Long-Run Wealth Distribution with Random Shocks

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Abstract

A standard neoclassical convergence model in which wealth accumulation is subject to random shocks is examined. The focus is on the limiting, or ergodic, distribution of wealth. This distribution satisfies a Fredholm integral equation. Direct mathematical solution is not possible. However results obtained characterize the limiting distribution of the logarithms of wealth values as a single-peaked distribution. It is asymmetric with the left-hand tail more heavily weighted. It follows that models which treat wealth transition as purely random lead to qualitatively different outcomes from those implied by the neoclassical convergence model augmented by random shocks.

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0.1 Convergence

The Neoclassical convergence model has been influential in recent years. In its basic form it says that all countries tend to converge to a common level of capital and output per head. The theory leads to a relationship similar to:

$$k_{t+1} = f\left[k_t\right] \tag{1}$$

where k_t is the logarithm of wealth (or income), and there is a unique stable value of $k = k*$, such that $k* = f\left[k*\right]$. The function $f\left[\cdot\right]$ will be assumed to be strictly concave. For stability one must have:

$$\left[\frac{\partial f\left[k\right]}{\partial k}\right]_{k=k*} < 1 \tag{2}$$

Then (2), together with strict concavity, allow there to be a second, unstable, equilibrium for a value of $k < k*$; in which case $k = -\infty$ will be a third (stable) equilibrium.

Such multiple equilibrium will not be encountered if the function $f\left[\cdot\right]$ is defined from either a Ramsey optimal saving model, or from a constant saving share Solow-Swan growth model. Even in these instances, it may be that:

$$Lim_{k\rightarrow-\infty}\left[f(k)\right] = -\infty \tag{3}$$

In which case there is an unstable zero wealth poverty trap equilibrium.

In empirical studies the usual practice is to work with a particular linearized version of the relation (1), viz:

$$k_{t+1} - k_t = f\left[k_t\right] - k_t = \alpha - \beta \cdot k_t + \epsilon_t \tag{4}$$

and to regress $k_{t+1} - k_t$ on k_t. With $0 < \beta < 1$, the normal finding, equation (2) says that on average poor countries grow faster than rich countries. This has been called $\beta-$*convergence.* That is not the same as $\sigma-$*convergence*, which means that the variance of the population of k values declines over time[1].

Friedman (1992) accused the approach based on regressions like (4) of "Galton's Fallacy" on the ground that non-persistent random fluctuations in k_t allow the possibility of a positive coefficient β even if the population variance of k were to show no downward trend. This is correct, if not as well expressed as it might be. The reference to random fluctuations in k_t suggests errors in variables, a famous Friedman theme, but not the one at issue here.

The point could be expressed more directly by saying that if (4) is the true model, then:

$$k_{t+1} = \alpha - (\beta - 1) \cdot k_t + \epsilon_t \tag{5}$$

in which case, should we observe a value of k_t far from the population mean, either k_{t-1} was far from the population mean, or $\epsilon(t-1)$ was exceedingly small. In either case the expected value of k_{t+1} conditional on k_t is $\alpha - (\beta - 1) \cdot k_t$ which will be closer to the population mean. However nothing is implied for $\sigma-$ convergence. Indeed it can be shown, as will be seen below, that a process such as (5) is consistent with a population density of k values which is invariant over time in the sense that it reproduces itself next period, although individual values will vary, partly systematically, showing β-convergence, and partly randomly, due to stochastic realisations of ϵ_t.

Quah (1993) noted independently that a Galton's Fallacy problem exists. Quah considers income as a pure Markov process. According to this

[1]For a clear exposition of the two concepts of covergence, and empirical discussion, see Sala-i-Martin (1996).

author, the process does not show convergence according to the data. Rather countries divide into two groups, rich and poor. See also Quah (1996a) and (1996b). There is a tendency to convergence within the two groups. There are also non-negligible probabilities that a country will shift from one group to the other. Indeed Bangladesh will become richer than the USA with probability 1, if one just waits long enough. This is a feature of any model which includes serially uncorrelated random shocks.

The original convergence theorists - such as Barro - on the one hand; and Friedman and Quah each one, on the other, all differ significantly in their views, and they differ from Galton too. Galton noted regression towards the mean of human heights. That entails that children of exceptionally tall fathers tend to have heights closer to the population mean. Galton's fallacy was to infer that the variance of heights is declining, in the sense that the population of heights is moving to a common mean. Consider two extreme cases. The incomes of individual agents are drawn from a common distribution, independently by time and agent. The variance of the distribution declines through time, so that inequality tends to decrease. Convergence however is stronger than the decline in overall inequality, as there is regression to the mean. An agent with a very low drawing at t is highly likely to do better at $t+1$. Alternatively, assume that position in the distribution is fixed once and for all by the initial drawing. Inequality falls over time but there is little convergence.

Thus if (4) describes the world there will be β-convergence. There may or may not be σ-convergence. Suppose, for instance, that we start with all k values very close together. The regression coefficient β may be positive and < 1. Yet k values will tend to diverge, because the scattering effects due to stochastic shocks will predominate. Equally, if k values are initally extremely diverse, we will see both β-convergence and σ-convergence.

A crucial role is always played by *transition probabilities*; that is probabilities, or probability densities, attaching to an individual unit which is in one state being in another state next period. Such transition probabilities may be computed from an underlying model specification. Or they may simply be assumed, as is done in many Markov models to be discussed below.

The theorists are theorists: they derive a relationship from theory and try to estimate a coefficient. They have to admit that there is a random element, because one cannot do econometrics without making that assumption. However the idea that all transition is random, as in Markov models, is contrary to the theorist's prior view of the problem. Ideally estimation should be based on the prior theoretical view, and should also avoid biases or other problems. The theory should be helpful in deciding what those problems might be.

0.2 Markov Stochastic Process Models

Champernowne (1953) divides income into discrete ranges. He shows that if the lengths of the intervals are in geometric progression and the transition probabilities depend only on the number of interval divisions crossed (including zero) then the limit distribution satisfies Pareto's law. Notice that if one worked in the logarithm of incomes, the geometric progression assumption would equate to assuming equal intervals in log income. Wold and Whittle (1957) are closest to the approach of the present paper in that they assume an accumulation equation for wealth - compound growth in their case. This accumulation process does not by itself exhibit a tendency to convergence. What stops estates growing without bound is that random mortality intervenes, when an estate is divided equally among n inheritors. These authors find that a Pareto distribution to be an asymptotic equilibrium of their process. Steindl (1972) generalizes the Wold and Whittle model by substituting a general lifetime probability function for the exponential lifetime specification of the original.

Quah (1993) and (1996a) can be seen as testing a non-theoretical prior, according to which the transition process is purely random. He rejects that model but not in the direction of global convergence; rather he finds local convergence. What does the Galton fallacy mean for Quah? Consider observations of income y taken at two times, called 1 and 2 without loss of generality. Suppose that y_1 and y_2 are drawn from a distribution with joint probability density:

$$P\left[y_1, y_2\right] \tag{6}$$

for which y_1 and y_2 have a common mean y_0. Quah notes that even if the covariance of y_1 and y_2 is positive, there are various possibilities for the ergodic limit of y_t when y_{t+1} values are obtained by repeatedly drawing y_{t+1} from the distribution:

$$P\left[y_{t+1}, y_t\right] \tag{7}$$

including convergence, partial convergence and no convergence at all.

Friedman proposes a solution to the Galton's fallacy problem, which is to regress $k_{t+1} - k_t$ on k_{t+1}, where he finds a weak fit and a low coefficient. If the problem were as described by Quah, this would not help at all.

To keep matters simple, suppose that the true model is:

$$k_{t+1} = \alpha + \beta \cdot k_t + \epsilon_t \tag{8}$$

where $0 < \beta < 1$. Barro regresses $k(t+1) - k_t$ on k_t; while Friedman regresses $k(t+1) - k_t$ on k_{t+1}.[2].

From (8):

$$k_{t+1} - k_t = \alpha - (1-\beta) \cdot k_t + \epsilon_t \tag{9}$$

Also, from (8):

$$k_t = \frac{k_{t+1} - \alpha - \epsilon_t}{\beta} \tag{10}$$

Therefore:

$$k_{t+1} - k_t = \frac{\alpha}{\beta} - \frac{1-\beta}{\beta} k_{t+1} + \frac{\epsilon_t}{\beta} \tag{11}$$

Thus if (8) is the data generating process, with $\beta = 0.97$, which, notice, is a high level of persistence, a Barro regression as (4) will find the coefficient on k_t to be -0.03, while Friedman's regression will find the coefficient on k_{t+1} to be $-\frac{\mathbf{0.03}}{\mathbf{0.97}} = -0.03093$, almost the same; and both the intercept and the variance of errors will be nearly the same. With β close to 1, Friedman's device of regressing the growth rate on final level should make little difference.

Friedman, however, has in mind smaller values of β than the 0.93 value taken above. Obviously, with β much smaller, the two regressions diverge considerably. Friedman shows this happening for the OECD countries over the period 1950 to 1979. Not surprisingly for this long period, the persistence coefficient is not so large. With $\beta = \cdot 97$, the 29-year persistence coefficient would be $\cdot 41$. With such a value it will be seen, comparing equations (9) and (11), that Friedman's final year income on growth regression will involve a larger intercept and greater variance around the regression line. This is exactly what Friedman finds.

[2]Note that Friedman is discussing income, where the present paper mainly has wealth in mind.

The theory predicts that $f[k_t]$ will be concave, probably strictly concave, while often a linear form is tested. Also, a random element is included in the empirics but not taken into account in the theory. This paper attempts a theoretical analysis of a model in which individual "countries" are accumulating wealth according to the rule (1), but are subject to random shocks which throw them off course each period. The accumulation rule may reflect the fact of the random shocks.

In general the interdependence of agents' decisions needs to be taken into account, which makes things very complicated - see Bliss (1995). However the present analysis concentrates on the ergodic wealth distribution of infinitely many agents, their distribution measured by a density function. In that case interdependence is effectively reduced to steady-state price values, which may be subsumed into the shape of $f[k_t]$.

0.3 Analysis

The process:

$$k_{t+1} = f[k_t] + \epsilon_t \tag{12}$$

will be exmained, where k_t will usually be taken to be the logarithm of wealth. The random error term ϵ_t is independent of k_t and $\epsilon_{t'}$ for $t' \neq t$, and has mean zero. Specifying the stochastic process in terms of the logarithm of wealth brings the advantage that the range of ϵ_t is unconfined, so that, for instance, ϵ_t might be normally distributed without allowing wealth to become negative. One might want to permit wealth to be negative, even down to $-\infty$, in which case k_t could be interpreted as wealth itself, not the logarithm of wealth. This interpretation is left to the reader and will not be noted explicitly in what follows[3].

The stochastic process (12) generates a *Fredholm Equation* of the second kind[4] for the equilibrium density of the logarithm of wealth:

$$\Lambda[k] = \int_{-\infty}^{+\infty} \pi[k - f[\kappa]] \cdot \Lambda[\kappa] \, d\kappa \tag{13}$$

where $\pi[\cdot]$ is the density of the random effect ϵ_t. The integral on the right-hand side of (11) is the sum of all transitions from κ to k weighted by the probability that the initial value is κ, which is $\Lambda[\kappa]$, and the probability of a transition to k which is the probability that ϵ_t takes the value $k - f[\kappa]$. Placing the same function $\Lambda[\cdot]$ on both sides of (13) identifies the ergodic fixed point outcome.

This derivation is somewhat similar to the so-called *Theory of Breakage* which leads to the equation:

$$F_j(x) = \int_u H_j\left[\frac{x}{u}\right] dF_{j-1}[u] \tag{14}$$

for which see Aitchison and Brown (1957), pp.26-7.

The process:

$$k_{t+1} = f[k_t + \epsilon_t] \tag{15}$$

generates another *Fredholm Equation, viz:*

[3] Negative wealth raises complicated modelling issues. Plainly small net debts raise no great problems as they can be worked off over time. This is because wealth here may be present balance sheet net worth, not including the present value of future income. Large net debts on the other hand raise problems as they may be unserviceable. When random shocks take agents into the region of unserviceable indebtedness, some kind of bankruptcy regime operates.

[4] See Hildebrand (1961) p. 381-2. In section 4.5 of the same chapter the author explains the connection between this type of equation and the joint effect of many causes.

$$\Lambda[k] = \int_{-\infty}^{+\infty} \pi\left[f^{-1}[k] - \kappa\right] \cdot \Lambda[\kappa]\, d\kappa \tag{16}$$

which is quite similar.

To keep things simple, we concentrate on the Fredholm Equation (13).

0.4 Some Results

Theorem 1 *The set of functions satisfying (13) is convex*[5]*. Hence the set of functions which integrate to 1 and also satisfy (13) is convex.*
Proof: Is immediate. If $\Lambda^1[k]$ and $\Lambda^2[k]$ both satisfy (11), then:

$$\lambda \cdot \Lambda^i[k] = \int_{-\infty}^{+\infty} \pi[k - f[\kappa]] \cdot \lambda \cdot \Lambda^i[\kappa]\, d\kappa \tag{17}$$

for $i = 1$ or 2, and for any value of λ.□

In analysing the distribution of k values, it is sometimes convenient to work in terms of the cumulative distributions. Hence $\Delta(k)$ is the proportion of the population with wealth not greater than k. Clearly $\Delta(-\infty) = 0$ and $\Delta(\infty) = 1$. Then (13) translates to:

$$\Delta[k_{t+1}] = \int_{-\infty}^{+\infty} \Theta[f[\kappa] - k_{t+1}] \cdot \Delta[\kappa]\, d\kappa \tag{18}$$

where $\Theta(\cdot)$ is the cumulative distribution of ϵ_t.

Notice that the effect on the distribution of wealth in moving from one period to the next is the sum of two separate transformations. First each k value maps to $f[k]$. We call this *f-transformation.* Next all values are scattered by the addition of random shocks ϵ_t. We call this *scattering.* Consider the first step. Before f-transformation:

[5]To say that the set of functions is convex is not, of course, to say that the functions are convex functions.

$$\Lambda[k] = \frac{d\Delta[k]}{dk} \tag{19}$$

Whereas after f-transformation:

$$\Gamma[k] = \Delta\left[f^{-1}[k]\right] \tag{20}$$

where $\Gamma[k]$ is the cumulative distribution of k after transformation. Then:

$$\Lambda[k] = \frac{d\Gamma[k]}{dk} = \frac{\Lambda[k]}{f'[k]} \tag{21}$$

where $f'[k]$ is the derivative of $f[k]$ with respect to k. This defines how the accumulation function affects the distribution of wealth in the absence of random effects.

Theorem 2 *If the distribution $\Lambda[k]$ has a regular maximum at k_0, then the transformed distribution $\frac{\Lambda[k]}{f'[k]}$ has a maximum at $k^{\tilde{}} > k_0$.*
Proof: A regular maximum in this context means that:

$$\frac{d\Lambda[k]}{dk} \cdot [k - k_0] < 0 \tag{22}$$

for $k \neq k_0$. Maximizing $\frac{\Lambda[k]}{f'[k]}$ with respect to k gives:

$$\frac{\Lambda^1[k]\, f^1[k] - \Lambda[k]\, f^2[k]}{f^1[k]^2} = 0 \tag{23}$$

where superscripts denote derivatives. For a value of k for which $\Lambda[k]$ takes a maximum (23) will be positive. Hence the result. □

Analysis of a Fredholm distribution can proceed either from the integral equation (13), or from the equation defining the process itself (12). The next section adopts the latter approach.

0.5 Direct Analysis from the Stochastic Process Equation

Applying mathematical expectation operator to (12) gives:

$$Ek = Ef[k] < f[Ek] \tag{24}$$

where the final inequality holds when $f[k]$ is strictly concave. Subtracting (24) from (12) and rearranging gives:

$$E[k - Ek]^2 > E\{[f[k_t] - f[Ek] + \epsilon_t]\}^2 = E\{f[k_t]\}^2 + E\{[\epsilon_t]\}^2 \tag{25}$$

If $f[k_t]$ is quadratic, (12) becomes:

$$k_{t+1} = a + b \cdot k_t - c \cdot k_t^2 + \epsilon_t \tag{26}$$

$$Ek = \frac{a + c \cdot E[k_t^2]}{1 - b} \tag{27}$$

Note that:

$$E\left[(k - Ek)^2\right] = E\left[k^2 - 2kEk + (Ek)^2\right] = E\left[k^2\right] - (Ek)^2 \tag{28}$$

So that:

$$Ek - c\frac{(Ek)^2}{1 - b} = \frac{a + c \cdot E\left[(k - Ek)^2\right]}{1 - b} \tag{29}$$

The left-hand side of (29) involves only the mean of k, while the right-hand side involves the variance of k.

0.6 Symmetric Bell-Shaped Distributions

The following argument touches particularly on the question of whether an ergodic equilibrium wealth distribution can be normal. As k is the logarithm of wealth, that is equivalent to asking whether wealth can be log-normally distributed in the limit.

Definition 1 *A density function of k, $D(k)$, will be said to be Symmetric Bell-Shaped (SBS) if:*
There exists a unique value k_0 such that $D(k)$ takes its maximum value.
The value of $D(k)$ is uniquely determined by $|k - k_0|$.
The value of $D(k)$ is decreases monotonically with $|k - k_0|$.

Definition 2 *The Fredholm equation (11) will be said to be standard if:*
The density function of errors $\pi[\cdot]$ is SBS with its maximum at zero.
The function $f[k_t]$ is strictly concave.

Theorem 3 *If the functions $f[\cdot]$ and $\pi[\cdot]$ are both continuous, then any Fredholm equation solution is continuous in k.*
Proof: Note that if $f[\cdot]$ is concave, it must be continuous, but this result does not require concavity. Consider an infinite sequence:

$$k_1, k_2, ..., k_n, ... \tag{30}$$

tending to k_0. It may be confirmed by inspection that:

$$Lim_{n\to\infty} \int_{-\infty}^{+\infty} \pi[k(n) - f[\kappa]] \cdot \Delta[\kappa]\, d\kappa = \int_{-\infty}^{+\infty} \pi[k_0 - f[\kappa]] \cdot \Delta[\kappa]\, d\kappa \tag{31}$$

provided that $\pi[\cdot]$ is continuous.□

Recall that the mapping of a Fredholm distribution solution into itself consists of the sum of two separate steps. The first is the effect of the f-transformation; the second is scattering.

The statement of the next theorem is most easily understood in terms of the cumulative distribution $\Gamma[k]$. The distribution has an infinite tail if there exist no finite value of k such that $\Gamma[k]$ takes either the value 0 or 1. In the former case the distribution will be said to have an infinite left-hand tail; in the latter case an infinite right-hand tail.

Theorem 4 *If the distribution $\pi[\cdot]$ has an infinite left- or right-hand tail, a Fredholm equation solution will have respectively an infinite left- or right-hand tail. A Fredholm equation solution may have an infinite left-hand tail without $\pi[\cdot]$ having an infinite left-hand tail if:*

$$k - f(k) \tag{32}$$

is bounded above for all negative k by A, where $-A$ is within the range of the left-hand tail of $\pi[\cdot]$.

Proof: The first statement is obvious. Whatever value $f[k_t]$ may take, if the distribution $\pi[\cdot]$ has an infinite tail, positive probability attaches to ϵ_t taking any value whatsoever in that direction; hence positive probability, and therefore distribution density, attaches to k_{t+1} taking any value in that direction. The second statement is also plain. As long as the difference $k - f(k)$ cannot exceed the reach of the left tail of $\pi[\cdot]$, density lost by the effect of f transformation $\square$

Even if $k - f(k)$ is bounded above within the range of the finite left-tail of $\pi[\cdot]$, for all negative k sufficiently large in absolute value, it might not be so bounded for $k < k*$ and closer to $k*$. Is it then possible for an equilibrium density to be split into two disjoint segments: a high wealth segment, including positive density for $k*$; and a low wealth segment from which agents cannot escape because random shocks always push them back down? We call such a distribution *disjoint*. The next theorem answers the question in the negative sense.

Theorem 5 *An equilibrium density cannot be disjoint.*

Proof: Suppose, contrary to the theorem, that an equilibrium density is disjoint. Then there exists a range of values of k, $[k^-, k^+]$, such that $\Lambda[k] = 0$ for these values, while positive density attaches to values of k just below k^-. Consider the value $k^{\hat{}} = f^{-1}[k^-]$. As $k^{\hat{}} < k^-$, positive density attaches to $k^{\hat{}}$. Therefore $f-$transformation will take density to k^-. When scattering is taken into account, some even higher k values will result, and positive density will attach to values of k just above k^-, contrary to assumption.$\square$

The next result is called a Lemma because it serves Theorem 6 directly.

Lemma 6 *A distribution is SBS after scattering only if it is SBS before scattering.*

Proof: Scattering does not affect the mean because:

$$Ek_{t+1} = Ek_t + E\epsilon_t = Ek_t \tag{33}$$

Suppose that $D(k)$ is a distribution with mean k_0, and asymmetric about that value, but that the image of that distribution after scattering is SBS. In that case the image must be symmetric about k_0. Call the image distribution $I(k)$. Then:

$$I(k) = \int_{-\infty}^{+\infty} \pi[k-\kappa] \cdot D[\kappa]\, d\kappa \tag{34}$$

because $D(k)$ is not symmetric but $\pi[\cdot]$ is symmetric about zero, the above cannot be equal to:

$$\begin{aligned} I(k_0+k) &= \int_{-\infty}^{k_0} \pi[k_0+k-\kappa] \cdot D[\kappa]\, d\kappa + \int_{k_0}^{k_0+k} \pi[k_0+k-\kappa] \cdot D[\kappa]\, d\kappa & (35) \\ &+ \int_{k_0+k}^{\infty} \pi[k_0+k-\kappa] \cdot D[\kappa]\, d\kappa B & (1) \end{aligned}$$

This demonstrates the Lemma.$\square$

Theorem 7 *If the problem is standard, no Fredholm equation solution can be SBS.*

Proof: Because of Lemma 1, if a Fredholm equation solution is to be SBS, the transformation $k \to f[k]$. must map an SBS distribution into an SBS distribution. For only in that case can the subsequent scattering generate an SBS distribution. However Theorem 2 states that the transformation $k \to f[k]$ increases the value of k at which the distribution takes its maximum value. For an SBS distribution the maximum value is taken at the mean. Scattering does not affect the mean. Therefore supposing a Fredholm solution to be SBS implies that its mean is higher than itself, which is a contradiction.$\square$

0.7 Single Peakedness

The above results tell us more about what a Fredholm solution is not than about what it is. Theorem 6 rules out SBS. The argument indicates what shape to expect should the distribution be single peaked. The $k \to f[k]$ transformation increases the value of k at which the distribution takes its maximum. Scattering must therefore lower that value, which it cannot do in the case of an SBS distribution. It can do it for a distribution skewed to the negative side, as there would then be more mass to the left of the maximum to generate random jumps pulling the maximum to the left. So for the single peak case the distribution is broadly characterized as asymmetric relative to log-normality. How do we know that a Fredholm solution will be single-peaked? This section provides the method which enables that property to be derived. Note that staring at the Fredholm equation itself hardly resolves the issue. Should there be a trough in the distribution that will tend to produce a trough in the image, simply because there will be less mass around that trough, or rather its image under the f-transformation, to feed density via small error values.

To show that there cannot be multiple peaks we again take advantage of the fact that the mapping of a wealth density into itself is the result of the sum of two separable transformations - f-transformation plus scattering. The way in which we rule out multiple peaks is by examining the effect of the two steps on extrema. The next theorem examines the effect of scattering by itself.

Theorem 8 *The effect of scattering on its own is to increase $\Lambda(k)$ in the neighbourhood of strict minima and to decrease $\Lambda(k)$ in the neighbourhood of strict maxima.*

Proof: The result is shown for a maximum. The proof for a minimum is the same.

For a maximum of $\Lambda(k)$, we must have:

$$\frac{d\Lambda(k)}{dk} = 0 \tag{36}$$

Now choose k, and θ , which affects the extent of scattering, to maximize:

$$\int_{-\infty}^{+\infty} \Lambda(\kappa) \cdot \pi \left[\theta\left(k-\kappa\right)\right] d\kappa \tag{37}$$

to obtain:

$$\int_{-\infty}^{+\infty} \Lambda(\kappa) \cdot \theta \cdot \pi' \left[\theta\left(k-\kappa\right)\right] d\kappa = 0 \tag{38}$$

where the prime denotes the first derivative. Maximizing with respect θ to gives:

$$\int_{-\infty}^{+\infty} \Lambda(\kappa) \cdot k \cdot \pi' \left[\theta\left(k-\kappa\right)\right] d\kappa = 0 \tag{39}$$

Now it is clear that (38) and (39) are the same condition after k and θ have been placed outside the integrals and eliminated. This implies that maximizing or minimizing density entails minimizing scattering. The Theorem follows.$\square$

Theorem 9 *If $f(k)$ is strictly concave, a Fredholm equation solution is single peaked.*

Proof: Should a Fredholm equation solution not be single peaked it must be the case that it has a local minimum between two local maxima. Let the values of k corresponding to these extrema be respectively $k^{\max 1}$, $k^{\min}$ and $k^{\max 2}$. After f-transformation the distribution $\frac{\Lambda[k]}{f'[k]}$ will similarly have a maximum, a minimum and a maximum at respectively $\widetilde{k}^{\max 1}$, $\widetilde{k}^{\min}$ and $\widetilde{k}^{\max 2}$.

By definition:

$$\Lambda\left(k^{\max 1}\right) > \Lambda\left(\widetilde{k}^{\max 1}\right) \tag{40}$$

$$\Lambda\left(k^{\min}\right) < \Lambda\left(\widetilde{k}^{\min}\right) \tag{41}$$

$$\Lambda\left(k^{\max 2}\right) > \Lambda\left(\widetilde{k}^{\max 2}\right) \tag{42}$$

Because of the effects of scattering shown in Theorem 8, f-transformation must increase maximum values of the density and decrease a minimum value, so that scattering may undo these effects. Therefore:

$$\frac{\Lambda\left(\widetilde{k}^{\max 1}\right)}{f'\left(\widetilde{k}^{\max 1}\right)} > \Lambda\left(k^{\max 1}\right) \tag{43}$$

$$\frac{\Lambda\left(\widetilde{k}^{\min}\right)}{f'\left(\widetilde{k}^{\min}\right)} < \Lambda\left(k^{\min}\right) \tag{44}$$

$$\frac{\Lambda\left(\widetilde{k}^{\max 2}\right)}{f'\left(\widetilde{k}^{\max 2}\right)} > \Lambda\left(k^{\max 2}\right) \tag{45}$$

Then (40) and (43); (41) and (44); and (42) and (45); respectively imply $f'\left(\widetilde{k}^{\max 1}\right) < 1$; $f'\left(\widetilde{k}^{\min}\right) > 1$; and $f'\left(\widetilde{k}^{\max 2}\right) < 1$. This contradicts the concavity of $f(k)$.$\square$

0.8 Concluding Remarks

The long history of the analysis of income or wealth distributions, going back to Pareto, includes different approaches. One is purely empirical. The shape of the distribution is examined and the fitness of simple mathematical specifications is investigated. Another approach is to start with postulates concerning the process which generates the distribution and then to investigate mathematically what is the limiting distribution which results. Yet the limiting distribution does not have to be the object of concern. The shorter term conditional transfer process can itself be the focus of investigation. Indeed the neoclassical convergence theorists can only do that, because for them the limiting distribution is trivial, being a state in which all countries - or individuals in the case of a personal distribution - are at the common limit point $k*$. When the transfer process is taken to be random there are wider possibilities than when it is modelled using economic theory.

The present paper marries two different traditions. They are the neoclassical approach, according to which wealth accumulation is systematic and deliberate; and the random shocks approach, according to which wealth accumulation is purely haphazard. As would be expected, such a model is complicated, and direct mathematical solution is hardly possible. Even so, we have been able to obtain a series of results which together effectively characterize the limiting distribution of the logarithms of wealth values. It is a single-peaked distribution which may or may not have infinite tails, and it is asymmetric with the left-hand tail more heavily weighted.

An implication of these results, particularly Theorem 9, is that Quah's finding, according to which there are two disjoint regions of attraction in a distribution, cannot be reconciled with the neoclassical model augmented by the addition of shocks.

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