# A Note on Measuring Preference Structuration[1]

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The concept of preference structuration not only provides possible escape-routes from social-choice-theoretic impossibility problems, but also points towards ways of formalizing notions of 'pluralism', 'consensus' and 'issue-dimensionality'. The present note introduces two methods of (operationally) measuring preference structuration, giving attention to both their conceptual characteristics and their computational feasibility. The method to be advocated, called the 'fractionalization' approach, combines well-known social-choice-theoretic criteria of preference structuration (such as single-peakedness or value-restriction) with the frequently used Rae-Taylor (1970) and Laakso-Taagepera (1979) approaches towards measuring the level of fractionalization, and the effective number of components, in a system.

## 1. The Problem

Since Duncan Black's seminal work (1948), it is known that Condorcet's paradox and many related social-choice-theoretic problems can be traced back to a 'lack of structure' within the relevant profile of personal preference orderings. Black himself proved that *single-peakedness* (jointly with the (harmless) technical condition that the number of individuals is odd) is a *sufficient* condition for the existence of Condorcet winners, and a well-known corollary of Black's insight is that Arrow's impossibility result (1951) fails to hold if the domain of admissible preference profiles is restricted to the set of all single-peaked profiles of personal preference orderings. Later on, several other conditions of preference structuration were proposed, for example Inada's condition of *single-cavedness* (1964), Inada's condition of *separability into two groups* (1964), Ward's condition of *latin-square-lessness* (1965), and Sen's condition of (triple-wise) *value-restriction* (1966/1982). The most demanding condition -- a limiting case -- of preference structuration is the condition of *unanimity*, i.e. the condition that all personal preference orderings in a relevant profile be identical. Triple-wise value-restriction can be shown to be the most general of all these structuration conditions (for an account of how the different conditions are logically interrelated, see Sen, 1966/1982). Moreover, each of these conditions shares with single-peakedness the property that it is *sufficient*, though not *necessary*, for the existence of Condorcet winners, for avoiding Arrow's impossibility result and, as easily provable, for avoiding the Gibbard-Satterthwaite result on strategic manipulability (Gibbard, 1973; Satterthwaite, 1975).

Apart from pointing towards possible escape-routes from social-choice-theoretic impossibility problems, the concept of preference structuration provides a formal framework for addressing the question of how much 'pluralism' or 'consensus' is contained in a profile of personal preference orderings. In particular, the concept of structuration enables us to say not merely that a profile consists of identical orderings and thus reflects agreement, or that it consists of a diverse range of orderings and thus reflects disagreement, but the concept of structuration also provides the language for a more fine-grained analysis of the *level* and *type* of agreement or disagreement reflected in a profile. The condition of single-peakedness, for instance, enables us to distinguish between (what we might call) *agreement/disagreement at a substantive level* and *agreement/disagreement at a meta-level*. Agreement at a substantive level would be the

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case of identical personal preference orderings. Disagreement at a substantive level would be the case of diverse personal preference orderings. Agreement at a meta-level would be the case of a single-peaked profile of personal preference orderings; if a profile is single-peaked, then all orderings contained in that profile can be systematically aligned along a common 'structuring dimension'. Disagreement at a meta-level, finally, would be the case of non-single-peakedness and thus the absence of a common 'structuring dimension'. Clearly, disagreement at a substantive level is compatible with agreement at a meta-level. In this terminology, agreement at a substantive level is not *necessary*, whilst agreement at a meta-level is *sufficient*, for successful collective decision making in accordance with Arrow's famous minimal conditions (i.e. transitivity of social orderings, the weak Pareto principle, independence of irrelevant alternatives, non-dictatorship, replacing universal domain with a suitable structuration condition).

These considerations should underline the analytic value of the concept of structuration, whether we are directly concerned with social-choice-theoretic issues, or whether our aim is primarily to analyse the level of 'pluralism' or 'consensus' or the 'number of issue-dimensions' contained in a profile of personal preference orderings. Theorists of democracy, for instance, may want to find out whether certain communication and deliberation processes can increase the level of preference structuration (e.g. Miller, 1992; Dryzek & List, 1999): the deliberative opinion polls conducted by Fishkin (1991) provide data about people's preferences both before and after a period of deliberation, but to do the requisite analysis operational measures of preference structuration are required.

The present note discusses two approaches to measuring the level of structuration in a profile of personal preference orderings. First we will discuss a simple and easily operationalizable method, which we will call the 'maximal structured component' approach. Then a more general, but computationally more demanding, method will be proposed, to be called the 'fractionalization' approach. The proposed method combines the above cited social-choice-theoretic criteria of structuration with the Rae-Taylor (1970) and Laakso-Taagepera (1979) approaches towards measuring the level of fractionalization, and the effective number of components, in a system (see also Taagepera & Grofman, 1981).

## 2. The 'Maximal Structured Component' Approach

Let $N = \{1, 2, ..., n\}$ be the relevant set of people, and let $X = \{x_1, x_2, ... x_k\}$ be the relevant set of alternatives. We assume that $n > 1$ and $k > 2$.

Let $R_i$ be person $i$'s personal preference ordering on $X$. $x_1 R_i x_2$ is interpreted to mean "from the perspective of person $i$, option $x_1$ is at least as good as option $x_2$". Each $R_i$ also induces a strong ordering $P_i$ and an indifference relation $I_i$, defined as follows:

$x_1 P_i x_2$ if and only if $x_1 R_i x_2$ and not $x_2 R_i x_1$
$x_1 I_i x_2$ if and only if $x_1 R_i x_2$ and $x_2 R_i x_1$

We shall say that an individual $i \in N$ is *concerned* with respect to a set of alternatives $Y \subseteq X$ if $i$ is not indifferent between *all* the alternatives in $Y$.

Let $\{R_i\}_{i \in N}$ be the corresponding profile of personal preference orderings across all people in $N$. Given a subset $M \subseteq N$, $\{R_i\}_{i \in M}$ is defined to be the sub-profile of $\{R_i\}_{i \in N}$ containing precisely the personal preference orderings of all members of $M$.

For each $M \subseteq N$, we can ask whether or not $\{R_i\}_{i\in M}$ satisfies a given structuration condition (S). (S) could be chosen to represent any of the above cited structuration conditions; we will here confine ourselves to formally defining *single-peakedness*, the most well-known one of these conditions, and *triplewise value-restriction*, the most general one.

If (S) is the condition of *single-peakedness*, then $\{R_i\}_{i\in M}$ satisfies (S) if there exists at least one strict linear ordering $W$ of the alternatives in $X$ (a 'dimension') such that, for all $i\in M$ and all $x, y, z \in X$ (such that $i$ is concerned with respect to $\{x, y, z\}$)[2], *if* ($xW\,y$ and $yW\,z$) or ($zW\,y$ and $yW\,x$) (i.e. "$y$ is 'between' $x$ and $z$ on the 'dimension' defined by $W$"), *then* $xR_iy$ implies $yP_iz$.

If (S) is the condition of *triplewise value-restriction*, then $\{R_i\}_{i\in M}$ satisfies (S) if, for each triple $x, y, z \in X$, there exists one alternative $\{x, y, z\}$ and one value in $\{best, worst, medium\}$ such that, for all $i\in M$ (such that $i$ is concerned with respect to $\{x, y, z\}$), the alternative does not have that value in the ordering $R_i$ (in the case of $R_i$ reflecting indifference, an alternative can have more than one value).

A simple and easily operationalizable measure of structuration can be developed on the basis of an idea that was first proposed by Niemi and Wright (1987). Niemi and Wright suggested measuring the "degree of unidimensionality" in a profile of personal preference orderings in terms of the "proportion of the preference orders [that] are consistent with single-peakedness".

Let $n_{max(S)}(\{R_i\}_{i\in N})$ be the maximal size of a subset $M$ of $N$ such that $\{R_i\}_{i\in M}$ satisfies (S). We shall call $\{R_i\}_{i\in M}$ a *maximal structured component* of $\{R_i\}_{i\in N}$. Our *primary index of structuration* is the quotient of $n_{max(S)}(\{R_i\}_{i\in N})$ to the size of $N$ (i.e. $n$). Formally, let

(1) $$i_{(S)}(\{R_i\}_{i\in N}) = \frac{max\{|M| : M\subseteq N \;\&\; \{R_i\}_{i\in M}\; satisfies\; (S)\}}{|N|}.$$

Then $i_{(S)}(\{R_i\}_{i\in N})$ is the maximal proportion of individuals in $N$ whose preference orderings are consistent with condition (S), for example, in the case of single-peakedness, whose preference orderings have no more than one peak with respect to a single common 'dimension' $W$. If (S) is the condition of single-peakedness, a simple algorithm of linear complexity in $n$ is available for computing $i_{(S)}(\{R_i\}_{i\in N})$.[3]

[2] If we omit (as in the standard definition of single-peakedness) the bracketed condition, condition (S) becomes more demanding. Moreover, if we use this more demanding variant of (S), a representation of the form (4) below may not always exist. This problem can be solved by allowing $N_r$ in (4) to be a 'residual' set for which $\{R_i\}_{i\in N_r}$ fails to satisfy (S). The same remark applies to the bracketed condition in our definition of triplewise value-restriction.

[3] The algorithm for determining $i_{single\text{-}peaked}(\{R_i\}_{i\in N})$ can be schematically summarized as follows:

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Begin;
    get the input N, X and {R_i}_{i∈N};
    define i_max := 0;
    for every logically possible strict ordering W of the alternatives in X do
        (define i_W := 0;
         for i:=1 to n do
             if R_i has at most one peak with respect to W then define i_W := i_W + 1;
         if i_W > i_max then define i_max := i_W);
    produce the output i_max/|N|;
end.
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Unless $i_{(S)}(\{R_i\}_{i\in N}) = 1$, it is not the case that the personal preference orderings of all members of $N$ are simultaneously consistent with (S). $i_{(S)}(\{R_i\}_{i\in N})$ itself is insensitive to the level of structuration amongst the preference orderings *outside* any maximal structured component of $\{R_i\}_{i\in N}$. Consider the following two cases:

(2) $N = N_1 \cup N_2$, where $N_1 \cap N_2 = \varnothing$ and $N_1, N_2 \neq \varnothing$ such that
  (i) $\{R_i\}_{i\in N_1}$ is a maximal structured component of $\{R_i\}_{i\in N}$, and
  (ii) $\{R_i\}_{i\in N_2}$ is a maximal structured component of $\{R_i\}_{i\in N \setminus N_1}$;

(3) $N = N_1 \cup N_2 \cup N_3$, where $i \neq j$ implies $N_i \cap N_j = \varnothing$ and $N_1, N_2, N_3 \neq \varnothing$ such that
  (i) $\{R_i\}_{i\in N_1}$ is a maximal structured component of $\{R_i\}_{i\in N}$ (and $N_1$ is chosen so as to maximize $n_{max(S)}(\{R_i\}_{i\in N \setminus N_1})$),
  (ii) $\{R_i\}_{i\in N_2}$ is a maximal structured component of $\{R_i\}_{i\in N \setminus N_1}$, and
  (iii) $\{R_i\}_{i\in N_3}$ is a maximal structured component of $\{R_i\}_{i\in (N \setminus (N_1 \cup N_2))}$.

In both cases $N$ can be partitioned into two or more (maximal) subsets for which the corresponding sub-profiles of $\{R_i\}_{i\in N}$ satisfy (S). We assume that the largest component of the partition has the same size in both cases, and hence $i_{(S)}(\{R_i\}_{i\in N})$ is the same in case (2) and case (3). Intuitively, however, case (3) exhibits less structure than case (2), and our primary index of structuration does not capture this intuition.

More generally, suppose that $N$ can be partitioned into a set of disjoint non-empty subsets such that, for each component of the partition, the corresponding sub-profile of $\{R_i\}_{i\in N}$ satisfies condition (S). Formally,

(4) $N = N_1 \cup N_2 \cup ... \cup N_r$, where $i \neq j$ implies $N_i \cap N_j = \varnothing$ and $N_1, N_2, ..., N_r \neq \varnothing$, such that, for each $j \in \{1, 2, ..., r\}$, $\{R_i\}_{i\in N_j}$ satisfies (S).

Then our primary index of structuration focuses only on the largest component of a partition of the form (4) and asks how large this largest component can possibly be.

One solution to this problem is to define *secondary, tertiary, ... indices of structuration* by focusing, respectively, on the second largest, third largest, ... components of a partition of the form (4) and asking how large these components can possibly be. To state these definitions formally, we must first identify a specific partition of the form (4) as 'maximal' in a relevant sense. Given the set of all partitions of $N$ of the form (4), we can define a lexical ordering on this set as follows: a partition $N = N_1 \cup N_2 \cup ... \cup N_r$ *lexically precedes* a partition $N = N'_1 \cup N'_2 \cup ... \cup N'_{r'}$ if only if the $r$-tuple $<|N_1|, |N_2|, ..., |N_r|>$ is lexically greater than the $r'$-tuple $<|N'_1|, |N'_2|, ..., |N'_{r'}|>$, where $<a_1, a_2, ..., a_r>$ is *lexically greater* than $<b_1, b_2, ..., b_{r'}>$ if and only if there exists $i \leq min(r, r')$ such that $a_i > b_i$ and, for all $j < i$, $a_i = b_i$. Now a partition $N = N_1 \cup N_2 \cup ... \cup N_r$ of the form (4) will be called a *maximal structured component partition* if it is not lexically preceded by any other partition of the form (4).

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Regarding the complexity of this algorithm, note that for computing $i_{single\text{-}peaked}(\{R_i\}_{i\in N})$ no more than $k!*n$ steps are required. (In fact, $(k!/2)*n$ steps are sufficient, since, for each pair of orderings that are mirror images of each other, it is sufficient to consider only one.)

In terms of a maximal structured component partition $N = N_1 \cup N_2 \cup ... \cup N_r$, our primary index of structuration satisfies

(5) $i_{(S)}(\{R_i\}_{i\in N}) = |N_1| / |N_1 \cup N_2 \cup ... \cup N_r| = |N_1| / |N|$.

Analogously, our *secondary index of structuration* can be defined to be

(6) $|N_2| / |N_2 \cup ... \cup N_r| = |N_2| / (|N|-|N_1|)$,

and a *tertiary index* can be defined to be

(7) $|N_3| / |N_3 \cup ... \cup N_r| = |N_3| / (|N|-|N_1|-|N_2|)$.

The secondary and tertiary indices are, respectively, the maximal proportion of individuals in $N$ but outside $N_1$ whose preference orderings are consistent with condition (S) and the maximal proportion of individuals in $N$ but outside $N_1 \cup N_2$ whose preferences orderings are consistent with (S).

Again, unless the secondary (tertiary) index equals 1, it is not the case that the personal preference orderings of *all* members of $N \setminus N_1$ (of $N \setminus (N_1 \cup N_2)$) are simultaneously consistent with (S), and the secondary (tertiary) index itself is insensitive to the level of structuration amongst those preference orderings that are in $\{R_i\}_{i\in N \setminus N_1}$ (in $\{R_i\}_{i\in N \setminus (N_1 \cup N_2)}$) but outside any maximal structured component of $\{R_i\}_{i\in N \setminus N_1}$ (of $\{R_i\}_{i\in N \setminus (N_1 \cup N_2)}$) In analogy to examples (2) and (3) above, examples can be constructed to illustrate that this insensitivity may be at odds with our intuitions about structuration. If we extend the sequence of primary, secondary, tertiary, ... indices further, the same insensitivity problem will recur at each stage in this process. Of course, it is possible to respond to this problem by considering this sequence in its entirety (if (S) is the condition of single-peakedness or any condition entailed by single-peakedness, the length of this sequence is bounded above by the number of different logically possible 'structuring dimensions', i.e. $k!/2$), and to use this sequence to define a *lexical ordering* on the set of all logically possible profiles of personal preference orderings. For each logically possible profile $\{R_i\}_{i\in N}$ we can consider the corresponding $r$-tuple <*primary index*, *secondary index*, ..., *r-ary index*>, where $r$ is chosen such that the $r$-ary index of structuration for $\{R_i\}_{i\in N}$ equals 1. We can then define a profile $\{R_i\}_{i\in N}$ to be more structured than a profile $\{R'_i\}_{i\in N}$ if the $r$-tuple corresponding to $\{R_i\}_{i\in N}$ is lexically greater than the $r'$-tuple corresponding to $\{R'_i\}_{i\in N}$. If giving *lexical priority* to maximal structured components conforms to our intuitions about preference structuration, this may be an acceptable -- and in practice easily operationalizable -- method of measuring structuration[4]. However, I will now propose another method, which applies well-known methods of measuring fractionalization to the problem of measuring preference structuration.

## 3. The 'Fractionalization' Approach

Given a partition of the form (4), the 'maximal structured component' approach always focuses on individual components of this partition and asks how large each of these

[4] An alternative way of avoiding the insensitivity problems of the 'maximal structured component' approach would be to define an index of structuration in terms of the ratio of the number of (non-empty) subsets $M \subseteq N$ for which $\{R_i\}_{i\in M}$ *satisfies* (S) to the number of logically possible (non-empty) subsets of $N$. However, a problem of this index might be its computational complexity: the number of logically possible (non-empty) subsets of $N$, i.e. $2^n$-1, grows exponentially with the number of persons in $N$. (I am indebted to Robert Luskin for this suggestion.)

components can possible be. But apart from the size each individual component, the number of components may be of interest too. Specifically, we may ask what the minimal number of components required to partition $N$ in the form (4) is, formally

(8) $d_{(S)}(\{R_i\}_{i\in N}) = min\{r : \exists N_1, N_2, ..., N_r \textit{ such that } (4) \textit{ is satisfied}\}$

(Again, if (S) is the condition of single-peakedness or any condition entailed by single-peakedness, this number is bounded above by $k!/2$).

If (S) is the condition of single-peakedness, $d_{(S)}(\{R_i\}_{i\in N})$ could be interpreted as the 'formal dimensionality' or 'number of formal structuring dimensions' of the preference profile $\{R_i\}_{i\in N}$.

However, while $d_{(S)}(\{R_i\}_{i\in N})$, unlike $i_{(S)}(\{R_i\}_{i\in N})$, is sensitive to the (minimal) *number* of components of a partition of the form (4), it is insensitive to the *size* of the different components.

To define a measure of structuration which combines both desired sensitivities, we shall use the Rae-Taylor (1970) and Laakso-Taagepera (1979) methods of measuring the level of fractionalization, and the effective number of components, in a system.

The intuition underlying these methods can be introduced as follows: the students at an educational institutions may have a different perception of the average class size from the administration. For a total of $|N| = n$ students partitioned into $r$ disjoint non-empty classes, $N_1, N_2, ..., N_r$, the administration perceives an average class size of ${}^1/_r(|N_1| + |N_2| + ... + |N_r|)$, while the students perceive an average class size of $(|N_1|*|N_1| + |N_2|*|N_2| + ... + |N_r|*|N_r|) / (|N_1| + |N_2| + ... + |N_r|) = {}^1/_n(|N_1|^2 + |N_2|^2 + ... + |N_r|^2)$, since, for each $j \in \{1, 2, ..., r\}$, a class size of $|N_j|$ is perceived by the $|N_j|$ students in class $N_j$ (see Taagepera & Grofman, 1981). The former average, the *arithmetic mean*, is highly sensitive to the total number of classes in a partition and thus to the addition or removal of tiny, residual components. Operationally, the arithmetic mean may therefore lack robustness in the light of a potential empirical uncertainty regarding such components. By contrast, the latter average, the *grass-roots average*, has the characteristic of weighting each component according to its size and is therefore relatively robust with regard to changes in a partition affecting only small components. It is this combination of a sensitivity to the size and number of components in a system with a robustness to changes affecting affect only very small components that makes the grass-roots average theoretically and operationally attractive as a basis for measuring fractionalization.

Formally, given a partition of $N$ into $r$ disjoint non-empty classes, $N_1, N_2, ..., N_r$, the grass-roots average of the component size is

(9) ${}^1/_{|N|}\sum_{j \in \{1, 2, ..., r\}}|N_j|^2$,

and the grass-roots average of the fractional shares of the $r$ components, also called the *Herfindahl-Hirschmann index of concentration*, is

(10) $\mathrm{HH} = \sum_{j \in \{1, 2, ..., r\}}(|N_j|/|N|)^2$,

an index that ranges from arbitrarily close to zero (for extremely low concentration) to one (for complete concentration).

The frequently used *Rae-Taylor index of fractionalization* is simply

(11) RT = 1 - HH,

ranging from zero (for a single-component partition) to arbitrarily close to one (for extreme fractionalization).

The *Laakso-Taagepera index of the effective number of components in a system* is based upon the intuition that the reciprocal of HH can be interpreted as the number of components that would be required to generate the given value of HH in a partition with perfectly equally sized components. Hence this reciprocal can be seen as a measure of the 'effective number' of components in the partition, formally

(12) LT = 1 / HH = 1 / (1 - RT).

Now the 'fractionalization' approach towards measuring preference structuration is based upon the idea that, for any partition of the form (4), we can immediately compute the corresponding values of HH, RT and LT. However, a profile of personal preference orderings, $\{R_i\}_{i\in N}$, does not determine a unique partition of the form (4). In order to reconstrue HH, RT and LT as meaningful measures of structuration, we therefore need to single out a 'privileged' partition of the requisite form. As in our definition of $d_{(S)}(\{R_i\}_{i\in N})$ above, the most obvious way to do this is to select a partition of the form (4) which *minimizes* the level of fractionalization according to HH, RT and LT. Formally, we can use HH to define an index of structuration as follows:

(13) $structure_{(S)}(\{R_i\}_{i\in N}) = max\{\sum_{j\in\{1, 2, ..., r\}}(|N_j|/n)^2 : N_1, N_2, ..., N_r \textit{ satisfy } (4)\}$.

In accordance with our intuitions, $structure_{(S)}(\{R_i\}_{i\in N}) = 1$ if and only if $\{R_i\}_{i\in N}$ satisfies condition (S). If the value of $structure_{(S)}(\{R_i\}_{i\in N})$ is close to zero, by contrast, this means that $\{R_i\}_{i\in N}$ is badly structured with respect to condition (S), in the sense that any partition of the form (4) would be highly fractionalized.

Similarly -- and in particular if (S) is the condition of single-peakedness or (S) has a suitable interpretation in terms of dimensionality --, we can interpret the effective number of components in a minimally fractionalized partition of the form (4), measured by HH or LT, as the 'effective number of formal dimensions' in the preference profile $\{R_i\}_{i\in N}$, formally

(14) $dim_{(S)}(\{R_i\}_{i\in N}) = min\{1 / [\sum_{j\in\{1, 2, ..., r\}}(|N_j|/n)^2] : N_1, N_2, ..., N_r \textit{ satisfy } (4)\}$.

Again, in accordance with our intuitions, $dim_{(S)}(\{R_i\}_{i\in N}) = 1$ if and only if $\{R_i\}_{i\in N}$ satisfies condition (S), i.e. if and only if $\{R_i\}_{i\in N}$ is perfectly 'unidimensional' according to condition (S).

A computational disadvantage of $structure_{(S)}(\{R_i\}_{i\in N})$ and $dim_{(S)}(\{R_i\}_{i\in N})$ may be that the number of logically possible partitions to be considered for determining the relevant maximal or minimal values of HH or LT, respectively, is potentially very large.

It is, however, possible to combine the 'fractionalization' approach with the basic idea of the 'maximal structured component' approach so as to construct computationally simplified (and intuitively appealing) variants of $structure_{(S)}(\{R_i\}_{i\in N})$ and $dim_{(S)}(\{R_i\}_{i\in N})$. If we attach intuitive significance to maximal structured component partitions, as defined in the section **2.**, then we can simply use the values of HH and LT for a maximal structured component partition as our measures of structuration and effective formal dimensionality in a profile $\{R_i\}_{i\in N}$.

(15) $structure'_{(S)}(\{R_i\}_{i\in N}) = \{\sum_{j\in\{1,2,...,r\}}(|N_j|/n)^2 : N = N_1 \cup N_2 \cup ... \cup N_r$ *is a maximal structured component partition with respect to* $\{R_i\}_{i\in N}\}$.

(16) $dim'_{(S)}(\{R_i\}_{i\in N}) = \{1 / [\sum_{j\in\{1,2,...,r\}}(|N_j|/n)^2] : N = N_1 \cup N_2 \cup ... \cup N_r$ *is a maximal structured component partition with respect to* $\{R_i\}_{i\in N}\}$.

An algortithm of linear complexity in $n$ is available for computing $structure'_{(S)}(\{R_i\}_{i\in N})$ and $dim'_{(S)}(\{R_i\}_{i\in N})$.[5]

Clearly, $structure'_{(S)}(\{R_i\}_{i\in N}) \leq structure_{(S)}(\{R_i\}_{i\in N})$ and $dim'_{(S)}(\{R_i\}_{i\in N}) \geq dim_{(S)}(\{R_i\}_{i\in N})$. An open question to be pursued further is what, for each choice of condition (S), the precise relation between $structure'_{(S)}(\{R_i\}_{i\in N})$ and $structure_{(S)}(\{R_i\}_{i\in N})$ (and, equivalently, between $dim'_{(S)}(\{R_i\}_{i\in N})$ and $dim_{(S)}(\{R_i\}_{i\in N})$) is. However, to the extent to which we regard maximal structured component partitions as 'privileged', we may regard $structure'_{(S)}(\{R_i\}_{i\in N})$ and $dim'_{(S)}(\{R_i\}_{i\in N})$ as attractive solutions to the problem of measuring preference structuration.

## 4. Concluding Remarks

As indicated at the beginning of this note, there are at least two substantive motivations for designing formal measures of preference structuration: one such motivation is a concern with the practical significance of social-choice-theoretic impossibility problems, and a second motivation is a more general concern with questions of 'pluralism', 'consensus' and 'issue-dimensionality'. While the proposed measures of preference structuration may be useful from the perspective of both motivations, two caveats need to be added, the first regarding the social-choice-theoretic significance of the proposed measures, and the second regarding their usefulness for capturing the ideas of 'pluralism', 'consensus' and 'issue-dimensionality'.

First, given the admissible choices for condition (S), we know that $structure'_{(S)}(\{R_i\}_{i\in N}) = 1$ (or, equivalently, $dim'_{(S)}(\{R_i\}_{i\in N}) = 1$) is a *sufficient* (but not *necessary*) condition for avoiding the standard social-choice-theoretic impossibility results. Intuitively, we also expect the frequency of cycles and social-choice-theoretically 'problematic' preference configurations to decrease with an increase in $structure'_{(S)}(\{R_i\}_{i\in N})$ (or, equivalently, with a decrease in $dim'_{(S)}(\{R_i\}_{i\in N})$). However, to test this intuition, further work is required. Gehrlein (1983) computed the proportion of profiles without a Condorcet winner amongst the set of all logically possible profiles of personal preference orderings, given fixed values of $n$ and $k$. Gehrlein's research design could be modified to compute, for each interval $[a, b] \subseteq [0, 1]$ (and fixed values of $n$ and $k$), the proportion of 'problematic' profiles amongst the set of all profiles

[5] Note that, if (S) is the condition of single-peakedness, the algorithm stated in note (3) can be modified to determine a maximal structured component partition of $N$, namely in the following way. Using a variant of this algorithm, $(k!/2)*n$ steps are sufficient to determine all potential choices of $N_1$ (such that $|N_1|$ is maximal). There are *at most* $k!/2$ potential choices of $N_1$ (corresponding to the $k!/2$ possible 'dimensions' $W$). For each potential choice of $N_1$, using our algorithm again, $(k!/2)*(n-|N_1|)$ steps are sufficient to determine all potential choices of $N_2$ (such that $|N_2|$ is maximal). Moreover, given a fixed $N_1$, there are *at most* $(k!/2)$-1 potential choices of $N_2$ (corresponding to the $(k!/2)$-1 remaining 'dimensions' $W$). Next, for each potential choice of $N_2$, $(k!/2)*(n-|N_1|-|N_2|)$ steps are sufficient to determine all potential choices of $N_3$, and, given fixed $N_1$ and $N_2$, there are *at most* $(k!/2)$-2 potential choices of $N_3$. Continuing, we find that *no more than* $(1+(k!/2)!)*((k!/2)*n)$ steps are required to determine a maximal structured component partition $N = N_1 \cup N_2 \cup ... \cup N_r$.

$\{R_i\}_{i\in N}$ such that *structure'*$_{(S)}(\{R_i\}_{i\in N}) \in [a, b]$ (in these terms, Gehrlein's work can be interpreted as the special case in which $[a, b] = [0, 1]$). Using this design, we could ask whether higher levels of structuration according to *structure'*$_{(S)}(\{R_i\}_{i\in N})$ are indeed conducive to solving social choice problems.

Secondly, suppose that we have identified a partition $N = N_1 \cup N_2 \cup ... \cup N_r$ that satisfies (4), possibly a maximal structured component partition. This means that, for each $j$, $\{R_i\}_{i\in N_j}$ satisfies the structuration condition (S). For instance, if (S) is the condition of single-peakedness, this means that, for each $j$, there exists a common 'dimension' $W$ with respect to which the preference orderings of everyone in $N_j$ have no more than one peak. However, this is first and foremost a purely *formal* statement about the 'geometrical' representability of the preferences of each person in $N_j$. In particular, such a formal statement does not entail anything about how each person in $N_j$ *actually conceptualizes* (his or her preference over) the options. For instance, there typically exists more than one 'dimension' $W$ with respect to which a given personal preference ordering has no more than one peak. This means that, while the observation that a sub-profile $\{R_i\}_{i\in M}$ of $\{R_i\}_{i\in N}$ satisfies condition (S) can be taken as a formal *symptom* of something like 'agreement at a meta-level', it may ultimately need to be supplemented with suitable semantic considerations, in order to give a substantive (and not merely formal) account of the level of 'consensus' or 'pluralism' within the relevant group of people. In the case of single-peakedness, asking whether a 'structuring dimension' $W$ has an obvious interpretation or rationalization may be a good starting-point. In particular, the larger the number of people whose preference orderings are structured by the same formal 'frame' (by virtue of satisfying condition (S)), the more plausibly we might try explain this common structure in 'semantic' terms.

Undoubtedly, further work is required. But this note hopes to emphasize the interest of social-choice-theoretic concepts for formally approaching issues of 'pluralism', 'consensus' and 'issue-dimensionality' and for devising suitable operationalizable measures of preference structuration.

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