Causal transmission in reduced-form models

Vassili Bazinas & Bent Nielsen [1]
Department of Economics, University of Oxford

July 2015

Abstract

We propose a method to explore the causal transmission of a catalyst variable through two endogenous variables of interest. The method is based on the reduced-form system formed from the conditional distribution of the two endogenous variables given the catalyst. The method combines elements from instrumental variable analysis and Cholesky decomposition of structural vector autoregressions. We give conditions for uniqueness of the causal transmission.

1 Introduction

In general, it is difficult to deduce the causal ordering of two observed variables from their joint distribution. However, if we can assume that a third variable is causal, it may be possible to deduce how the effect of this third variable will transmit between the two variables of interest. By conditioning on a *catalyst*, the joint distribution of a bivariate system can be used to infer a *causal transmission*. Our approach allows for different catalysts transmitting through the same two variables in different ways. We formulate this for a general distributional setup.

Philosophers and scientists argue that some background of causal knowledge is required in order to construct new causal facts. The view "no causes in, no causes out" Cartwright (1989) expresses the concern that we cannot jump from theory to cause without some causal facts in hand. Pearl (2000) similarly underlines the importance of distinguishing between causal and associational concepts, as every causal conclusion relies on a causal assumption that is untested in observational studies. In contrast, Granger (1969) causality is an example of an associational concept seeking to infer correlations from data without a causal assumption. Causal analysis goes one step further by inferring correlations under changing conditions.

Our method can be thought of as unifying instrumental variable analysis and recursive ordering of structural vector autoregressions. Instrumental variable analysis will in general not order the endogenous variables. Rather we use it to identify a structural relation uniquely. Cholesky decomposition orders endogenous variables, but the ordering is not unique. By carrying out a Cholesky decomposition in the presence of an instrument there is scope for a unique ordering which is interpretable as a causal transmission. In this situation we will refer to the instrument as a catalyst.

The catalyst w may transmit causally through the variables y, z. It is possible that w transmits through z to y or through y to z or, of course, that there is no ordering of the variables. We present two sets of testable conditions. A first set of conditions are needed for establishing that the catalyst w transmits through z to y, say, in a unique

[1] B. Nielsen is supported by the Institute for Economic Modeling at the Oxford Martin School.

fashion. A second set of conditions are needed for showing that w actually affects y. The econometric framework is a reduced-form based on the conditional distribution of y, z given w. The theory is formulated for general densities but with special attention to the most common cases, which include the bivariate normal distribution and mixtures of a univariate normal distribution with a logit or probit distribution.

The ideas presented here are related to graphical modeling. Graphical model theory seeks to represent dependence structures of statistical models in terms of graphs. There are many quirks to statistical models and distribution theory, which in turn results in a variety of types of graphs; most are directed and some require normality, see Lauritzen (1996) and Wermuth and Sadeghi (2012) for details. Directed graphs and causality are closely related, see Cox and Wermuth (2004). However, the causal transmission idea presented here deviates from that literature in important ways. First, we are concerned with the discovery of transmission, so we postpone the use of directed graphs as far as possible. Secondly, we establish results for general densities. Thirdly, our causal transmission requires a variety of both (conditional) independence and dependence properties. As a result, we only make limited use of graphs as illustrations and carefully define the properties of those graphs that we use.

Our notion of causal transmission bears many similarities to super exogeneity as introduced in Engle et al. (1983). A variable z is super exogenous for y with respect to w under two conditions: first, a structural invariance property, which is by and large similar to our causal transmission; second, weak exogeneity of z for y, which is an estimation property. We do not need the estimation property, which gives a more flexible framework for studying causal transmission.

In §2 we define and explore causal transmissions. In §3 we generalize the idea to situations with multiple catalysts which transmit through the variables of interest in different ways and we offer a structural interpretation that combines Cholesky decomposition and instrumental variable estimation. An empirical illustration using a UK monetary data set follows in §4. Proofs are given in an appendix.

2 Causal transmission

Causal relations are asymmetric by nature: influence flows one way and cannot be reversed. We analyze a joint conditional probability model for two endogenous variables given a catalyst. We start by exploring ordering within a bivariate normal setup. Subsequently, we give results for unique ordering and non-trivial transmission in a general bivariate distribution setup. From this we define causal transmission.

2.1 The uniqueness problem

Suppose we are interested in an economic relationship between two endogenous, or modeled, variables (y, z) given a third variable w. The third variable w is deemed exogenous. Thus, we are interested in the conditional distribution $\mathsf{f}(y, z|w)$. Under normality the joint, conditional distribution is given by:

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \gamma_{yw} \\ \gamma_{zw} \end{pmatrix} w + \begin{pmatrix} \epsilon_y \\ \epsilon_z \end{pmatrix}, \tag{2.1}$$

where the innovations are normally distributed, with positive definite variance:

$$\begin{pmatrix} \epsilon_y \\ \epsilon_z \end{pmatrix} \overset{\mathsf{D}}{=} \mathsf{N}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{pmatrix} \right\}. \tag{2.2}$$

There are two different ways of ordering y and z, corresponding to two Cholesky decompositions. First, we can condition y on z to obtain the equations

$$y = \gamma_{yz} z + \gamma_{yw\cdot z} w + \epsilon_{y\cdot z}, \tag{2.3}$$

$$z = \gamma_{zw} w + \epsilon_z, \tag{2.4}$$

with derived parameters $\gamma_{yz} = \sigma_{yz}/\sigma_{zz}$ and $\gamma_{yw\cdot z} = \gamma_{yw} - \gamma_{yz}\gamma_{zw}$, and independent, normal innovations $\epsilon_{y\cdot z}$, ϵ_z with variances $\sigma_{yy\cdot z} = \sigma_{yy} - \sigma_{zy}^2/\sigma_{zz}$, σ_{zz}. Secondly, we can condition z on y to obtain the equations

$$z = \gamma_{zy} y + \gamma_{zw\cdot y} w + \epsilon_{z\cdot y}, \tag{2.5}$$

$$y = \gamma_{yw} w + \epsilon_y, \tag{2.6}$$

where $\gamma_{zy} = \sigma_{zy}/\sigma_{yy}$ and $\gamma_{zw\cdot y} = \gamma_{zw} - \gamma_{zy}\gamma_{yw}$, and the independent, normal innovations $\epsilon_{z\cdot y}$, ϵ_y with variances $\sigma_{zz\cdot y} = \sigma_{zz} - \sigma_{yz}^2/\sigma_{yy}$, σ_{yy}. Without further information the two orderings are equivalent.

An ordering arises from the equations (2.3)-(2.4) under the restrictions $\gamma_{yw\cdot z} = 0$ and $\gamma_{zw} \neq 0$. The equations (2.3)-(2.4) then reduce to

$$y = \gamma_{yz} z + \epsilon_{y\cdot z}, \tag{2.7}$$

$$z = \gamma_{zw} w + \epsilon_z. \tag{2.8}$$

This ordering of (y, z) is unique in the sense that it is not possible to have $\gamma_{yw\cdot z} = 0$ and $\gamma_{zw} \neq 0$ so that (2.3)-(2.4) reduce to (2.7)-(2.8), while $\gamma_{zw\cdot y} = 0$ in (2.5)-(2.6). We prove this result for general distributions in §2.2.

2.2 Result for general distributions

For a general joint, conditional distribution $\mathsf{f}(y, z|w)$, we explore testable restrictions that ensure a unique and non-trivial chain from w through z to y. In §2.3 we interpret w as a *catalyst* that initiates a unique *causal transmission* through z to y.

2.2.1 Unique transmission

The natural generalization of the result for normal distributions is a Markov property. Generally, the joint density of $(y, z|w)$ can be decomposed as

$$\mathsf{f}(y, z|w) = \mathsf{f}(y|z, w)\mathsf{f}(z|w) = \mathsf{f}(z|y, w)\mathsf{f}(y|w). \tag{2.9}$$

At this point, there is no natural ordering of the bivariate system. The uniqueness result is inspired by the normal example. It presents a condition under which we can rule out the possibility that both $\mathsf{f}(y|z, w) = \mathsf{f}(y|z)$ and $\mathsf{f}(z|y, w) = \mathsf{f}(z|y)$. In other words, we give a condition that ensures a Markov chain from w to y through z, while excluding a Markov chain from w to z through y.

Theorem 2.1 *Suppose the density* $\mathsf{f}(y, z|w)$ *has support on a product space, and it is positive on this support. Suppose that, for all* y, z,

$$\mathsf{f}(y|z, w) = \mathsf{f}(y|z). \tag{2.10}$$

Then, it holds for all y, z *in a set with positive probability that*

$$\mathsf{f}(z|w) \neq \mathsf{f}(z) \tag{2.11}$$

$$\Rightarrow \qquad \mathsf{f}(z|y, w) \neq \mathsf{f}(z|y). \tag{2.12}$$

The requirement in Theorem 2.1 that the support is a product space is satisfied in a range of common situations, for instance in a normal setup. It allows for the less interesting case where y or z is atomic. If z is atomic, then condition (2.11) always fails. If y is atomic, then conclusion (2.12) reduces to (2.11).

Theorem 2.1 gives conditions for a unique Markov structure among the variables. Condition (2.10) implies

$$\mathsf{f}(y, w|z) = \mathsf{f}(y|z)\mathsf{f}(w|z). \tag{2.13}$$

Theorem 2.1 shows that the conditions (2.10), (2.11) imply (2.12), and therefore there is no Markov structure from w through y to z, that is

$$\mathsf{f}(z, w|y) \neq \mathsf{f}(z|y)\mathsf{f}(w|y). \tag{2.14}$$

In other words, the conditional model for y, z given w allows for two possible Markov structures, but we can distinguish these through testable assumptions.

The next step is a requirement that the Markov structure is non-trivial.

Definition 2.1 *Consider the conditional distribution of* y, z *given* w *with the Markov structure* $\mathsf{f}(y, z|w) = \mathsf{f}(y|z)\mathsf{f}(z|w)$. *If* $\mathsf{f}(y|z) \neq \mathsf{f}(y)$ *and* $\mathsf{f}(z|w) \neq \mathsf{f}(z)$, *we have a* ***non-trivial Markov structure****, which we represent by the undirected graph* w—z—y.

We note that Theorem 2.1 implies that the two non-trivial Markov structures w—z—y and w—y—z cannot hold simultaneously. We summarize this as follows.

Theorem 2.2 *Suppose the density* $\mathsf{f}(y, z|w)$ *has support on a product space, and it is positive on this support. Suppose that, for all* y, z,

$$\mathsf{f}(y|z, w) = \mathsf{f}(y|z) \qquad , \tag{2.15}$$

and that, for all y, z *in a set with positive probability,*

$$\mathsf{f}(z|w) \neq \mathsf{f}(z) \qquad \text{and} \qquad \mathsf{f}(y|z) \neq \mathsf{f}(y). \tag{2.16}$$

Then we have a unique and non-trivial Markov structure w—z—y.

The non-trivial Markov structure w—z—y is symmetric in y and w. Thus, in itself it does not give a direction from w to y. A non-trivial Markov structure does not, in general, imply that w and y are dependent, so w may affect z without affecting y. We explore this next and return to the directional issue and causality in §2.3.

2.2.2 Non-trivial transmission

Suppose we have a non-trivial Markov structure w—z—y. In a causal context it is of intest if w actually has an effect on y through z. Indeed, the Markov structure w—z—y allows the possibility that y and w are independent. From a causal view point, this is not so exciting. We will seek to characterise when the effect is non-trivial.

Definition 2.2 *Consider a non-trivial Markov structure w—z—y. There is a **non-trivial transmission** between w and y when $\mathsf{f}(y|w) \neq \mathsf{f}(y)$.*

When exploring the transmission between w and y, it is useful to introduce the notation $\perp\!\!\!\perp$ for independence, so that $y \perp\!\!\!\perp w$ when $\mathsf{f}(y|w) = \mathsf{f}(y)$. Note that the conditional distribution of y given w is a compound distribution of the form

$$\mathsf{f}(y|w) = \int \mathsf{f}(y|z)\mathsf{f}(z|w)dz. \tag{2.17}$$

The integral can be interpreted as summation if the dominating measure dz is discrete.

We start with a sufficient condition for a trivial transmission.

Lemma 2.1 *Suppose $\mathsf{f}(y, z|w) = \mathsf{f}(y|z)\mathsf{f}(z|w)$. Then $y \perp\!\!\!\perp z$ or $z \perp\!\!\!\perp w \Rightarrow y \perp\!\!\!\perp w$.*

In general, the sufficient condition is not necessary. From a causal transmission perspective, we are interested in exploring when the sufficient condition is necessary. Indeed, with a non-trivial Markov structure w—z—y, we have the conditional dependence $y \not\perp\!\!\!\perp z$ and $z \not\perp\!\!\!\perp w$. Our question is when this implies $y \not\perp\!\!\!\perp w$. We give some examples.

When z is binary the question relates to collapsibility of contingency tables. Dawid (1980, Theorem 8.3) attributes the following result to Yule.

Lemma 2.2 *Suppose w—z—y with binary z. Then $y \not\perp\!\!\!\perp w$.*

Moving away from binary z, we find the same result for some common distributions.

Lemma 2.3 *Suppose w—z—y with normal $(y, z|w)$ satisfying (2.1). Then $y \not\perp\!\!\!\perp w$.*

Lemma 2.4 *Suppose w—z—y with binary y so that $(y|z)$ is logit,* $\text{logit}\{\mathsf{f}(y = 1|z)\} = \gamma_{yz}z$ *or probit,* $\mathsf{f}(y = 1|z) = \Phi(\gamma_{yz}z)$*, while $(z|w)$ is normal,* $\mathsf{N}(\gamma_{zw}w, \sigma_{zz})$*. Then $y \not\perp\!\!\!\perp w$.*

However, the sufficient condition is not necessary in general. An example follows.

Example 2.1 *Suppose w—z—y, We construct an example where $y \not\perp\!\!\!\perp z$ and $z \not\perp\!\!\!\perp w$, yet $y \perp\!\!\!\perp w$. Let w, y be binary, while z takes three values. Describe the conditional distributions $\mathsf{f}(z|w)$ and $\mathsf{f}(y|z, w) = \mathsf{f}(y|z)$ by the transition matrices*

0	1	2	$z \mid w$
4/8	3/8	1/8	0
4/8	2/8	2/8	1

0	1	$y \mid z$
1/4	3/4	0
2/4	2/4	1
2/4	2/4	2

The conditional distribution $\mathsf{f}(y|w)$, computed as the product of the transition matrices, satisfies $\mathsf{f}(y|w) = \mathsf{f}(y)$, that is

0	1	$y \mid w$
3/8	5/8	0
3/8	5/8	1

2.3 Causal interpretation

Theorem 2.2 gave testable conditions ensuring that the conditional distribution $\mathsf{f}(y, z|w)$ reduces to a non-trivial Markov structure w—z—y. This was followed in §2.2.2 by a variety of conditions ensuring a non-trivial transmission between w and y. In the following, we give this a causal interpretation. We will think of the variable w as taking a value that is determined outside the system (y, z). This value then transmits through the system as described by the conditional distribution $\mathsf{f}(y, z|w)$.

Definition 2.3 *Consider variables w, z, y. Assume that for each realisation of w, then $\mathsf{f}(y, z|w)$ describes the distribution of outcomes of y, z. Let w represent an intervention on the system. Then we say that w is a* ***catalyst****.*

Definition 2.4 *Consider non-trivial Markov structure w—z—y with non-trivial transmission between w and y and where w is a catalyst. Then we have a* ***causal transmission*** *of the catalyst w to y through z. This is represented by the notation $w \to z \to y$.*

Our definition of a catalyst is related to the notion of a causal effect by (Pearl, 2000, p. 70). Pearl's definition is formulated for directed acylical graphs that assume a direction. In contrast, Definitions 2.3, 2.4 consider the testable and undirected Markov structure w—z—y and merely gives it a causal and directional interpretation. Thus, the important distinction between our exposition and the existing literature is the objective of characterizing potential transmission of catalysts using testable assumptions as far as possible. Catalysts will not always be obvious but can potentially be discovered through examination of observational data. Essentially, we are seeking to discover natural experiments and straddle the boundary between observational and experimental frameworks. Definition 2.4 has the feature that we are agnostic about the causal relationship between the endogenous variables when a catalyst is not present.

We consider three special cases: a normal model and two types of logit/probit-normal mixtures.

Example 2.2 *Suppose $(y, z|w)$ has a bivariate normal distribution as in* (2.3)-(2.4) *or* (2.5)-(2.6) *with a positive definite covariance matrix. If $\gamma_{yw\cdot z} = 0$ while $\gamma_{zw} \neq 0$, then Theorem 2.1 implies a unique Markov structure. If in addition $\gamma_{yz} \neq 0$, then Theorem 2.2 and Lemma 2.3 imply a non-trivial Markov structure w—z—y and a non-trivial transmission between w and y. When w is interpretable as a catalyst, then $w \to z \to y$.*

Example 2.3 *Suppose y is binary and $(y, z|w)$ satisfies a logit-normal mixture model or a probit-normal mixture model. That is, the conditional distribution $(y|z, w)$ satisfies*

$$\text{logit}\{\mathsf{f}(y = 1|z, w)\} = \gamma_{yz} z + \gamma_{yw\cdot z} w \quad \textit{or} \quad \mathsf{f}(y = 1|z, w) = \Phi(\gamma_{yz} z + \gamma_{yw\cdot z} w),$$

while $(z|w)$ is $\mathsf{N}(\gamma_{zw} w, \sigma_{zz})$. If $\gamma_{yw\cdot z} = 0$ while $\gamma_{zw} \neq 0$, then Theorem 2.1 implies a unique Markov structure. If in addition $\gamma_{yz} \neq 0$, then Theorem 2.2 and Lemma 2.4 imply a non-trivial Markov structure w—z—y and a non-trivial transmission between w and y. When w is interpretable as a catalyst, then $w \to z \to y$.

We note that in this situation $\mathsf{f}(y|z, w)$ is much easier to work with than $\mathsf{f}(z|y, w)$. Due to Theorem 2.1, we only need to check the first instance to narrow the potential orderings of the system $(y, z|w)$.

Example 2.4 *Suppose z is binary and $(y|z,w)$ is $\mathsf{N}(\gamma_{yz}z+\gamma_{yw\cdot z}w,\sigma_{yy\cdot z})$. If $\gamma_{yw\cdot z}=0$ while $\mathsf{f}(z|w)\neq\mathsf{f}(z|w)$ then Theorem 2.1 implies a unique Markov structure. If, in addition, $\gamma_{yz}\neq 0$ then Theorem 2.2 and Lemma 2.2 imply a non-trivial Markov structure w—z—y and a non-trivial transmission between w and y. When w is interpretable as a catalyst, then $w\to z\to y$.*

3 Structural considerations

The causal transmission concept unites ideas from Cholesky decompositions within structural vector autoregressions with ideas from instrumental variable estimation. We explore how causal transmission arises as a special case in those two settings. We draw comparisons with the more restrictive concept of super exogeneity before delving into a structural interpretation when multiple catalysts are available.

3.1 Cholesky decomposition

Sims (1980) used vector autoregressions to address the haphazard accumulation of restrictions to achieve identification in the large simultaneous equation models of the time. This approach has evolved into the frequently-used structural vector autoregressive (SVAR) approach, where a structural model is identified from the reduced-form. In its basic form, this involves a recursive ordering of the variables. We will discuss how Cholesky decomposition relates to causal transmission.

It is well-known that, while useful, recursive orderings are not unique. Causal transmission takes its starting point in recursive orderings but uses a catalyst to establish a unique ordering. If we ignore dynamic features we can explore this using the setup in §2.1. The reduced-form system for the variables y,z given w is then given by (2.1). Pre-multiplying that system by a square matrix A gives a structural model

$$\begin{pmatrix} a_{1y} & a_{1z} \\ a_{2y} & a_{2z} \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}=\begin{pmatrix} b_{1w} \\ b_{2w} \end{pmatrix}w+\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{3.1}$$

where $e=A\epsilon$ has covariance $\Omega_e=A\Sigma_\epsilon A'$ and where Σ_ϵ is the covariance matrix in (2.2). A structural model of this general form is not identifiable from the reduced-form model. We therefore consider two Cholesky decompositions where A is triangular and Ω is diagonal. The first possibility is

$$A=\begin{pmatrix} 1 & a_{1z} \\ 0 & 1 \end{pmatrix}, \qquad \Omega_e=\begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{pmatrix}, \tag{3.2}$$

which is identifiable from (2.3), (2.4), when $a_{1z}=-\gamma_{yz}=-\sigma_{yz}/\sigma_{zz}$, while $\omega_{11}=\sigma_{yy\cdot z}$ and $\omega_{22}=\sigma_{zz}$. The second possibility is

$$A=\begin{pmatrix} 1 & 0 \\ a_{2z} & 1 \end{pmatrix}, \qquad \Omega_e=\begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{22} \end{pmatrix}, \tag{3.3}$$

which is identifiable from (2.5), (2.6), when $a_{2z}=-\gamma_{zy}=-\sigma_{zy}/\sigma_{yy}$, while $\omega_{11}=\sigma_{yy}$ and $\omega_{22}=\sigma_{zz\cdot y}$. The Cholesky forms (3.2), (3.3) are observationally equivalent.

Using the causal transmission analysis we may find, for instance, that $w \to z \to y$ in the reduced-form model. This is consistent with the first Cholesky form (3.2) with the additional restriction that $b_{1w} = 0$, that is:

$$\begin{pmatrix} 1 & a_{1z} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ b_{2w} \end{pmatrix} w + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{3.4}$$

where the errors e_1 and e_2 are independent. This model is asymmetric. It shows how economic shocks in z can transmit to the structural relation $y + a_{1z}z$. Subtly, the asymmetry is captured by w rather than the errors e_1, e_2, which have a symmetric role. Thus, the interpretation of this structural model is that it shows how, typically, large shocks of the type w move through the economy, which is also subject to, typically, small shocks of the type e_1, e_2. For instance, w may represent the onset of the financial crises or a major government intervention, while the shocks e_1, e_2 represent the minor, daily pulling and pushing forces in the economy. Thus, the structural assumption we need for this analysis is that w is a catalyst. The remaining features of the causal transmission $w \to z \to y$ are testable and discoverable from reduced-form analysis. In §3.4 we extend this analysis to a situation with multiple catalysts.

3.2 Instrumental variable estimation

The traditional simultaneous equations model has no causal direction. Instead, the focus is to estimate the behavioral equations with the aid of instruments. We discuss this in the context of a simple demand and supply example, with a focus on the demand curve.

The demand curve expresses the (log) quantity q^d that would be demanded at a given (log) price level p

$$q^d = a_d + b_d p + e_d, \tag{3.5}$$

or, reversing the equation, the price one would be willing to pay for a given quantity

$$p = \frac{a_d}{b_d} + \frac{1}{b_d} q^d - e_d. \tag{3.6}$$

Thus, the model has no causal direction.

If an observed instrument w changes the supply, we can estimate the demand function from observations of traded quantities q and prices p. Ignoring intercepts, we can, for instance, estimate the reduced-form system (2.1), that is

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \gamma_{qw} \\ \gamma_{pw} \end{pmatrix} w + \begin{pmatrix} \epsilon_q \\ \epsilon_p \end{pmatrix}, \tag{3.7}$$

where $\mathsf{f}(\epsilon_q, \epsilon_p | w) = \mathsf{f}(\epsilon_q, \epsilon_p)$ is normal. From the reduced-form we can derive an equation that does not depend on the instrument w

$$q = \frac{\gamma_{qw}}{\gamma_{pw}} p + u \qquad \text{where} \qquad u = \epsilon_q - \frac{\gamma_{qw}}{\gamma_{pw}} \epsilon_p, \tag{3.8}$$

when $\gamma_{pw} \neq 0$. From a reduced-form perspective, the error term u in equation (3.8) has the property that it is independent of the instrument w, so that $\mathsf{E}(uw) = 0$ with respect to the distribution $\mathsf{f}(\epsilon_q, \epsilon_p|w)$. Thus, we need the identifying assumption that the demand error e_d in (3.5) or (3.6) is uncorrelated with the instrument under a suitable economic expectation. In this just-identified setup, the ratio of least squares estimators for γ_{qw} and γ_{pw} is known as indirect least squares, two-stage least squares, or limited information maximum likelihood.

The above argument is symmetric in price and quantity. We do not distinguish between situations where the actual trades are driven by considerations to price or to quantity. However, by imposing the testable restriction that u and ϵ_p are independent or, equivalently, that $\gamma_{qw}/\gamma_{pw} = \mathsf{Cov}(\epsilon_q, \epsilon_p)/\mathsf{Var}(\epsilon_p)$, the model becomes directional. If the instrument w can be viewed as a catalyst, it transmits causally through p to the traded quantity q.

3.3 Super exogeneity

Broadly speaking, causal transmission is a weaker concept than super exogeneity introduced by Engle et al. (1983): if we have causal transmission and weak exogeneity, we get super exogeneity.

To appreciate the distinction we need to recall the definition of weak exogeneity. In a time series setup write the joint, conditional density of $(y_t, z_t|w_t, past)$ as $\mathsf{f}_{\xi,\lambda}(y_t, z_t|w_t, past) = \mathsf{f}_{\xi}(y_t|z_t, w_t, past)\mathsf{f}_{\lambda}(z_t|w_t, past)$, with parameters ξ, λ varying in a parameter space Θ. If the parameter space is a product space $\Theta = \Xi \times \Lambda$ so $\xi \in \Xi$ and $\lambda \in \Lambda$, then z is weakly exogenous for ξ. Then the joint likelihood can be optimized by maximizing the conditional, partial likelihood for $(y_t|z_t, past)$ and the marginal, partial likelihood for $(z_t|past)$ separately. If, in addition, $\mathsf{f}_{\xi}(y_t|z_t, w_t, past) = \mathsf{f}_{\xi}(y_t|z_t, past)$ while $\mathsf{f}_{\lambda}(z_t|w_t, past) \neq \mathsf{f}_{\lambda}(z_t, past)$ we get, essentially, super exogeneity of z for y with respect to w. We note that these additional conditions are the conditions (2.10), (2.11) of the uniqueness Theorem 2.1.

Causal transmission is a weaker than super exogeneity in the sense that the directions of causal transmission and weak exogeneity need not match. To see this write

$$\mathsf{f}_{\xi,\lambda}(y, z|w) = \mathsf{f}_{\xi}(y|z, w)\mathsf{f}_{\lambda}(z|w) = \mathsf{f}_{\zeta}(z|y, w)\mathsf{f}_{\eta}(y|w).$$

The model may include further restrictions that are not explicitly stated here, so that ξ, λ vary in a product space $\Xi \times \Lambda$, whereas ζ, η do not vary in a product space. In other words, z is weakly exogenous for y, but y is not weakly exogenous for z. At the same time we may have $w \to y \to z$, so that w transmits causally through y to z. This causal transmission is not compatible with weak exogeneity, so we do not have super exogeneity. Some more subtle differences between the two concepts are that super exogeneity of z for y with respect to w does neither require that $\mathsf{f}_{\xi}(y|z) \neq \mathsf{f}_{\xi}(y)$ nor $\mathsf{f}_{\xi,\lambda}(y|w) \neq \mathsf{f}_{\xi,\lambda}(y)$.

3.4 Multiple causal transmissions

The concept of causal transmission generalizes to multiple catalysts that may flow through the system in different ways. For notational convenience we present this by augmenting the linear, normal system (2.1) with two distinct catalysts w_1, w_2, so that:

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \gamma_{y1} \\ \gamma_{z1} \end{pmatrix} w_1 + \begin{pmatrix} \gamma_{y2} \\ \gamma_{z2} \end{pmatrix} w_2 + \begin{pmatrix} \epsilon_y \\ \epsilon_z \end{pmatrix}, \tag{3.9}$$

where $\mathsf{f}(\epsilon_y, \epsilon_z|w) = \mathsf{f}(\epsilon_y, \epsilon_z)$ is normal as in (2.2). The variables w_1, w_2 are observable and may represent two types of shocks to the economy at different points in time.

We now set up the two possibilities for ordering y, z through conditioning. Conditioning y on z gives

$$y = \gamma_{yz} z + \gamma_{y1\cdot z} w_1 + \gamma_{y2\cdot z} w_2 + \epsilon_{y\cdot z}, \tag{3.10}$$
$$z = \gamma_{z1} \ w_1 + \gamma_{z2} \ w_2 + \epsilon_z, \tag{3.11}$$

where $\epsilon_{y\cdot z}$, ϵ_z are independent and $\gamma_{yz} = \sigma_{yz}/\sigma_{zz}$, while conditioning z on y gives

$$z = \gamma_{zy} y + \gamma_{z1\cdot y} w_1 + \gamma_{z2\cdot y} w_2 + \epsilon_{z\cdot y}, \tag{3.12}$$
$$y = \gamma_{y1} \ w_1 + \gamma_{y2} \ w_2 + \epsilon_y, \tag{3.13}$$

where $\epsilon_{z\cdot y}$, ϵ_y are independent and $\gamma_{zy} = \sigma_{zy}/\sigma_{yy}$. Assuming w_1, w_2 are catalysts, we get two causal transmission hypotheses

$$\mathsf{H}_1: \quad \gamma_{y1\cdot z} = 0 \cap \gamma_{z1} \neq 0 \cap \gamma_{yz} \neq 0 \quad \Rightarrow \quad w_1 \to z \to y, \tag{3.14}$$
$$\mathsf{H}_2: \quad \gamma_{z2\cdot y} = 0 \cap \gamma_{y2} \neq 0 \cap \gamma_{zy} \neq 0 \quad \Rightarrow \quad w_2 \to y \to z. \tag{3.15}$$

When the hypotheses H_1 and H_2 are both satisfied we get causal transmissions in opposite directions, which we represent by superimposing two directed graphs

$$\mathsf{H}_1 \cap \mathsf{H}_2 \quad \Rightarrow \quad w_1 \longrightarrow z \overset{\longrightarrow}{\dashleftarrow} y \dashleftarrow w_2 \tag{3.16}$$

The joint restrictions imposed by $\mathsf{H}_1 \cap \mathsf{H}_2$ are possibly best expressed in terms of the original system (3.9) as:

$$\mathsf{H}_1 \cap \mathsf{H}_2: \quad \gamma_{y1} = \frac{\sigma_{yz}}{\sigma_{zz}} \gamma_{z1}, \quad \gamma_{z2} = \frac{\sigma_{zy}}{\sigma_{yy}} \gamma_{y2}, \quad \gamma_{z1} \neq 0, \quad \gamma_{y2} \neq 0, \quad \sigma_{yz} \neq 0.$$

Written in a vector format, we have the reduced-form model

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \sigma_{yz}/\sigma_{zz} \\ 1 \end{pmatrix} \gamma_{z1} w_1 + \begin{pmatrix} 1 \\ \sigma_{zy}/\sigma_{yy} \end{pmatrix} \gamma_{y2} w_2 + \begin{pmatrix} \epsilon_y \\ \epsilon_z \end{pmatrix}, \tag{3.17}$$

Following the considerations in §3.1, the corresponding structural model is

$$\begin{pmatrix} 1 & -\gamma_{yz} \\ -\gamma_{zy} & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \delta_{21} \end{pmatrix} w_1 + \begin{pmatrix} \delta_{12} \\ 0 \end{pmatrix} w_2 + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{3.18}$$

where $\gamma_{yz} = \sigma_{yz}/\sigma_{zz}$ and $\gamma_{zy} = \sigma_{zy}/\sigma_{yy}$ are multipliers for the catalysts, while $\delta_{21} = (1 - \rho^2)\gamma_{z1}$ and $\delta_{12} = (1 - \rho^2)\gamma_{y2}$, with $\rho^2 = \sigma_{yz}^2/(\sigma_{yy}\sigma_{zz})$. The innovations of the structural equation (3.18) satisfy

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \overset{\mathsf{D}}{=} \mathsf{N} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{yy\cdot z} & -\sigma_{yz}(1 - \rho^2) \\ -\sigma_{zy}(1 - \rho^2) & \sigma_{zz\cdot y} \end{pmatrix} \right\}, \tag{3.19}$$

with correlation $-\rho = -\sigma_{yz}(1-\rho^2)/(\sigma_{yy\cdot z}\sigma_{zz\cdot y})^{1/2}$. We have identified a structural model with respect to catalysts w_1 and w_2 without imposing any ad hoc restrictions on the causal ordering through the covariance matrix. These catalysts are orthogonal to each other in the structural model, in the sense that w_1 is omitted from the first structural equation and w_2 is omitted from the second structural equation. Structure is, therefore, identified as a linear relationship that remains invariant to large shocks. Rather than imposing structure to identify orthogonal shocks, we use shocks to identify structure. Instead of having a structural model that is ordered for an entire sample, we are only concerned with ordering during periods when large interventions take place. We note that if the parameters γ_{yz}, γ_{zy} of the system (3.18) were unrelated to the covariance parameters σ_{yy}, σ_{yz}, σ_{zz} in (3.19) we would have a just-identified and undirected, bivariate simultaneous equations model.

Causal transmissions in both directions depending on the type of shock seems compatible with the discussion of shocks in macroeconomics. In many situations, we use indicator variables to represent large external shocks to the economy. When large external shocks arrive in quick succession, it may be difficult to separate the effect of the individual shocks. A pertinent example is the beginning of the financial crisis in 2007-2008 when oil shocks, financial collapse, and large fiscal and monetary policy interventions occurred in quick succession. We envisage that it would be possible to disentangle the effect of these shocks by lining these up, individually, with shocks at other points in time.

3.5 Comments

As we have seen, causal transmission is related to traditional approaches such as recursive ordering induced by a Cholesky decomposition and instrumental variable estimation.

As a causal concept, causal transmission is modest in scope: all causal orderings are relative to particular interventions with no attempt to give an overall causal ordering of the variables of interest, y, z. The concept is more modest than the causal inference interpretation of quasi-experiments, where the difference of potential and realized outcomes is estimated using an instrumental variable approach and the causal language from random control trials is applied, see Imbens (2014). In practice, the consequence is that it becomes clearer that results can only be extrapolated to future interventions insofar as those interventions are comparable with the interventions in the sample.

The type of interventions are somewhat different in nature from the traditional shocks of structural vector autoregressive analysis. Here we focus on large interventions as opposed to the sample variance of the residuals, noting that the residuals simply represent the unmodelled part of the data. The approach reflects a view that it is difficult to disentangle many minor shocks to the economy.

Causal transmission is also restrictive in the sense that we need both a catalyst and a Markov structure. At the same time it is potentially possible to discover catalysts empirically. An empirical example follows in §4.

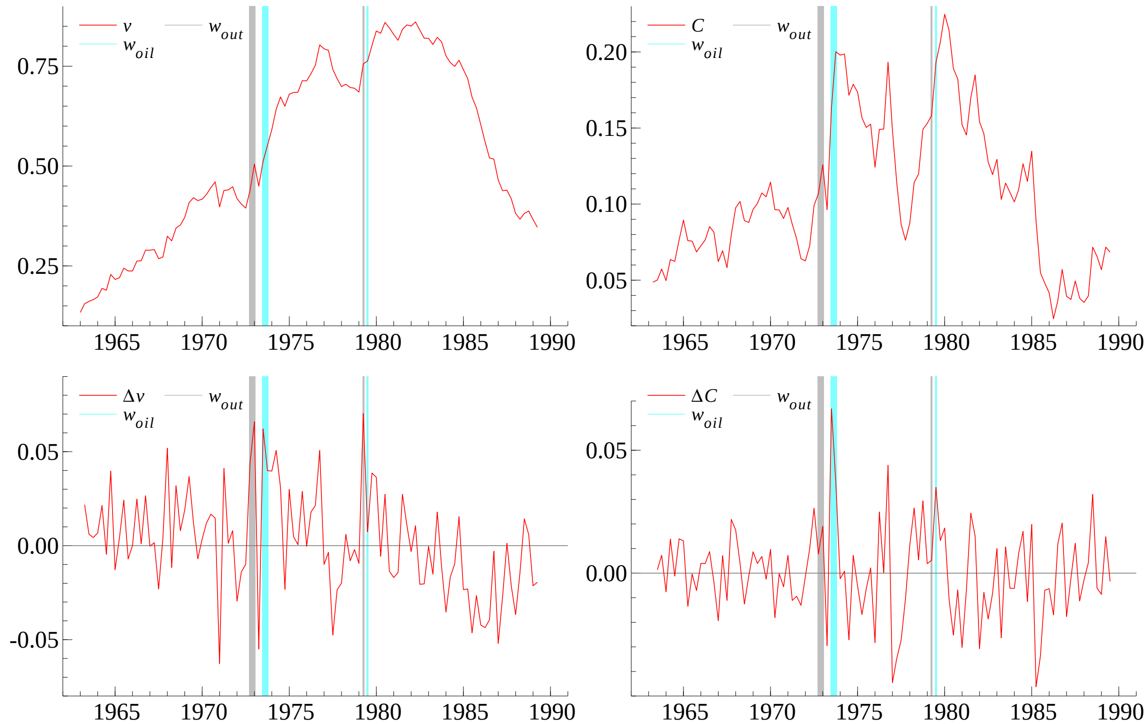


Figure 4.1: Levels and first-differences of the variables in the system (v, C) plotted with the selected outliers (w_{out}, w_{oil}) from Hendry and Mizon (1993).

4 Empirical example

We illustrate the causal transmission using the simplified bivariate model of money demand for the UK in Hendry and Nielsen (2007). This has the convenient features of being bivariate, reasonably well-specified, and with two catalysts operating in opposite directions. The data are formed from quarterly observations of log M1 money m, log real total final expenditure x, its log deflator p, and a constructed net interest rate R_n taken from Hendry and Mizon (1993) over the period from 1963:2 to 1989:2. To simplify the analysis, we convert the four variables into a bivariate system, modelling the velocity of circulation of money v and the cost of holding money C through

$$v_t = x_t - m_t + p_t, \qquad C_t = \Delta p_t + R_{n,t}.$$

We show how the the results from the previous sections may be applied in practice to identify multiple causal transmissions. Subsequently, we provide impulse responses for the interventions that are identified. Finally, we address the Lucas critique that asserts that an econometric model may be unstable under changing conditions.

The graphics and subsequent computations were carried out in MATLAB (MATLAB, 2014) and OxMetrics (Hendry and Doornik, 2014).

Table 4.1: Three models estimated over the period 1963:4 to 1989:2. Standard errors reported in parentheses.

	Joint Model		**Conditional Models**		**Structural Model**	
	Δv_t	ΔC_t	$\Delta v_t \vert \Delta C_t$	$\Delta C_t \vert \Delta v_t$	Δv_t	ΔC_t
Δv_t	–	–	–	0.433 (0.075)	–	0.402 (0.068)
ΔC_t	–	–	0.606 (0.104)	–	0.579 (0.093)	–
Δv_{t-1}	−0.343 (0.095)	−0.048 (0.081)	−0.314 (0.082)	0.100 (0.074)	−0.317 (0.091)	0.092 (0.076)
ΔC_{t-1}	0.086 (0.117)	0.046 (0.099)	0.058 (0.100)	0.009 (0.085)	0.064 (0.108)	0.003 (0.090)
v_{t-1}	−0.097 (0.014)	−0.005 (0.012)	−0.094 (0.012)	0.037 (0.013)	−0.095 (0.012)	0.035 (0.013)
C_{t-1}	0.529 (0.071)	−0.077 (0.060)	0.575 (0.062)	−0.306 (0.065)	0.575 (0.066)	−0.293 (0.068)
1	−0.004 (0.006)	0.009 (0.005)	−0.009 (0.005)	0.011 (0.004)	−0.009 (0.005)	0.011 (0.004)
$w_{out,t}$	0.051 (0.012)	0.013 (0.010)	0.044 (0.010)	−0.010 (0.009)	0.039 (0.010)	0
$w_{oil,t}$	0.030 (0.012)	0.051 (0.010)	−0.001 (0.011)	0.038 (0.009)	0	0.039 (0.008)
$\widehat{\sigma}_{vv}^{1/2}$	0.019	–	–	–	–	–
$\widehat{\sigma}_{CC}^{1/2}$	–	0.016	–	–	–	–
$\widehat{\rho}$	0.512		–		0.482	
$\widehat{\sigma}_{vv\cdot C}^{1/2}$	–	–	0.017	–	0.016	–
$\widehat{\sigma}_{CC\cdot v}^{1/2}$	–	–	–	0.014	–	0.014
likelihood	559.31				558.74	

4.1 The unrestricted reduced-form

Figure 4.1 shows v_t, C_t in levels and differences. The transformed data series are non-stationary, but their first-order differences have a more stationary appearance. The plots also show two dummy variables $w_{out,t}$, $w_{oil,t}$ representing large fiscal expansions in 1972:4–1973:1 and 1979:2 as well as the oil price shocks in 1973:3–4 and 1979:3. They will later be interpreted as catalysts.

The dummy variables are taken from Hendry and Mizon (1993). They were originally found through a residual analysis as large outliers. By including dummies for these particular observations the remaining observations appear to match a normal reference distribution, and the model passes standard specification tests including recursive tests. At the same time these dummies have interpretation as interventions and are in this respect related to the historical narrative approach of Romer and Romer (2010).

The initial specification is a second-order vector autoregressive model including the two dummy variables $w_{out,t}$ and $w_{oil,t}$. The estimated model is the joint model reported in equilibrium-correction form in the first two columns of Table 4.1.

Specification tests are reported in Table 4.2. The residual specification tests include

Table 4.2: Specification tests for unrestricted joint model. p-values reported in brackets.

Test	$\chi^2_{\mathsf{norm}}[2]$	$\mathsf{F}_{\mathsf{ar}(1-5)}[5,91]$	$\mathsf{F}_{\mathsf{arch}(1-4)}[4,95]$	$\mathsf{F}_{\mathsf{het}}[10,92]$	$\max Chow$
Δv_t	2.2 [0.33]	0.4 [0.86]	1.2 [0.32]	0.6 [0.81]	8.4 [0.29]
ΔC_t	1.9 [0.39]	1.9 [0.10]	2.1 [0.08]	1.5 [0.15]	12.3 [0.04]

a cumulant based test χ^2_{norm} for normality, a test F_{ar} for autoregressive temporal dependence (Godfrey, 1978), a test $\mathsf{F}_{\mathsf{arch}}$ for autoregressive conditional heteroscedasticity (Engle, 1982), a test $\mathsf{F}_{\mathsf{het}}$ for heteroscedasticity (White, 1980), and a test $\max Chow$ based on the maximum of recursive 1-step-ahead Chow (1960) forecast test statistics. We will benefit from this recursive test in §4.6. The above references only consider static or stationary models, but the specification tests also apply for non-stationary autoregressions, see Kilian and Demiroglu (2000) for χ^2_{norm}, Nielsen (2006) for F_{ar}, and Nielsen and Whitby (2015) for $\max Chow$. We see that the specification for the velocity equation is very good, while the specification for the cost equation is less good, but tolerable. The two Chow tests take their maximum values in 1971:1 and in 1976:4. These dates correspond to the decimalization of the Pound and the debt intervention by the International Monetary Fund. Overall, these tests indicate that we cannot reject the model and that the innovations are independent, identically normal.

The dummy variables play a dual role in the subsequent analysis. First, we need the dummy variables to achieve a reasonable specification of the econometric model. Without these the residuals appear too irregular and we cannot perform valid inference. The chosen statistical model is based on the normal distribution and the observations captured by the dummy variables are outliers relative to this reference distribution. Second, the dummy variables help us to distinguish between large and small shocks. The large shocks occur infrequently and they are often interpretable as catalysts.

The above specification analysis indicate that the largest shocks after the oil crises and output expansions are the the decimalization of the Pound in 1971:1 and the turmoil around the IMF intervention in 1976:4. In terms of fit, the results in Table 4.2 do not suggest that it is necessary to include dummies to represent these events. This could be followed up with a sensitivity analysis for the inference we draw about about the oil shocks and the output expansion. For instance, does it make a difference to include a dummy for the decimalization? At the same time we could include dummies for the decimalization and the IMF intervention to explore the transmission of those events. In other words, if we are concerned with a particular macroeconomic intervention we can to some extent search for similar interventions in the past and explore their transmission.

4.2 Causal transmission in UK money demand data

We now explore causal transmission. Table 4.1 reports the unrestricted reduced-form model is columns 1 and 2. This is a model for v_t, C_t given dummies and the past.

The effect of the oil price shocks can be explored by conditioning v_t on C_t and follow Example 2.2. The conditional equation for v_t given C_t and the marginal equation for C_t are reported in columns 3 and 2, respectively, in Table 4.1. The coefficient for $w_{oil,t}$

is insignificant in the conditional equation, but significant in the marginal equation. Theorem 2.1 shows there exists a unique Markov structure, so that v_t and $w_{oil,t}$ are conditionally independent given C_t. Further, the coefficient for ΔC_t is significant in the conditional equation. Theorem 2.2 then shows the Markov structure is non-trivial so $w_{oil,t}$—C_t—v_t. Lemma 2.3 then shows that the transmission between $w_{oil,t}$ and v_t is non-trivial. Correspondingly, the coefficient for $w_{oil,t}$ is significant in the marginal v_t equation. From an economic perspective, it seems reasonable to interpret the oil shocks as catalysts so that $w_{oil,t} \to C_t \to v_t$. The interpretation is that large oil price shocks move prices and in turn the velocity.

To illustrate the uniqueness result we now consider the conditional equation for C_t given v_t in column four of Table 4.1. Here, $w_{oil,t}$ is significant, so we cannot have a Markov structure from $w_{oil,t}$ through v_t to C_t. This is in line with Theorem 2.1.

Turning to the output shock we condition C_t on v_t. The conditional equation for C_t given v_t and the marginal equation for v_t are reported in columns 4 and 1, respectively. We follow Example 2.2 again. The output dummy $w_{out,t}$ is significant in the marginal equation and insignificant in the conditional equation. Moreover, velocity, Δv_t, is significant in the conditional equation. Theorems 2.1, 2.2 then show a non-trivial Markov structure $w_{out,t}$—v_t—C_t. Lemma 2.3 shows that the transmission is non-trivial. Interpreting $w_{out,t}$ as a catalyst we then have $w_{out,t} \to v_t \to C_t$. Economically, large fiscal expansions may impact the velocity of money without having an impact on inflation straight away. The conclusion is, however, less clear than the causal transmission of the oil shocks. Indeed, in line with the discussion in §2.2.2, we check if the fiscal shock $w_{out,t}$ actually has a non-negligible effect on the cost of holding money. The coefficient in the C_t equation has a t-statistic of 1.3, which at best shows marginal significance. Thus, we may very well have $w_{out,t} \to v_t \to C_t$, but evidence for this transmission is weaker than the evidence for the transmission of the oil shocks.

4.3 Imposing multiple catalysts

The two causal transmissions $w_{oil,t} \to C_t \to v_t$ and $w_{out,t} \to v_t \to C_t$ can be imposed individually. These are the hypotheses H_1, H_2 of (3.14), (3.15). Imposing both gives $w_1 \longrightarrow z \underset{\dashleftarrow}{\dashrightarrow} y \dashleftarrow w_2$ as described in §3.4. This is a system of seemingly unrelated regressions. When maximizing the likelihood we chose to parametrize it in terms of $\sigma_{yy\cdot z}$, $\sigma_{zz\cdot y}$, ρ and derive standard errors for γ_{yz} and γ_{zy} using the δ-method.

The restricted model is reported in columns 5 and 6 of Table 4.1 in the structural form derived from §3.4. The likelihood ratio statistic for the two restrictions is $2(559.31 - 558.74) = 1.14$, which is not significant when compared to a χ^2_2 distribution. The structural estimates largely match those of the conditional models in Table 4.1. Writing the model in structural form, it becomes very clear that the dummies $w_{out,t}, w_{oil,t}$ affect distinct linear combinations of the endogenous variables. The first structural equation is interpretable as the monetary quantity relation, showing how money demand reacts to output shocks, while the second structural equation is interpretable as a cost-push relation showing how money demand is driven by price shocks.

4.4 Cointegration

The velocity and cost of holding money variables are non-stationary and should possibly be subjected to a cointegration analysis. This is compatible with causal transmission.

Following the maximum likelihood setup of Johansen (1995), the cointegration model with rank one is given by the equilibrium-correction model

$$\begin{pmatrix} \Delta v_t \\ \Delta C_t \end{pmatrix} = \begin{pmatrix} \alpha_v \\ \alpha_C \end{pmatrix} (\beta_v v_{t-1} + \beta_C C_{t-1} + \beta_1) \\ + \begin{pmatrix} \gamma_{vv} & \gamma_{vC} \\ \gamma_{Cv} & \gamma_{CC} \end{pmatrix} \begin{pmatrix} \Delta v_{t-1} \\ \Delta C_{t-1} \end{pmatrix} + \begin{pmatrix} \gamma_{v1} & \gamma_{v2} \\ \gamma_{C1} & \gamma_{C2} \end{pmatrix} \begin{pmatrix} w_{out,t} \\ w_{oil,t} \end{pmatrix} + \begin{pmatrix} \epsilon_{v,t} \\ \epsilon_{C,t} \end{pmatrix} \quad (4.1)$$

The model with multiple causal transmissions and cointegration imposed has a likelihood 555.44. For present purposes, we merely consider the likelihood ratio test for the cointegration restriction within the model with multiple causal transmission imposed. The test statistic is $2(558.74 - 555.44) = 6.60$, which should be compared to a 95% critical value of 9.1, see Johansen (1995, Table 15.2).

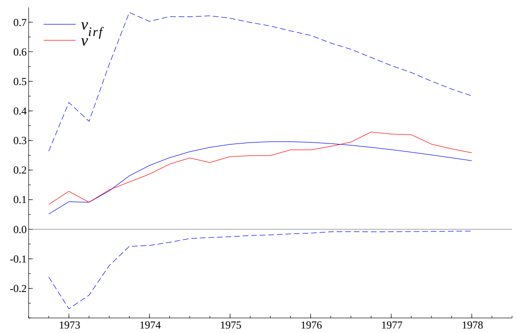
With a unit cointegration rank, the coefficients to v_{t-1}, C_{t-1} are proportional accross the equations. This results in the cointegrating relation $v_{t-1} = 6.239 C_{t-1}$, which is interpretable as long-run money demand. The adjustment coefficient in the conditional equation for v_t given C_t is a modest 9.5% per quarter, whereas the adjustment in the marginal equation for C_t is insignificant. We note that in a model without multiple causal transmissions imposed, the constraint $\alpha_C = 0$ would be a hypothesis of weak exogeneity Johansen (1995, §8), but the weak exogeneity is broken when imposing the cross-equation restrictions implied by causal transmission.

4.5 Impulse responses

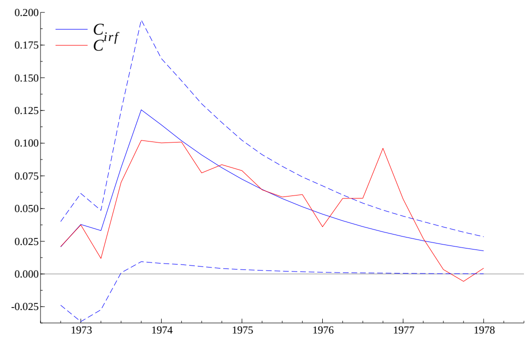
We now carry out an impulse response analysis with respect to the economic shocks represented by $w_{out,t}$ and $w_{oil,t}$. We reconstruct empirical scenarios and compare our results to the data. This offers a distinct advantage over impulse responses created by placing identifying restrictions on the covariance matrix. Figures 4.2a, 4.2b explore the period around the first oil crisis, where the fiscal expansions in 1972:4–73:1 are followed by the oil shock in 1973:3–4. Likewise, Figures 4.2c, 4.2d explore the period around the second oil crisis, where the fiscal expansions in 1979:2 are followed by the oil shock in 1979:3. In both cases, we provide joint impulse responses and compare these to real data over a five-year horizon in Figure 4.2. All joint impulses perform remarkably well compared to the scenario under consideration. What is more, the impulse response functions do not decline in performance across each scenario, indicating a temporal stability in causal transmission. This is addressed further in §4.6.

4.6 Lucas critique

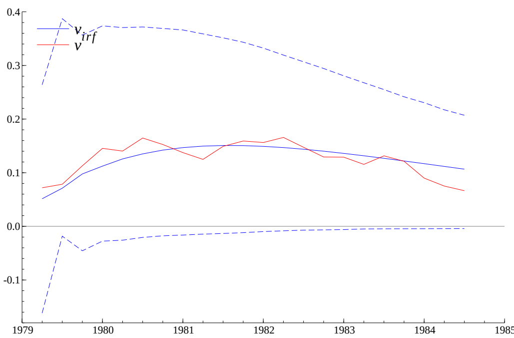
Major shocks like the oil crises and fiscal expansions change the policy environment and, in turn, may influence the behavior of individual agents. It has long been a concern whether this results in instability for the parameters of an economic model, rendering it



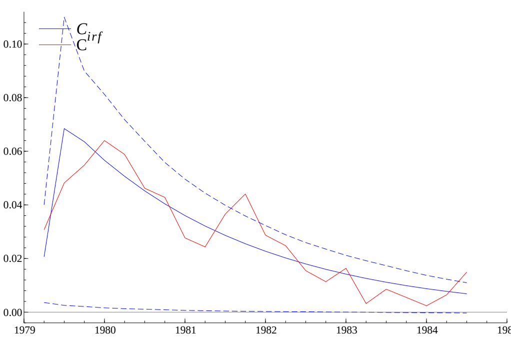
(a) Joint impulse response for v.



(b) Joint impulse response for C.



(c) Joint impulse response for v.



(d) Joint impulse response for C.

Figure 4.2: Impulse responses matched with the data for the early and late 1970s output and oil shocks. Dashed lines are simulated 90% confidence bands.

useless for analyzing the effect of implementing the policy. This is known as the Lucas (1976) critique, although the concern goes back to Frisch and Haavelmo.

Engle et al. (1983) argue that super exogeneity is a sufficient condition for valid policy analysis in the context of a well-specified econometric model that passes recursive specification tests. As remarked in §3.3, super exogeneity is essentially a combination of causal transmission and weak exogeneity. We illustrate the use of causal transmission in policy analysis by performing a recursive analysis of the money data.

Previously, $w_{oil,t}$ was constructed as the sum of impulse indicators across the two oil crises. Now, we construct dummies $w_{oil1,t}, w_{oil2,t}$ for 1973:3–4 and 1979:3, respectively, so that $w_{oil,t} = w_{oil1,t} + w_{oil2,t}$. We re-estimate the equations for $(v_t|C_t)$, (C_t) reported in Table 4.1 over subsamples 1963:4–1977:2 and 1963:4-1989:3 using the split oil dummy. The results are reported in Table 4.3. It is clear that the transmission of the first catalyst $w_{oil1} \to C \to v$ does not differ in a statistically significant way from the transmission of the second catalyst $w_{oil2} \to C \to v$. Deconstructing the catalyst w_{out} provides similar evidence for the stability of the causal transmission of the output shocks. The search for causal transmission in well-specified models therefore seems compatible with the Lucas critique. This does, of course, go hand in hand with the fact that the model in Table 4.1 passes recursive specification tests.

Table 4.3: Estimation for UK M1 data based on a sub-sample and the full sample.

	1963:4-1977:2		1963:4-1989:2	
	$\Delta v_t \vert \Delta C_t$	ΔC_t	$\Delta v_t \vert \Delta C_t$	ΔC_t
ΔC_t	0.542 (0.189)	–	0.605 (0.105)	–
Δv_{t-1}	−0.359 (0.123)	−0.060 (0.094)	−0.311 (0.087)	−0.031 (0.084)
ΔC_{t-1}	−0.004 (0.185)	−0.092 (0.142)	0.057 (0.101)	0.036 (0.099)
v_{t-1}	−0.097 (0.032)	0.007 (0.025)	−0.094 (0.013)	−0.003 (0.012)
C_{t-1}	0.626 (0.145)	−0.146 (0.111)	0.574 (0.063)	−0.084 (0.061)
1	−0.012 (0.008)	0.012 (0.006)	−0.009 (0.005)	0.009 (0.005)
$w_{out,t}$	0.042 (0.015)	0.016 (0.011)	0.044 (0.010)	0.013 (0.010)
$w_{oil1,t}$	0.001 (0.018)	0.058 (0.011)	0.000 (0.014)	0.055 (0.012)
$w_{oil2,t}$	–	–	−0.002 (0.018)	0.043 (0.017)

5 Concluding remarks

Causal transmission has been introduced to capture the idea that large economic shocks may transmit gradually through the macroeconomy.

There are three ingredients to the definition of causal transmission of catalyst w through z to y. First, we need a non-trivial Markov structure w—z—y, that is the Markov structure $\mathsf{f}(y, z|w) = \mathsf{f}(y|z)\mathsf{f}(z|w)$ needs to be non-trivial in the sense that y, z are dependent and z, w are dependent. Secondly, we need a non-trivial transmission between w, y, that is w, y are dependent. Thirdly, we need a causal assumption for the catalyst w. When these conditions are satisfied we write $w \to z \to y$. We have shown how this definition can be extended to the transmission of two unrelated catalysts.

Causal transmission is defined for general densities and it does not require normality. The first two conditions to the definition of causal transmission are testable using observational data. In standard models, the first condition of a non-trivial Markov structure implies the second condition of a non-trivial transmission. These standard models include normal models and mixtures of normal and logit/probit models.

Causal transmissions also require a catalyst. As in instrumental variable analysis the catalyst can be found as a natural experiment prior to the empirical analysis or it may be discoverable from the empirical analysis of observational data. In this way, Hendry and Santos (2010) give an algorithm for discovering super-exogeneity. This algorithm generalizes the robustified least-squares approach used by Hendry and Mizon (1993) in their UK money analysis. A theory for analyzing such algorithms is gradually emerging. Indeed, a statistical theory for robustified least-squares is presented in Johansen and Nielsen (2015). The causal transmission relies on an economic interpretation of the catalyst. In this way it relates to the narrative approach of Romer and Romer (2010).

The analysis is inspired by Bårdsen et al. (2012). They present 3-year ahead quarterly forecasts from March 2007 generated from their macro-econometric model for Norway. The forecasts track actual unit labour cost, inflation, and unemployment rather well, but fail spectacularly on the short-term interest rate. In 2008, policymakers in Norway and abroad changed the policy rate dramatically in response to the financial crisis, creating a large shift of the short-term interest rate. It appears that this had the causal impact of offsetting potential big shifts in the labour market in such a way that the macro-econometric model produces good forecasts of unit labour cost, inflation, and unemployment. It is plausible that the effects seen in the forecasts of the Norwegian macro-econometric models of Bårdsen et al. (2012) could be described as a combination of a major financial shock and a subsequent policy reaction calibrated to offset the financial shock in the labour market. In future work, we will seek to generalize the ideas presented here to facilitate such an analysis to guide economic policy.

A Proofs

The result in Theorem 2.1 hinges on the equivalence in the following lemma, of which the left to right implications is a related to Lauritzen (1996, Proposition 2.1).

Lemma A.1 *Suppose* $\mathsf{f}(y,z|w)$ *has support on a product space and that it is positive on this support. Then, for all* y,z

$$\left.\begin{array}{rcl}\mathsf{f}(y|z,w) & = & \mathsf{f}(y|z) \\ \mathsf{f}(z|y,w) & = & \mathsf{f}(z|y)\end{array}\right\} \Leftrightarrow \mathsf{f}(y,z|w) = \mathsf{f}(y,z). \tag{A.1}$$

Proof of Lemma A.1. Since the density is positive on a product space, then the marginal densities are also positive.

$\Rightarrow$: By the definition of conditional densities, the first statement on the left hand side of (A.1), and the definition of conditional densities

$$\mathsf{f}(y,z|w) = \mathsf{f}(y|z,w)\mathsf{f}(z|w) = \mathsf{f}(y|z)\mathsf{f}(z|w) = \mathsf{f}(y,z)\mathsf{f}(z|w)/\mathsf{f}(z).$$

Swap y,z and use the second statement on the left hand side of (A.1) to get

$$\mathsf{f}(y,z|w) = \mathsf{f}(y,z)\mathsf{f}(y|w)/\mathsf{f}(y). \tag{A.2}$$

Equating the two expressions we get

$$\mathsf{f}(y|w) = \mathsf{f}(y)\mathsf{f}(z|w)/\mathsf{f}(z).$$

Fixing z this shows that $\mathsf{f}(y|w) = c\mathsf{f}(y)$ for some constant $c = \mathsf{f}(z|w)/\mathsf{f}(z)$, which must be one so that the densities $\mathsf{f}(y|w)$, $\mathsf{f}(y)$ integrate to unity. Insert this in (A.2) to get the desired right hand side of (A.1).

$\Leftarrow$: We prove the first left right hand side statement. Note that

$$\mathsf{f}(y|z,w) = \mathsf{f}(y,z|w)/\mathsf{f}(z|w).$$

The right hand side of (A.1) shows $\mathsf{f}(y,z|w) = \mathsf{f}(y,z)$. Integrate over y to get $\mathsf{f}(z|w) = \mathsf{f}(z)$. Insert these statements above to get

$$\mathsf{f}(y|z,w) = \mathsf{f}(y,z)/\mathsf{f}(z) = \mathsf{f}(y|z),$$

as desired. The other left hand side statement is proved in a similar fashion. ■

Proof of Theorem 2.1. Condition (2.10) shows $\mathsf{f}(y|z,w)=\mathsf{f}(y|z)$. Thus

$$\mathsf{f}(y,z|w)=\mathsf{f}(y|z,w)\mathsf{f}(z|w)=\mathsf{f}(y|z)\mathsf{f}(z|w). \tag{A.3}$$

First, rearrange to get

$$\mathsf{f}(y|z,w)=\mathsf{f}(y,z|w)/\mathsf{f}(z|w)=\mathsf{f}(y|z). \tag{A.4}$$

Then, note that Condition (2.11) has $\mathsf{f}(z|w)\neq\mathsf{f}(z)$. Insert this in (A.3) to get

$$\mathsf{f}(y,z|w)\neq\mathsf{f}(y|z)\mathsf{f}(z)=\mathsf{f}(y,z). \tag{A.5}$$

Now apply Lemma A.1. The first statement on the left of (A.1) holds through (A.4), while the right hand side fails through (A.5). Thus, the second statement on the left hand side of (A.1) fails as desired. ■

Proof of Theorem 2.2. Combine Theorem 2.1 and Definition 2.1. ■

Proof of Lemma 2.1. Consider the compound distribution integral (2.17).
If $\mathsf{f}(z|w)=\mathsf{f}(z)$, then $\mathsf{f}(y|w)=\int\mathsf{f}(y|z)\mathsf{f}(z)dz=\int\mathsf{f}(y,z)dz=\mathsf{f}(y)$.
If $\mathsf{f}(y|z)=\mathsf{f}(y)$, then $\mathsf{f}(y|w)=\int\mathsf{f}(y)\mathsf{f}(z|w)dz=\mathsf{f}(y)\int\mathsf{f}(z|w)dz=\mathsf{f}(y)$. ■

Proof of Lemma 2.2. Proof by contradiction. When z is binary the compound integral (2.17) reduces to $\mathsf{f}(y|w)=\mathsf{f}(y|z=0)p_w+\mathsf{f}(y|z=1)(1-p_w)$. If $y\perp\!\!\!\perp w$ then $D_{w,w^\dagger}=\mathsf{f}(y|w)-\mathsf{f}(y|w^\dagger)=0$ for all $y,w,w^\dagger$. Combine to get $D_{w,w^\dagger}=\{\mathsf{f}(y|z=0)-\mathsf{f}(y|z=1)\}(p_w-p_{w^\dagger})=0$. With the independence structure y and w vary in a product space. Thus $p_w=p_{w^\dagger}$ so $z\perp\!\!\!\perp w$ or $\mathsf{f}(y|z=0)=\mathsf{f}(y|z=1)$ so $y\perp\!\!\!\perp z$. ■

Proof of Lemma 2.3. Referring to equations (2.3), (2.4) the Markov assumption implies $0=\gamma_{yw\cdot z}=\gamma_{yw}-\gamma_{yz}\gamma_{zw}$. Thus, if $\gamma_{yz}\neq 0$ and $\gamma_{zw}\neq 0$ then $\gamma_{yw}\neq 0$. ■

Proof of Lemma 2.4. We show that $\mathsf{f}(y=0|w)$ is strictly decreasing in w if and only if $\gamma_{yz}\neq 0$ and $\gamma_{zw}\neq 0$. The partial derivatives of the normal density $\mathsf{f}(z|w)$ and the logit/probit probabilities $\mathsf{f}(y=0|w)$ satisfy, using D as partial derivative symbol,

$$\begin{aligned} & D_w\mathsf{f}(z|w) &=& (-\gamma_{zw}/\sigma_z^2)D_z\mathsf{f}(z|w), \\ \text{logit:}\quad & D_z\mathsf{f}(y=0|z) &=& -\gamma_{yz}\mathsf{f}(y=0|z)\left\{1-\mathsf{f}(y=0|z)\right\}, \\ \text{probit:}\quad & D_z\mathsf{f}(y=0|z) &=& -\gamma_{yz}\phi(\gamma_{yz}z), \end{aligned}$$

which are bounded. We can then differentiate the probability $\mathsf{f}(y=0|w)$ and use integration by parts to get

$$\begin{aligned} D_w\mathsf{f}(y=0|w) &= \int_{-\infty}^{\infty}\mathsf{f}(y=0|z)D_w\mathsf{f}(z|w)dz \\ &= \frac{-\gamma_{zw}}{\sigma_z^2}\int_{-\infty}^{\infty}\mathsf{f}(y=0|z)D_z\mathsf{f}(z|w)dz \\ &= \frac{\gamma_{zw}}{\sigma_z^2}\int_{-\infty}^{\infty}\mathsf{f}(z|w)D_z\mathsf{f}(y=0|z)dz, \end{aligned}$$

which is zero if and only if $\gamma_{yz}\gamma_{zw}=0$. ■

References

Bårdsen, G., Kolsrud, D., and Nymoen, R. (2012). Forecast robustness in macroeconometric models. Working paper, Norwegian University of Science and Technology.

Cartwright, N. (1989). *Nature's Capacities and Their Measurement*. Clarendon Press, Oxford.

Chow, G. C. (1960). Tests of equality between sets of coefficients in two linear regressions. *Econometrica*, 28:591–605.

Cox, D. W. and Wermuth, N. (2004). Causality: a statistical view. *International Statistical Review*, 72:285–305.

Dawid, A. P. (1980). Conditional independence for statistical operations. *Annals of Statistics*, 8:598–617.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, 50:987–1108.

Engle, R. F., Hendry, D. F., and Richard, J.-F. (1983). Exogeneity. *Econometrica*, 51:277–304.

Godfrey, L. G. (1978). Testing against general autoregressive and moving average error models when the regressors include lagged dependent variables. *Econometrica*, 46:1293–1302.

Granger, C. W. J. (1969). Investigating Causal Relations by Econometric Models and Cross-Spectral Methods. *Econometrica*, 37:424–38.

Hendry, D. and Mizon, G. (1993). Evaluating dynamic econometric models by encompassing the var. In Phillips, P. C., editor, *Models, Methods and Applications of Econometrics*, pages 272–300. Blackwell, Oxford.

Hendry, D. F. and Doornik, J. A. (2014). *Empirical Model Discovery and Theory Evaluation: Automatic Selection Methods in Econometrics*. MIT Press, Cambridge, MA.

Hendry, D. F. and Nielsen, B. (2007). *Econometric Modeling: A Likelihood Approach*. Princeton University Press, Princeton-Oxford.

Hendry, D. F. and Santos, C. (2010). An automatic test of super exogeneity. In Bollerslev, T., Russell, J., and Watson, M. W., editors, *Volatility and Time Series Econometrics: Essays in Honor of Robert F. Engle*, pages 164–193. Oxford University Press, Oxford.

Imbens, G. W. (2014). Instrumental variables: an econometrician's perspective. *Statistical Science*, 29:323–358.

Johansen, S. (1995). *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models*. Oxford University Press, Oxford.

Johansen, S. and Nielsen, B. (2015). Outlier detection algorithms for least squares time series. *Scandinavian Journal of Statistics*. To appear.

Kilian, L. and Demiroglu, U. (2000). Residual based tests for normality in autoregressions: asymptotic theory and simulations. *Journal of Business & Economic Statististics*, 18:40–50.

Lauritzen, S. L. (1996). *Graphical Models*. Oxford University Press, Oxford.

Lucas, R. J. (1976). Econometric policy evaluation: A critique. *Carnegie-Rochester Conference Series on Public Policy, Elsevier*, 1:19–46.

MATLAB (2014). *Version 8.4.0 (R2014b)*. The MathWorks Inc., Natick, MA.

Nielsen, B. (2006). Order determination in general vector autoregressions. In H. C. Ho, C. K. I. . T. L. L., editor, *Time series and related topics: In memory of Ching-Zong Wei*, pages 93–112. IMS Lecture Notes and Monograph Series 52.

Nielsen, B. and Whitby, A. (2015). A joint chow test for structural instabiliy. *Econometrics*, 3:156–186.

Pearl, J. (2000). *Causality: Models, Reasoning, and Inference*. Cambridge University Press, Cambridge.

Romer, C. D. and Romer, D. H. (2010). The macroeconomic effects of tax changes: Estimates based on a new measure of fiscal shocks. *American Economic Review*, 100:763–801.

Sims, C. A. (1980). Macroeconomics and Reality. *Econometrica*, 48:1–48.

Wermuth, N. and Sadeghi, K. (2012). Sequences of regressions and their independences. *TEST*, 21:215–252.

White, H. (1980). A heteroskedastic-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica*, 48:817–838.