agent's demand complex cell with the vertices $(0,0),(1,2)$, and $(2,1)$. So if we make any small translation to either agent's valuation, then equilibrium fails: for any prices close to $(2,2)$ the perturbed agent's demand must be some subset of these vertices, and it is easy to see that then $(2,2)$ cannot be an aggregate demand.

Example A.21. In both Exs. A.15 and A.16, we can calculate $\Gamma^4(A^1,A^2)$ to be 3 (indeed $\Gamma^4_2(A^1,A^2)=3$; and $\Gamma^4_k(A^1,A^2)=0$ for $k\neq 2$). The weights of the individual's 2-cells that meet at $(1,5,4,2)$ are both 1. This illustrates again that the condition of Thm. 5.12 is sufficient, but *not* necessary, for existence of equilibrium when $n\geq 4$.

A.5 Proofs and Additional Examples for Section 6

Example A.22. (**Circular Ones Model, cf. Bartholdi et al. (1980)**). Consider "complements" consumers, each of whom is only interested in a single, specific, pair of goods, such that these pairs form a cycle. Thus there are n kinds of consumers and n goods, and we can number both goods and consumers $1,\ldots,n$, such that every consumer of kind $i<n$ demands goods i and $i+1$, which it sees as perfect complements, while consumers of kind n demand goods n and 1. It is easy to check that:

$$\det\begin{pmatrix} 1 & 0 & 0 & & & 0 & 1 \\ 1 & 1 & 0 & & & 0 & 0 \\ 0 & 1 & 1 & & & 0 & 0 \\ \cdot & 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & 1 & 0 \\ 0 & 0 & 0 & & & 1 & 1 \end{pmatrix} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

So if n is odd, our Unimodularity Theorem tells us equilibrium does *not* always exist. Furthermore, the proof of Prop. 4.10 showed how to easily construct an explicit example of equilibrium failure for any non-unimodular demand type. Here we simply select a single agent of each kind, each of which values its desired pair at v, so that they are all indifferent between purchase and no purchase (and hence their facets all intersect) if every good's price is $v/2$.[63]

If n is even, the columns of this matrix are not linearly independent. However, if we exclude the i^{th} column, for any i, the remaining $n-1$ columns are then linearly independent, and can trivially be extended to n linearly independent vectors with determinant 1 by adding the column $\mathbf{e}^i$. So using Thm. 4.3, equilibrium *always* exists if n is even,

[63]So equilibrium fails if aggregate supply is exactly 1 unit of each good (the "middle of the parallelepiped") since the minimum and maximum aggregate demands are zero, and 2 units of each good, respectively, at this price. (It is easy to check failure of equilibrium for $x_i=1$, for all i, by contradiction. At least one good, w.l.o.g. good 1, would not be part of a pair being allocated together. So good 1 has value 0 to whoever receives it, hence $p_1\leq 0$. Therefore $p_2\geq v$, since otherwise consumer 1 would demand both goods 1 and 2. Therefore $p_3\leq 0$, since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore $p_4\geq v$, etc., so $p_j\leq 0$ if j is odd. But consumer n then wishes to buy goods n and 1, which is a contradiction.)

since the valuations are, trivially, concave.[64,65]

Proof of Prop. 6.1. (1): by definition, $\mathbf{x} \in D_u(\mathbf{p})$ if $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \le u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}' \in A$, with equality iff $\mathbf{x}' \in D_u(\mathbf{p})$ also. For any invertible matrix G, we may re-write $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') = \mathbf{p}^T G G^{-1}(\mathbf{x} - \mathbf{x}') = (G^T\mathbf{p})^T(G^{-1}\mathbf{x} - G^{-1}\mathbf{x}')$. If G is additionally unimodular, then $G^{-1}\mathbf{x}$ and $G^{-1}\mathbf{x}' \in \mathbb{Z}^n$. If we write $\mathbf{y} = G^{-1}\mathbf{x}$ and $\mathbf{y}' = G^{-1}\mathbf{x}'$ then $(G^T\mathbf{p})^T(\mathbf{y} - \mathbf{y}') \le G^*u(\mathbf{y}) - G^*u(\mathbf{y}')$ holds iff $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \le u(\mathbf{x}) - u(\mathbf{x}')$.

(2): since $\mathcal{L}_u = \{\mathbf{p} \in \mathbb{R}^n : |D_u(\mathbf{p})| > 1\}$, this follows from (1).

(3): if F is a facet of $\mathcal{L}_u$, then by (2), $G^T F = \{G^T\mathbf{p} : \mathbf{p} \in F\}$ is a facet of $\mathcal{L}_{G^*u}$. If $\mathbf{v}$ is normal to F, then $\mathbf{p}^T\mathbf{v}$ is constant for $\mathbf{p} \in F$, so $\mathbf{p}^T G G^{-1}\mathbf{v} = (G^T\mathbf{p})^T G^{-1}\mathbf{v}$ is constant for $G^T\mathbf{p} \in G^T F$. So $G^{-1}\mathbf{v}$ is a demand type vector for G^*u. As G has an integer inverse, the converse is also true. □

Example A.23. To illustrate Prop. 6.1, consider the example of Fig. 5. Create a new good, 3, from two units of good 1 plus one unit of good 2, and consider the economy in which the goods traded are 1 and 3. Note that we can recreate one unit of good 2 by buying one unit of good 3 and selling two units of good 1, and we can convert any bundle expressed in terms of goods 1 and 2 (as a column vector) to a bundle of goods 1 and 3 by pre-multiplying by $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$–this matrix plays the role of G^{-1}.

Observe that the "substitutes" agent of the original economy (who bought either $(1, 0)$ or $(0, 1)$ at price $(30, 20)$) corresponds to an agent in the new economy who would "buy" either $(1, 0)$ or $(-2, 1)$. We can interpret this as an agent with an endowment of -2 units of good 1 (a contract to sell), and who buys either three units of good 1 or one unit of good 3. Thus this agent treats goods 1 and 3 as 3:1 substitutes. Similarly, the "complements" agent of the original economy (who bought neither or both of goods 1 and 2) corresponds to an agent in the new economy with an endowment of -1 unit of good 1, who buys one unit of either good 1 or good 3 (so is indifferent between bundles $(0, 0)$ and $(-1, 1)$) that is, an agent who treats goods 1 and 3 as 1:1 substitutes. So this is a pure substitutes economy in which equilibrium fails.

Since the demand type of the original economy contained the columns of $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, which have determinant -2, the demand type in the new economy contains the columns of $\begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$, that is, $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, which also have determinant -2.

Proof of Prop. 6.2. Suppose there is always an equilibrium for every finite set of agents with concave valuations of type $G^{-1}\mathcal{D}$, and any relevant supply.

Let $\{u^j : j \in J\}$ be finitely many concave valuations of type $\mathcal{D}$ and let $\mathbf{x}$ be a relevant supply bundle. Then, by Prop. 6.1(3) the valuations $\{G^*u^j : j \in J\}$ are of demand type $G^{-1}\mathcal{D}$. It is obvious they are also all concave. By definition of pullback, $\mathbf{y} := G^{-1}\mathbf{x}$ is in the convex hull of the domain of their aggregate valuation, i.e. is a

[64] For example, the aggregate demand of 1 unit of each good is supported by price $v/2$ for every good, when there is exactly one consumer of each kind, each of which values its preferred pair at v.

[65] Sun and Yang (2011) and Teytelboym (2014) have independently used alternative methods to show these results for a version of this model; the even n case is a special case of the "generalised gross substitutes and complements" demand type that we discuss in Section 6.2; one can also use the relationship with matching (see Section 6.6), together with Pycia (2008), to obtain the $n = 3$ case.

relevant supply. By assumption, there exists a price $\mathbf{p}$ at which the agent with valuation $G^* u^j$ demands $\mathbf{y}^j$ and $\sum_{j=1}^{k} \mathbf{y}^j = \mathbf{y}$. Define $\mathbf{x}^j := G\mathbf{y}^j \in D_{u^j}(G^{-T}\mathbf{p})$ (see Prop. 6.1(1)). Then $\mathbf{x} = \sum_{j=1}^{k} \mathbf{x}^j \in D_{u^J}(G^{-T}\mathbf{p})$. As G is invertible, the converse is shown similarly. $\square$

Remark A.24. The LIP of a valuation is its associated "tropical hypersurface"; algorithms to calculate this are implemented by "gfan" (see Jensen, 2011), and "polymake" (see Gawrilow and Joswig, 2000). The primitive normal vectors are found in the process.

There exist polynomial time algorithms to test *total* unimodularity of a matrix (Schrijver, 2000, Ch. 20). A set of vectors is unimodular iff a basis change as described in Footnote 39 yields a totally unimodular matrix, so this allows us to easily check for unimodularity of a demand type.

Intersection 0-cells can be found using the "a-tint" extension of polymake (Hampe, 2014a). Hampe (2014b) shows these methods provide answers in reasonable time for $n \leq 10$ for harder problems than ours, if $|A|$ is relatively small.

The calculation of the "mixed volumes", $\Gamma_k^n(A^1, A^2)$, in general has high complexity (see Cox et al., 2005, Section 7.6). However, recent work is developing ever-faster algorithms (see, e.g., Chen et al., 2014, and Jensen, 2016; the latter uses methods of tropical geometry). More importantly, $\Gamma_k^n(A^1, A^2)$ is trivial to calculate in many cases that matter most in economics, including if every agent j's valuation is over all bundles containing at most d_j goods, or if every agent's valuation has the same domain and this domain's volume is easy to calculate (see Fact A.11).

Calculating cell weights requires finding the volumes of lattice polytopes (Defn. 5.6). This also has high complexity in general (Dyer and Frieze, 1988), but efficient approximate methods, such as those developed by Dyer et al. (1991), are likely to be applicable in our context. Finally, the "Quickhull" algorithm (Barber et al., 1996) helps check for discrete convexity in Step (8).