AN ELEMENTARY EQUILIBRIUM EXISTENCE THEOREM

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Summary: This paper gives a proof of the existence of competitive equilibrium under the added assumption that the excess demand function satisfies the weak axiom. In this case, a proof using the separating hyperplane theorem, and without using a fixed point theorem, is possible.

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1. Introduction

The classic proofs on the existence of competitive equlibrium employ Kakutani's fixed point theorem (see Debreu (1982) and its references). It is also known that this is the right theorem to use, provided excess demand functions have no structure apart from Walras' Law. This is because, as observed by Uzawa (1962), one could *prove* Kakutani's fixed point theorem by assuming a fundamental lemma used in equilibrium existence. (A proof of this result could also be found in Debreu (1982).)

Since Uzawa's observation, it has been established that utility maximization among agents in an economy imposes no structure on its aggregate excess demand function (see Sonnenschein (1973, 1974) and Debreu (1974)). In other words, properties like the strong or weak axiom, which may be expected to hold at the level of the individual agent, are lost in the process of aggregation. It follows that a general proof of equilibrium existence *must* employ fixed point theorems. However, fixed point theorems are not needed in the case when excess demand is known to possess some aggregate structure; in particular, the properties of gross substitubility and the weak axiom have been extensively studied. Both these properties are sufficient to ensure that the equilibrium price is unique (at least generically) and that it is stable with respect to Walras' tatonnement. To obtain these properties in the aggregate must necessarily imply some restriction on the way agents' preferences and endowments are distributed. A general equilibrium model with such restrictions, where the excess demand function satisfies gross substitubility has been developed by Grandmont (1992); for the weak axiom, models have been developed by Hildenbrand (1983), Marhuenda (1995), Quah (1997a, 1997b) and Jerison (1999) amongst others.

It is well known that if the excess demand function satisfies gross substitubility, then there is a very simple proof of equilibrium existence (see, for example, Hildenbrand and Kirman (1988)). We show in this paper that a simple proof employing the separating hyperplane theorem is also available when excess demand satisfies the weak axiom. This proof has the virtue that it separates very sharply the function of the geometric and continuity properties of excess demand. Provided an excess demand function satisfies the weak axiom, there will be some price vector with the following property: holding all other prices fixed, raising the price of good i leads to excess supply, and lowering it leads to excess demand. The existence of a price vector with this property does not require the continuity of the excess demand function, but it is quite easy to see that this price vector *is* an equilibrium price, i.e., has an excess demand of zero, provided the excess demand function is continuous. Indeed, continuity is needed at this step, and nowhere else.

It might be interesting to note that a similar process of simplification has also occurred in game theory. Von Neumann's (1928) original proof of the minimax theorem employed Brouwer's fixed point theorem, but the problem in fact has a nice geometrical structure that allows it to be quite intuitively solved with the separating hyperplane theorem (see Gale et al (1950)). Similarly, Hart and Schmeidler (1989) showed that the existence of correlated equilibria could be established with linear methods, even though the proof of the existence of Nash equilibrium uses Kakutani's fixed point theorem.

2. An Equilibrium Existence Theorem

In this section we give a proof of the existence of general equilibrium using the separating hyperplane theorem, under the added assumption that the economy's excess demand satisfies the weak axiom.

The set P in R^l is a *cone* if, whenever p is in P, λp is also in P, for all positive λ. A cone is said to be *pointed* if, whenever $p \neq 0$ is in P, $-p$ is not in P. A standard approach to the equilibrium existence problem involves the construction of a correspondence, $Z : P \to R^l$, where P is a convex and pointed cone in R^l. In this case, the economy has l goods, P is the set of price vectors, and Z is the excess demand (see Debreu (1982)). Typically Z will have a number of properties:

Property 1. Z satisfies Walras' Law, i.e., $p \cdot Z(p) = 0$ for all p in P.

Property 2. Z is a compact and convex valued, upper hemi-continuous correspondence.

If one is investigating an exchange economy, then $P = R^l_{++}$, and Z will typically have two other properties:

Property 3. Z is bounded below.

Property 4. Z satisfies the following boundary condition: if p_n in R^l_{++} tends to $\bar{p}$ on the boundary of R^l_{++}, with $\bar{p} \neq 0$, then $|Z(p_n)|$ tends to infinity. (Note that for any compact set S in R^l, we denote $\min\{|s| : s \in S\}$ by $|S|$.)

It is well known that if excess demand has Properties 1 to 4, then an equilibrium exists, i.e., there will be a price p^* in R^l_{++} such that $0 \in Z(p^*)$ (see Debreu (1982)). This result is usually established with Kakutani's fixed point theorem. We show here that another, quite instructive method, is available if Z also satisfies a weak form of the weak axiom. For an excess demand *function*, the weak axiom is usually defined as the following: if $p \cdot Z(p') \leq 0$ and $Z(p) \neq Z(p')$, then $p' \cdot Z(p) > 0$. The definition we give below is an extension to correspondences, and is also weaker than the usual definition when applied to functions.

DEFINITION: The correspondence $Z : P \to R^l$ satisfies the *weak weak axiom* if the following is true: whenever there exists z' in $Z(p')$ such that $p \cdot z' \leq 0$, then $p' \cdot Z(p) \geq 0$.

We will now set out to show that an excess demand correspondence satisfying Properties 1 to 4, and the weak weak axiom will have an equilibrium price. We begin with a lemma which guarantees that a *finite* set of excess demand vectors must have a supporting price.

LEMMA 2.1: Suppose that the correspondence $Z : P \to R^l$ satisfies Property 1 and the weak weak axiom. Then for any finite set $S = \{z_1, z_2, ..., z_n\}$ where z_i is an element of $Z(p_i)$, there is x^*, in the convex hull of $\{p_1, p_2, ...p_n\}$ such that $x^* \cdot S \geq 0$.

Proof: We proof by induction on n. If $n = 1$, choose $x^* = p_1$. If $n = 2$, then either $p_2 \cdot z_1$ or $p_1 \cdot z_2$ is non-negative. If it is the latter, choose, $x^* = p_1$.

Assume now that the proposition is true for n and assume that it is not true for $n + 1$. Consider the following constrained maximization problem:

$\max x \cdot z_k$ *subject to* x *satisfying the conditions:*

(a) $x \cdot z_i \geq 0$ *for* i *in* $I_k = \{1, 2, ...k - 1, k + 1, ..., n + 1\}$ *and*

(b) x *is in the convex hull of* $P_k = \{p_i : i \in I_k\}$.

By varying k, we have $n + 1$ problems of this sort.

Consider the case when $k = n + 1$. By the induction hypothesis, there is certainly x such that $x \cdot z_i \geq 0$ for all i in I_{n+1}, since this set has only n elements. Furthermore, the convex hull of P_{n+1} is compact, so the problem has at least one solution, which we denote by $\bar{x}_{n+1}$. Since we are proving by contradiction, we assume that $\bar{x}_{n+1} \cdot z_{n+1} < 0$.

We will now show that $\bar{x}_{n+1} \cdot z_i = 0$ for all i in I_{n+1}. If not, the set $J = \{i : \bar{x}_n \cdot z_i =$

$0\} \cup \{n+1\}$ has n elements or less, and so there is $\bar{y}$ with $\bar{y} \cdot z_i \geq 0$ for all i in J. Consider now the vector $\theta\bar{y} + (1-\theta)\bar{x}_{n+1}$, which is in the convex hull of $\{p_1, p_2, ..., p_{n+1}\}$, provided θ is in $[0, 1]$. Then

(i) $[\theta\bar{y} + (1-\theta)\bar{x}_{n+1}] \cdot z_i \geq 0$, for i in $J \setminus \{n+1\}$

(ii) $[\theta\bar{y} + (1-\theta)\bar{x}_{n+1}] \cdot z_i > 0$, for $i \notin J$ provided θ is sufficiently small

(iii) $[\theta\bar{y} + (1-\theta)\bar{x}_{n+1}] \cdot z_{n+1} \geq (1-\theta)\bar{x}_{n+1} \cdot z_{n+1} > \bar{x}_{n+1} \cdot z_{n+1}$.

This means that $\bar{x}_{n+1}$ does not solve the constrained maximization problem.

So the solution to this problem, $\bar{x}_{n+1}$, must satisfy $\bar{x}_{n+1} \cdot z_i = 0$ for i in I_{n+1} and $\bar{x}_{n+1} \cdot z_{n+1} < 0$. We can apply the same argument to a solution of the other problems. In this way, we obtain $\bar{x}_k$, for $k = 1, 2, ..., n+1$ with

(i) $\bar{x}_k \cdot z_i = 0$ for i in I_k and

(ii)$\bar{x}_k \cdot z_k < 0$.

Define $\bar{x} = [\sum_{i=1}^{n+1} \bar{x}_i]/(n+1)$; $\bar{x}$ is certainly in the convex hull of $\{p_1, p_2, ..., p_{n+1}\}$. Furthermore, $\bar{x} \cdot z_i < 0$, for $i = 1, 2, ..., n+1$. By the weak weak axiom, $p_i \cdot Z(\bar{x}) > 0$ for all i. Since $\bar{x}$ is in the convex hull of the p_is, we have $\bar{x} \cdot Z(\bar{x}) > 0$, which contradicts Walras' Law (Property 1). QED

Lemma 2.1 showed that any finite set of excess demand vectors has a supporting price. Our next objective is to apply this result to show that the entire range of Z has a supporting price, but before we do that, we review some basic results on cones. Suppose that A is a convex and pointed cone in R^l. Define the set $A^* = \{v \in R^l : v \cdot a < 0 \text{ for all } a \in A, a \neq 0\}$, and the set $A^0 = \{v \in R^l : v \cdot a \leq 0 \text{ for all } a \in A, \}$. The set A^0 is usually referred to as the *polar cone* or *negative polar cone* of A. A^* is defined similarly, except that the inequality is strict rather than weak.

LEMMA 2.2: Suppose that A is a convex and pointed cone. The following is true:

(i) $\text{cl}(A^*) = (\text{cl}A)^0$ and (ii) If A is closed, $(A^0)^0 = A$.

(We denote the closure of any set S by $\text{cl}S$.)

Proof: (i) If x is in $\text{cl}(A^*)$, there is x_n such that $x_n \cdot a < 0$ for all a in $A \setminus \{0\}$. Taking limits, we have $x \cdot \bar{a} \leq 0$ for all $\bar{a}$ in $\text{cl}A$. So x is in $(\text{cl}A)^0$.

We will now show that $(\text{cl}A)^0 \subseteq \text{cl}(A^*)$. In fact, we will show something a little stronger, that $A^0 \subseteq \text{cl}(A^*)$. If x is in A^0, by definition, $x \cdot a \leq 0$ for all a in A. Since A is convex and pointed, by the separating hyperplane theorem, there is $w \neq 0$ such that $w \cdot A > 0$, for all a in $A \setminus \{0\}$. Since $[x - (w/n)] \cdot a < 0$ for all a, $x - (w/n)$ is in A^*. Letting n go to infinity, we see that x is in $\text{cl}(A^*)$.

(ii) If a is in A, for all v in A^0, $v \cdot a \leq 0$, so a is certainly in $(A^0)^0$. On the other hand, if a is not in A, then by the separating hyperplane theorem, there is w such that $w \cdot a > w \cdot A$. (Note that the inequality is strict because A is closed and pointed.) This means that $w \cdot A \leq 0$; otherwise the right hand side of the inequality is unbounded above. So w is A^0. We also have $w \cdot a > 0$, so this means that a is not in $(A^0)^0$. QED

PROPOSITION 2.3: Suppose that the correspondence $Z : P \to R^l$ satisfies Property 1 and the weak weak axiom. Then there is p^* in the closure of P such that $(p^* - p) \cdot Z(p) \geq 0$ for all p in P.

Proof: We claim that $\text{co}Z \cap P^* = \phi$, where $\text{co}Z$ is the convex hull of the set $\{Z(p) \in R^l : p \in P\}$ and $P^* = \{v \in R^l : v \cdot p < 0 \text{ for all } p \in P, p \neq 0\}$. If not, we can find $\sum_{i=1}^K \beta_i z_i$ in P^*, where z_i is in $Z(p_i)$ for some p_i, and the β_is are non-negative numbers that add up to 1.

By Lemma 2.1, there is x in P, with $x \cdot z_i \geq 0$ for all i, and consequently, $x \cdot [\sum_{i=1}^{K} \beta_i z_i] \geq 0$, contradicting the definition of P^*. So our claim is true. The separating hyperplane theorem guarantees that there is $p^* \neq 0$ such that $p^* \cdot Z(p) \geq p^* \cdot P^*$. Since P^* is a cone, the right hand side of this inequality could be bounded above only if is non-positive, so we have $p^* \cdot Z(p) \geq 0$ for all p in P.

We also claim that p^* is in cl$P$. Since $p^* \cdot P^* \leq 0$, we also have $p^* \cdot (\text{cl}P^*) \leq 0$. By part (i) of Lemma 2.2, $p^* \cdot (\text{cl}P)^0 \leq 0$, so p^* is in $(P^0)^0$, which is equal to cl$P$ by part (ii) of the Lemma 2.2. QED

It is worth pointing out that p^* in Proposition 2.3 is really very close to an equilibrium price. To see that, let p be a price in P, with $p^i = p^{*i}$ for all i, except $i = k$. Proposition 2.3 tells us that $(p^{*k} - p^k)Z^k(p) < 0$. In other words, if p^k is greater than p^{*k} there will be excess supply of k; if it is lower, there will be excess demand of k. Note that we arrived at the existence of p^* relying exclusively on the geometric properties of Z. Continuity is not used at all. It is only needed to arrive at an equilibrium in the conventional sense.

LEMMA 2.4: Suppose that the correspondence $Z : P \to R^l$ satisfies Property 2 and that there exists a price p^* in the interior of the cone P such that $(p^* - p) \cdot Z(p) \geq 0$ for all p in P. Then $0 \in Z(p^*)$.

Proof: Suppose not; then 0 and $Z(p^*)$ are disjoint and convex sets, and so by the separating hyperplane theorem, there is $v \neq 0$ such that $v \cdot Z(p^*) < 0$. Note that the strict inequality is guaranteed by the compactness of $Z(p^*)$. Define $p = p^* - \lambda v$, for some positive number λ. Since p^* is in the interior of P, for λ sufficiently small p is also in P; furthermore, $(p^* - p) \cdot Z(p) = \lambda v \cdot Z(p)$ which is strictly negative provided λ is sufficiently small so that

$v \cdot Z(p) < 0$. The last condition is possible since, by Property 2, $Z(p)$ is compact and Z is upper hemi-continuous. QED

The next theorem gathers together our results so far to establish the existence of an equilibrium in the case when the excess demand correspondence satisfies properties typically attained in an exchange economy.

THEOREM 2.5: *Suppose that the correspondence* $Z : R^l_{++} \to R^l$ *satisfies Properties 1 to 4, and the weak weak axiom. Then there is a price* $p^* \gg 0$ *such that* $0 \in Z(p^*)$.

Proof: Proposition 2.3 guarantees that a supporting price p^* exists in R^l_+. If we can show that p^* cannot be on the boundary, then Lemma 2.4 guarantees that p^* is an equilibrium price. Assume, to the contrary, that $J = \{i : p^{*i} = 0\}$ is non-empty. We define the price vector p_n by $p_n^i = 1/n$ if i is in J, and $p_n^i = p^{*i}$ if i is not in J. So the sequence p_n tends to p^* on the boundary. Choose a sequence z_n, where z_n is in $Z(p_n)$. Then

$$\begin{aligned}
(p^* - p_n) \cdot z_n &= (p^* - p_n) \cdot z_n \\
&= \sum_{i=1}^{l} (p^{*i} - p_n^i) z_n^i \\
&= -\frac{1}{n} \left[\sum_{i \in J} z_n^i \right]
\end{aligned}$$

If this term is negative, we have a contradiction. Indeed it is, because $[\sum_{i \in J} z_n^i]$ is positive when n is sufficiently large. To see this, note that z_n^i is bounded below (Property 3), so in order for Walras' Law to be satisfied, z_n^i cannot tend to infinity if i is not in J; but the boundary condition (Property 4) requires $|z_n|$ to tend to infinity, so z_n^i must tend to infinity for some i in J. QED

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