Identification of the age-period-cohort model and the extended chain ladder model

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Summary

In this paper, we consider the identification problem arising in the age-period-cohort models, as well as in the extended chain ladder model. We propose a canonical parametrization based on the accelerations of the trends in the three factors. This parametrization is exactly identified. It eases interpretation, estimation and forecasting. The canonical parametrization is shown to apply for a class of index sets which have trapezoid shapes, including various Lexis diagrams and the insurance reserving triangles.
Some key words: Age-period-cohort model; Chain-ladder model; Identification.

1 Introduction

Consider the age-period-cohort model used in epidemiology and demography. It describes the logarithm of the mortality in an additive form, involving three interlinked time scales

$$\mu_{ij} = \alpha_i + \beta_j + \gamma_{i+j-1} + \delta, \tag{1}$$

where i is the cohort, j is the age, and $i+j-1$ is the period. The indices i, j vary bivariately in an index set $I \in N^2$. The parameters of the model $\alpha_i, \beta_j, \gamma_{i+j-1}, \delta$ describe the trends of the three factors in the model. It has long been appreciated that this parametrization is not identified. Holford (1983) therefore used generalized inverses when solving maximum likelihood equations, remarking that the choice of generalized inverse can have a large effect on the parameter estimates. A similar solution has implicitly been used in the insurance literature, see Zehnwirth (1994). Clayton & Schifflers (1987) suggested that the ratios of the relative risks are identifiable. On a logarithmic scale, they are the second differences, which will be the key ingredient in this paper. Carstensen (2007) represented the variation of the parameterization of (1) by adding and subtracting linear trends from $\alpha_i, \beta_j, \gamma_{i+j-1}, \delta$, which relate to a group theoretic description of the identification suggested here. He also pointed out that an ideal parametrization should be simple in both estimation and computation.

In this paper, we revisit the identification problem. We propose a canonical parametrization which includes the identifiable second differences suggested by Clayton & Schifflers (1987), and prove that it has an 1-1 correspondence with μ_{ij}, for all $i, j \in I$. It will be shown that the interpretation of such a parametrization is straight forward, and its design matrix can easily be deduced.

We shall consider the three leading cases of the age-period-cohort model related to the Lexis diagram as discussed by Keiding (1990). In the terminology of Keiding, the first principal set of dead is data from certain cohorts that die within a given age range. This is where the indices vary in an age-cohort rectangle. The other two cases are when the indices vary in an age-cohort trapezoid. The second principal set of dead studies the deaths of certain cohorts in a given period as in a longitudinal study. The third principal set of dead studies the death within a certain period and between a given age range as in a repeated cross-sectional study.

We shall also consider the extended chain ladder model used for reserving in non-life insurance. The issue in reserving is that claims relating to a given accident year may be reported many years after the accident. Thus, the available data in any given calendar year k is a simplex of size k with claims indexed by their accident year and by their reporting or development year. The accident year and the development year add up to the calendar year plus one. This simplex is referred to as a run-off triangle. The classical chain ladder model, discussed by for instance England and Verrall (2002), involves only two time scales relating to the accident and the development year. An extended chain ladder model parametrized using three time scales as in (1) has been introduced by Zehnwirth (1994) and Barnett & Zehnwirth (2000).

Initially in §2, we start by considering the identification problem in the simple case, where the indices i, j vary in a square. This corresponds to the 'first principal set of dead'. We shall establish a canonical parametrization ξ for models with such an index set. It will be seen that ξ is given by the second differences of α_i, β_j, γ_{i+j-1} and the three corner points μ_{11}, μ_{21}, μ_{12}. §3 shows the three corner points can be replaced by other points. §4 then extends this work to more general index sets, including the 'second and third principal sets of dead' and the insurance run-off triangles.

2 Identification for square index sets

Consider a simple square index set given by Definition 1. In this section, we propose a canonical parametrization for model (1) for this situation.

Definition 1 *I is a* ***square index set*** *if for some $k \in N$,*

$$I = \{(i, j); i, j = 1, \ldots, k\}. \tag{2}$$

For a square index set, the parameters of (1) are

$$\theta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_{2k-1}, \delta) \in R^{4k}. \tag{3}$$

Now, let $\mu = \{\mu_{ij} : (i, j) \in I\}$ as given by (1). The map from θ to μ is surjective, but not injective. As pointed out by Carstensen (2007), linear trends in α_i, β_j, γ_{i+j-1} can be added without changing the value of μ_{ij}. This can be phrased by θ being over-parameterized.

Clayton & Schifflers (1987) worked with a multiplicative formulation of (1) and suggested that ratios of ratios of the parameters would be invariant. In the linear setup (1), the linear trends can correspondingly be removed from α_i, β_j, γ_{i+j-1} by taking second differences like $\Delta^2\alpha_i = \alpha_i - 2\alpha_{i-1} + \alpha_{i-2}$. To generate a canonical parametrization ξ, we rewrite (1) in terms of these second differences, and three initial points. This can be done by introducing the telescopic sums $\alpha_i = \alpha_1 + \sum_{t=2}^{i} \Delta\alpha_t$ and $\Delta\alpha_t = \Delta\alpha_2 + \sum_{s=3}^{t} \Delta^2\alpha_s$, thus

$$\alpha_i = \alpha_1 + (i-1)\Delta\alpha_2 + \sum_{t=3}^{i}\sum_{s=3}^{t} \Delta^2\alpha_s,$$

Substitute this expression for α_i and similar expressions for β_j and γ_{i+j-1} into (1). Writing $\triangle\alpha_2 + \triangle\gamma_2 = \mu_{21} - \mu_{11}$ and $\triangle\beta_2 + \triangle\gamma_2 = \mu_{12} - \mu_{11}$, we get

$$\mu_{ij} = \mu_{11} + (i-1)(\mu_{21} - \mu_{11}) + (j-1)(\mu_{12} - \mu_{11}) + a_{ij}, \tag{4}$$

for all $i, j \in I$, where the term a_{ij} is given by

$$a_{ij} = \sum_{t=3}^{i}\sum_{s=3}^{t} \Delta^2\alpha_s + \sum_{t=3}^{j}\sum_{s=3}^{t} \Delta^2\beta_s + \sum_{t=3}^{i+j-1}\sum_{s=3}^{t} \Delta^2\gamma_s.$$

The expression for a_{ij} can be simplified further by exchanging the double sums, for instance, $\sum_{t=3}^{i}\sum_{s=3}^{t} \Delta^2\alpha_s$ equals $\sum_{s=3}^{i} \Delta^2\alpha_s(i-s+1)$, thus

$$a_{ij} = \sum_{s=3}^{i} \Delta^2\alpha_s(i-s+1) + \sum_{s=3}^{j} \Delta^2\beta_s(j-s+1) + \sum_{s=3}^{i+j-1} \Delta^2\gamma_s(i+j-s). \tag{5}$$

Based on formula (4), we define a parameter vector $\xi \in R^{4k-4}$ as

$$\xi = (\mu_{11}, \mu_{21}, \mu_{12}, \Delta^2\alpha_3, \ldots, \Delta^2\alpha_k, \Delta^2\beta_3, \ldots, \Delta^2\beta_k, \Delta^2\gamma_3, \ldots, \Delta^2\gamma_{2k-1}). \tag{6}$$

Theorem 1 shows that ξ gives a unique parameterization of μ. We therefore call it a canonical parameter.

For estimation purpose, a design matrix for the canonical parameter ξ can be deduced from (4). In the case of a square index set, I of dimension $k = 3$, the design matrix is given by

$$\begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \\ \mu_{22} \\ \mu_{13} \\ \mu_{31} \\ \mu_{23} \\ \mu_{32} \\ \mu_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 & 1 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mu_{11} \\ \mu_{21} - \mu_{11} \\ \mu_{12} - \mu_{11} \\ \Delta^2\alpha_3 \\ \Delta^2\beta_3 \\ \Delta^2\gamma_3 \\ \Delta^2\gamma_4 \\ \Delta^2\gamma_5 \end{pmatrix}.$$

Theorem 1 shows that ξ is unique in general. The uniqueness of ξ implies that the design matrix has full column rank, this can be checked by inspection when $k = 3$. The proof is provided in the Appendix.

Theorem 1 *Let $\mu = \{\mu_{ij}; (i,j) \in I\}$, where I is a square index set, and μ_{ij} satisfies (1). The parametrization ξ given by (6) satisfies*

(1) ξ is a function of θ.

(2) μ is a function of ξ, due to (4).

The parametrization of μ by ξ is exactly identified in that $\xi^\dagger \neq \xi^\ddagger$ implies $\mu(\xi^\dagger) \neq \mu(\xi^\ddagger)$.

The result could also be cast in terms of group theoretic arguments. We define as in Carstensen (2007) the group g as

$$g : \begin{pmatrix} \alpha_i \\ \beta_j \\ \gamma_k \\ \delta \end{pmatrix} \mapsto \left\{ \begin{array}{c} \alpha_i + a + (i-1)d \\ \beta_j + b + (j-1)d \\ \gamma_{i+j-1} + c - (i+j-2)d \\ \delta - a - b - c \end{array} \right\}, \tag{7}$$

where a, b, c, d are arbitrary constants. The parameter μ is a function of θ, which is invariant to g, that is, $\mu(\theta) = \mu\{g(\theta)\}$. Using invariance arguments as those in Cox & Hinkley (1974, §5.3), Theorem 1 shows that ξ is a maximal invariant function of θ under g.

The assigned parameter θ can be constructed from ξ using (7). For instance, if we choose $\alpha_1 = \beta_1 = \gamma_1 = \gamma_2 = 0$, then θ can be computed from ξ as

$$\begin{aligned} \alpha_i &= (i-1)(\mu_{21} - \mu_{11}) + \sum_{t=3}^{i} \sum_{s=3}^{t} \Delta^2 \alpha_s, \\ \beta_j &= (j-1)(\mu_{12} - \mu_{11}) + \sum_{t=3}^{j} \sum_{s=3}^{t} \Delta^2 \beta_s, \\ \gamma_{i+j-1} &= \sum_{t=3}^{i+j-1} \sum_{s=3}^{t} \Delta^2 \gamma_s, \\ \delta &= \mu_{11}. \end{aligned}$$

Formula (4) shows that these components add up to μ_{ij}. If other values for $\alpha_1, \beta_1, \gamma_1, \gamma_2$ are desired, corresponding linear trends can be added as set out in (7) by choosing appropriate values of levels a, b, c and slope d.

Since we can choose a, b, c, d arbitrarily, interpretation of the original parameters α_i, β_j, γ_{i+j-1} is difficult. The visual impression of the parameters α_i, β_j, γ_{i+j-1} will depend on the choice of a, b, c, d. For instance, by varying d a figure of the α_i-parameters can appear to be increasing or decreasing. Correspondingly the level, and hence the sign of the first differences $\Delta\alpha_i$ is arbitrary. The second differences $\Delta^2\alpha_i$, $\Delta^2\beta_j$, $\Delta^2\gamma_{i+j-1}$ do however, have a unique interpretation. The interpretation of such second differences, or accelerations, is standard in time series analysis. Likewise, any forecasting can be done more safely on the second differences rather than the levels. In applications, it would therefore be helpful to make graphs of the second differences.

In some applications the components α_i, β_j, γ_{i+j-1}, and δ themselves are not all that important, whereas the original parameters μ_{ij} are of main interest. Plots of the parameters μ_{ij} will be meaningful as μ_{ij} is a function of ξ via (4) and (5), and it is therefore identified. An example is when the object of interest is to forecast how many children there will be in different grades in the school system in the year 2010. In that case let i be cohort, j age, and plot μ_{ij} as a function of either age or cohort such that the period is $i + j - 1 = 2010$. Other examples could be how mortality of people of age 80 vary in terms of either the period or the cohort or how mortality of people born in 1930 vary in terms of either period or age. Similarly, in insurance the intrisic issue is to predict outstanding claims relating to a given accident year rather than to forecast the calendar parameters, γ_k say.

3 The Role of initial points

The choice of canonical parametrization is not unique. Any bijective mapping of ξ would also identify μ exactly. In particular, the three initial points in ξ given by (6) can be replaced by another set of three points without changing the content of Theorem 1.

The argument for changing the initial points is based on a manipulation of equation (4). It is convenient to introduce the matrix notation

$$Y = \begin{pmatrix} \mu_{i_1 j_1} \\ \mu_{i_2 j_2} \\ \mu_{i_3 j_3} \end{pmatrix}, \quad X = \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \end{pmatrix}, \quad A = \begin{pmatrix} a_{i_1 j_1} \\ a_{i_2 j_2} \\ a_{i_3 j_3} \end{pmatrix}, \quad B = \begin{pmatrix} b_{i_1 j_1} \\ b_{i_2 j_2} \\ b_{i_3 j_3} \end{pmatrix},$$

with a_{ij} as in formula (5) and $b_{ij} = (1, i-1, j-1)$. With this notation, it holds from (4) that

$$Y = BX + A, \tag{8}$$

which can be solved for X when B is invertible. We find that B is invertible when $\det(B) = i_2 j_3 - i_3 j_2 + i_3 j_1 - i_1 j_3 + i_1 j_2 - i_2 j_1$ is different from zero. As a consequence, X can be replaced by Y, which gives a new canonical parameter

$$\xi = (\mu_{i_1 j_1}, \mu_{i_2 j_2}, \mu_{i_3 j_3}, \Delta^2\alpha_3, \ldots, \Delta^2\alpha_k, \Delta^2\beta_3, \ldots, \Delta^2\beta_k, \Delta^2\gamma_3, \ldots, \Delta^2\gamma_{2k-1}). \tag{9}$$

The following Corollary to Theorem 1 holds.

Corollary 1 *Suppose μ_{ij} satisfies (1) on a square index set I and consider the parameter ξ given by (9). If the matrix B is invertible, then the conclusions of Theorem 1 remains true.*

A design matrix can be constructed from ξ as given by (9). This is done by combining (8) and (4). This shows that for all $(i, j) \in I$, it holds that

$$\mu_{ij} = b_{ij} X + a_{ij} = b_{ij} B^{-1} Y + (a_{ij} - b_{ij} B^{-1} A), \tag{10}$$

which is a linear function of ξ as defined in (9). The inverse of the matrix B is given by

$$B^{-1} = \frac{1}{\det(B)} \left\{ \begin{array}{ccc} (i_2-1)(j_3-j_2) & (i_1-1)(j_1-j_3) & (i_1-1)(j_2-j_1) \\ -(i_3-i_2)(j_2-1) & -(i_1-i_3)(j_1-1) & -(i_2-i_1)(j_1-1) \\ & & \\ j_2 - j_3 & j_3 - j_1 & j_1 - j_2 \\ & & \\ i_3 - i_2 & i_1 - i_3 & i_2 - i_1 \end{array} \right\}.$$

4 Identification for general index sets

In many situations, the parameterization (1) has an index set which is not a square as considered in §2. For instance, I can be a parallelogram in a Lexis diagram, or a simplex in an insurance run-off triangle. More generally, I could be any irregular shape, with one or more missing points. It is therefore useful to construct a canonical parametrization for (1) with a non-square index set.

A convenient generalization of the square index sets are index sets of rectangular shapes, where the period $i+j-1$ can be constrained to a certain interval. We will call such index sets generalized trapezoids and give a precise definition below. Working with such index sets it is immediately clear how to define a canonical parametrization from knowing the dimensions of the generalized trapezoid. The generalized trapezoid covers the most important situations encountered in practice, that is the three types of Lexis diagrams and the insurance run-off triangle.

Definition 2 *The index set I is a* ***generalized trapezoid*** *if for some $l, k, m \in N$, $h \in N_0$, and $h + m \leq l + k - 1$, then*

$$I = \{(i, j); i = 1, \ldots, k, j = 1, \ldots, l, i + j - 1 = h + 1, \ldots, h + m\}. \quad (11)$$

In the following, we illustrate with diagrams some applications of the general trapezoid. Fig. 1(a,b,c) shows examples of the three types of Lexis diagrams discussed by Keiding (1990), where the age and cohort add up to the period. The first principal set of dead gives a rectangular index set, whereas the second and third principal sets of dead are trapezoids. Fig. 1(d) gives an example of an insurance run-off triangle as discussed by Zehnwirth (1994) and Barnett & Zehnwirth (2000).

[Figure 1 about here]

For every generalized trapezoid I, we define the canonical parameter ξ from the dimensions l, k, m, h by restricting the three time scales. That is,

$$\xi = (\mu_{i_1,j_1}, \mu_{i_2,j_2}, \mu_{i_3,j_3}, \Delta^2\alpha_3, \ldots, \Delta^2\alpha_k, \Delta^2\beta_3, \ldots, \Delta^2\beta_l, \Delta^2\gamma_{h+3}, \ldots, \Delta^2\gamma_{h+m}), \quad (12)$$

where $(i_1, j_1), (i_2, j_2), (i_3, j_3) \in I$ and satisfy $\det(B) \neq 0$, the following corollary to Theorem 1 then holds.

Corollary 2 *Suppose μ_{ij} satisfies (1) on a generalized trapezoid index set I and consider the parameter ξ given by (9). If the matrix B is invertible, then Theorem 1 remains true.*

Corollary 2 is proved by analyzing formula (10), by showing that the terms $\Delta^2\gamma_3, \ldots, \Delta^2\gamma_{h+2}$ are not needed. To see this, we isolate the $\Delta^2\gamma$ terms in a_{ij} of (10). It is then seen that $\Delta^2\gamma_s$ with index $s < h+3$ is weighted by

$$w_s = (i + j - s) - b_{ij}B^{-1}(i_1 + j_1 - s, i_2 + j_2 - s, i_3 + j_3 - s)'.$$

The last vector can easily be written in terms of the matrix B giving

$$w_s = (i + j - s) - (1, i - 1, j - 1)B^{-1}B(2 - s, 1, 1)' = 0.$$

A design matrix can be constructed from ξ as given by (12). As in §3, this is done directly from the formula (10). The design matrix has a number of zero elements, for instance, we can show that $\Delta^2\gamma_{h+3}, \ldots, \Delta^2\gamma_{h+p}$, with $p \geq 3$ has weight zero if $i_1 + j_1 - 1, i_2 + j_2 - 1, i_3 + j_3 - 1$ and also $i + j - 1$ are all larger than $h + p$ following similarly procedure as the proof of Corollary 2.

Corollary 2 gives a sufficient condition only for the type of index sets where ξ in (12) is a canonical parameterization. Fig. 2(a), is an example of an index set which is not a generalized trapezoid. Fig. 2(b) shows an extended index set which is a generalized trapezoid. Corollary 2 gives a canonical parameter ξ for the latter set. This parameter ξ is also a canonical parameter for the original set. To see this, we decompose ξ into elements ξ_6 say, related to the 3 by 3 simplex and the second differences $\Delta\beta_4$, $\Delta\gamma_4$, $\Delta\gamma_5$. It turns out that there is a bijective mapping from those three elements to μ_{14}, μ_{24}, μ_{33}, which can be formulated as

$$\begin{pmatrix} \mu_{14} \\ \mu_{24} \\ \mu_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \Delta^2\beta_4 \\ \Delta^2\gamma_4 \\ \Delta^2\gamma_5 \end{pmatrix} + f(\xi_6),$$

where $f(\cdot)$ is some functions of ξ_6. The design matrix here has rank 3. However, if any one of the three points μ_{14}, μ_{24}, μ_{33} is missing, ξ would be over-parameterized.

[Figure 2 about here]

In the above example, a canonical parameter for a general index set was found by extending the set to a generalized trapezoid. This strategy will, however, not work in general. Fig. 2(c,d) shows some simple examples. In the first example, the index set has four points. By adding the point μ_{22}, a generalized trapezoid is found with canonical parameter ξ of dimension 5. This ξ over-parameterized the original set. In the second example, the same is seen when adding μ_{12}.

5 Discussion

In this paper, we have established a canonical parametrization ξ given by (12) for the age-period-cohort models and for the extended chain ladder model. The canonical parameter ξ is based on the identifiable second differences. It provides a base for easy estimation and forecasting.

While the canonical parameter is indeed unique, its interpretations is in terms of accelerations, which can be somewhat complicated to communicate. The level parameters α_i, β_j, and γ_{i+j-1} are not unique, and plots of them can be visually misleading as they evolve around arbitrarily chosen linear trends. In some applications one could instead communicate plots of the original parameter μ_{ij} for a fixed value of either i, j, or $i+j-1$.

Theorem 1 and Corollaries 1, 2 provide sufficient conditions on the type of index sets for which exact identification can be achieved. Fig. 2 (a,b) shows that identification can also be achieved for a more general class of somewhat irregular index sets.

Another sufficient condition for the permissible index sets can be based on a recursive argument. First, find a set $I_k \subset I$, which is a generalized trapezoid with canonical parameter ξ_k. A point $(i,j) \in I \setminus I_k$ can be added to I_k if it introduces at most one double difference that is not in ξ_k. Thus $I_{k+1} = \{I_k \bigcup (i,j)\} \subseteq I$ is exactly identified by ξ_{k+1}. It should be noted that Fig. 2(a) is an example, which cannot be obtained by this one-step recursive scheme. This set is obtained by adding three points $(1,4), (2,4), (3,3)$ to the identifiable 3 by 3 simplex.

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Appendix

Proof of Theorem 1

Theorem 1 is proven by induction using formula (4).

Trivial step: consider the first two diagonals, i.e. the initial three elements.

If one of $\mu^{\dagger}_{11} \neq \mu^{\ddagger}_{11}$, $\mu^{\dagger}_{12} \neq \mu^{\ddagger}_{12}$, or $\mu^{\dagger}_{21} \neq \mu^{\ddagger}_{21}$, then the statement is true.

Initial step: consider the third diagonal.

If $\mu^{\dagger}_{11} = \mu^{\ddagger}_{11}$, $\mu^{\dagger}_{12} = \mu^{\ddagger}_{12}$, $\mu^{\dagger}_{21} = \mu^{\ddagger}_{21}$, but $\Delta^2\gamma^{\dagger}_3 \neq \Delta^2\gamma^{\ddagger}_3$; then $\mu^{\dagger}_{22} \neq \mu^{\ddagger}_{22}$ by formula (4).

If $\mu^{\dagger}_{11} = \mu^{\ddagger}_{11}$, $\mu^{\dagger}_{21} = \mu^{\ddagger}_{21}$, $\mu^{\dagger}_{12} = \mu^{\ddagger}_{12}$, $\Delta^2\gamma^{\dagger}_3 = \Delta^2\gamma^{\ddagger}_3$, but $\Delta^2\beta^{\dagger}_3 \neq \Delta^2\beta^{\ddagger}_3$, or $\Delta^2\alpha^{\dagger}_3 \neq \Delta^2\alpha^{\ddagger}_3$, then $\mu^{\dagger}_{1,3} \neq \mu^{\ddagger}_{1,3}$, or $\mu^{\dagger}_{3,1} \neq \mu^{\ddagger}_{3,1}$ by formula (4).

Induction step: consider the diagonal $(r+1)$, where $r+1 = 4, \dots, 2k-1$.

Assume $\mu^{\dagger}_{11} = \mu^{\ddagger}_{11}$, $\mu^{\dagger}_{12} = \mu^{\ddagger}_{12}$, $\mu^{\dagger}_{21} = \mu^{\ddagger}_{21}$, and for $s = 3, \dots, r$, $\Delta^2\gamma^{\dagger}_s = \Delta^2\gamma^{\ddagger}_s$, $\Delta^2\beta^{\dagger}_s = \Delta^2\beta^{\ddagger}_s$, $\Delta^2\alpha^{\dagger}_s = \Delta^2\alpha^{\ddagger}_s$, but $\Delta^2\gamma^{\dagger}_{r+1} \neq \Delta^2\gamma^{\ddagger}_{r+1}$. Then $\mu^{\dagger}_{2,r} \neq \mu^{\ddagger}_{2,r}$ by formula (4).

We then can show $\mu^{\dagger}_{11} = \mu^{\ddagger}_{11}$, $\mu^{\dagger}_{12} = \mu^{\ddagger}_{12}$, $\mu^{\dagger}_{21} = \mu^{\ddagger}_{21}$, for $s = 3, \dots, r$, $\Delta^2\gamma^{\dagger}_s = \Delta^2\gamma^{\ddagger}_s$, $\Delta^2\beta^{\dagger}_s = \Delta^2\beta^{\ddagger}_s$, $\Delta^2\alpha^{\dagger}_s = \Delta^2\alpha^{\ddagger}_s$, $\Delta^2\gamma^{\dagger}_{r+1} = \Delta^2\gamma^{\ddagger}_{r+1}$, but $\Delta^2\beta^{\dagger}_{r+1} \neq \Delta^2\beta^{\ddagger}_{r+1}$, or $\Delta^2\alpha^{\dagger}_{r+1} \neq \Delta^2\alpha^{\ddagger}_{r+1}$, then we have $\mu^{\dagger}_{1,r+1} \neq \mu^{\ddagger}_{1,r+1}$, or $\mu^{\dagger}_{r+1,1} \neq \mu^{\ddagger}_{r+1,1}$ by formula (4).

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Fig. 1. Panels (a,b,c) show Lexis diagrams for first, second, and third, respectively principal set of dead. Panel (d) shows an insurance run-off triangle.

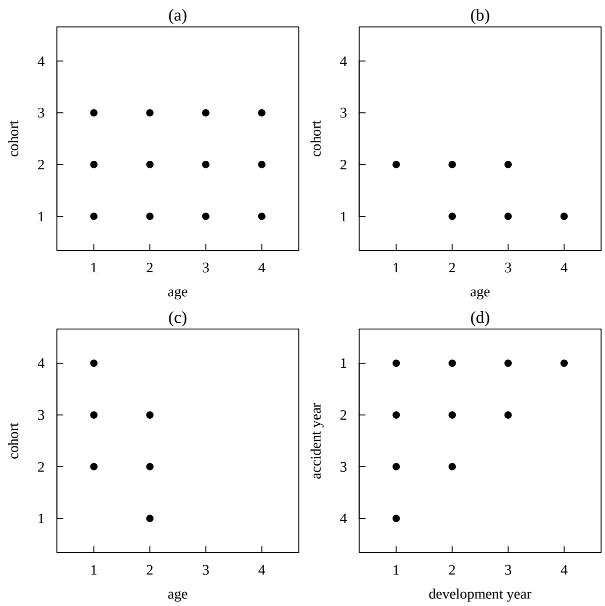

Fig. 2. Panel (a) shows an index set, which is not a generalized trapezoid. This set is extended to a generalized trapezoid in panel (b). The sets in (a) and (b) have the same canonical parameter. Panels (c,d) show examples of sets, which are not generalized trapezoids and where the associated generalized trapezoid has a canonical parameter of larger dimension that the set itself.